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ON THE EXPONENTIAL DIOPHANTINE EQUATION $x^y + y^x = z^z$

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Abstract. For any positive integer D which is not a square, let (u_1, v_1) be the least positive integer solution of the Pell equation $u^2 - Dv^2 = 1$, and let $h(4D)$ denote the class number of binary quadratic primitive forms of discriminant $4D$. If D satisfies $2 \nmid D$ and $v_1 h(4D) \equiv 0 \pmod{D}$, then D is called a singular number. In this paper, we prove that if (x, y, z) is a positive integer solution of the equation $x^y + y^x = z^z$ with $2 \mid z$, then maximum $\max\{x, y, z\} < 480000$ and both x, y are singular numbers. Thus, one can possibly prove that the equation has no positive integer solutions (x, y, z).

Keywords: exponential diophantine equation; upper bound for solutions; singular number

MSC 2010: 11D61

1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$ be the sets of all integers and positive integers, respectively. In recent years, the solutions of circulant exponential diophantine equations have been investigated in many papers (see [7], [8], [9], [14], [15], [16]). In 2013, using upper bounds of linear forms in p-adic logarithms, Zhang, Luo and Yuan in [15] proved that the equation

$$x^y + y^x = z^z, \quad x, y, z \in \mathbb{N}, \tag{1.1}$$

has only finitely many solutions (x, y, z), and all solutions (x, y, z) of (1.1) satisfy $z < 2.8 \times 10^9$. In addition, they proposed the following conjecture:

Conjecture. *The equation* (1.1) *has no solution* (x, y, z).

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Obviously, the upper bound given in [15] is far too large for any practical purpose. In 2014, Deng and Zhang in [5] proved that (1.1) has no solutions (x, y, z) with x and y being odd primes. Very recently, Wu in [13] proved that (1.1) has no solutions (x, y, z) with $2 \nmid z$. His proof relied upon a deep result concerning the existence of primitive divisors of Lucas and Lehmer numbers due to Bilu, Hanrot and Voutier, see [1].

In this paper we shall discuss the solutions of (1.1) with $2 \mid z$. This is the remaining and the more difficult part of (1.1). First we give a better upper bound for the solutions of (1.1) as follows:

Theorem 1.1. *All solutions (x, y, z) of (1.1) with $2 \mid z$ satisfy* $\max\{x, y, z\} < 480000$.

Let D be a positive integer which is not a square. It is well known that the Pell equation

$$u^2 - Dv^2 = 1, \quad u, v \in \mathbb{Z}, \tag{1.2}$$

has positive integer solutions (u, v). Further, let (u_1, v_1) be the least positive integer solution of (1.2), and let $h(4D)$ denote the class number of binary quadratic primitive forms of discriminant $4D$. If D satisfies

$$2 \nmid D, \quad v_1 h(4D) \equiv 0 \pmod{D}, \tag{1.3}$$

then D is called a singular number. We give a relationship between the solutions of (1.1) and singular numbers as follows:

Theorem 1.2. *If (x, y, z) is a solution of (1.1) with $2 \mid z$, then both x and y are singular numbers.*

Thus, combining the computational results of $h(4D)$ and v_1 (see [3], [10], [12]) with our theorems, one can possibly verify the above mentioned conjecture.

2. Proof of Theorem 1.1

Lemma 2.1. *Let a_1, a_2 be coprime nonzero integers with $a_1 \equiv a_2 \equiv 1 \pmod 4$, and let b_1, b_2 be positive integers. Further, let $\Lambda = a_1^{b_1} - a_2^{b_2}$, and let $v_2(\Lambda)$ denote the degree of 2 in Λ. If $\min\{|a_1|, |a_2|\} > 3$, then we have*

$$\begin{aligned} v_2(\Lambda) &< 19.5540(\log|a_1|)(\log|a_2|) \\ &\quad \times \left(\max\left\{12\log 2, 0.4 + \log(2\log 2) + \log\left(\frac{b_1}{\log|a_2|} + \frac{b_2}{\log|a_1|}\right)\right\}\right)^2. \end{aligned}$$

P r o o f. This is a special case of Theorem 2 of [4] for $p = 2$. Since $\min\{|a_1|,|a_2|\} > 3$ and $a_1 \equiv a_2 \equiv 1 \pmod 4$, we have $\min\{|a_1|,|a_2|\} > 3$. Therefore, we may choose that $E = 2$, $g = 1$ and $\log A_i = \log|a_i|$ for $i = 1,2$. Thus, by the theorem, we get

$$\begin{aligned} v_2(\Lambda) &\leqslant \frac{36.1g}{E^3(\log 2)^4}(\log A_1)(\log A_2) \\ &\quad \times \Bigl(\max\Bigl\{5, 6E\log 2, 0.4 + \log(E\log 2) + \log\Bigl(\frac{b_1}{\log A_2} + \frac{b_2}{\log A_1}\Bigr)\Bigr\}\Bigr)^2 \\ &< 19.5540(\log|a_1|)(\log|a_2|) \\ &\quad \times \Bigl(\max\Bigl\{12\log 2, 0.4 + \log(2\log 2) + \log\Bigl(\frac{b_1}{\log|a_2|} + \frac{b_2}{\log|a_1|}\Bigr)\Bigr\}\Bigr)^2. \end{aligned}$$

The lemma is proved. □

P r o o f of Theorem 1.1. By [15], if (x,y,z) is a solution of (1.1), then we have

$$\min\{x,y,z\} > 1 \tag{2.1}$$

and

$$\gcd(x,y) = \gcd(x,z) = \gcd(y,z) = 1. \tag{2.2}$$

We now assume that (x,y,z) is a solution of (1.1) with $2 \mid z$. By (2.2), we have

$$2 \nmid x, \quad 2 \nmid y. \tag{2.3}$$

Without loss of generality, we may assume that $x \leqslant y$. Then, by [5] and [15], we have

$$3 < x < z < y. \tag{2.4}$$

Further, since $z^z > x^y$ by (1.1), we get

$$y\log x < z\log z. \tag{2.5}$$

On the other hand, we see from (1.1) and (2.3) that

$$0 \equiv z^z \equiv x^y + y^x \equiv x + y \pmod 4. \tag{2.6}$$

Let

$$(a_1,a_2,b_1,b_2) = \begin{cases} (x,-y,y,x) & \text{if } x \equiv 1 \pmod 4, \\ (y,-x,x,y) & \text{if } x \equiv 3 \pmod 4. \end{cases} \tag{2.7}$$

By (2.6) and (2.7), we have $a_1 \equiv a_2 \equiv 1 \pmod 4$. Further, let $\Lambda = a_1^{b_1} - a_2^{b_2}$, and let $v_2(\Lambda)$ denote the degree of 2 in Λ. By (1.1) and (2.7), we have $\Lambda = x^y + y^x = z^z$ and $v_2(\Lambda) \geqslant z$. Therefore by (2.4), using Lemma 2.1, we get

$$\begin{aligned} z &< 19.5540(\log x)(\log y) \\ &\quad \times \left(\max\left\{12\log 2, 0.4 + \log(2\log 2) + \log\left(\frac{x}{\log x} + \frac{y}{\log y}\right)\right\}\right)^2. \end{aligned} \tag{2.8}$$

If $12\log 2 \geqslant 0.4 + \log(2\log 2) + \log(x/\log x + y/\log y)$, then we have $2000 > \mathrm{e}^{7.591} > y/\log y$ and $y < 25000$. Therefore, by (2.4), the theorem holds.

If $12\log 2 < 0.4 + \log(2\log 2) + \log(x/\log x + y/\log y)$, then from (2.8) we get

$$z < 19.5540(\log x)(\log y)\left(0.7271 + \log\left(\frac{x}{\log x} + \frac{y}{\log y}\right)\right)^2. \tag{2.9}$$

Notice that $r/\log r$ is increasing for any real number r with $r > \mathrm{e}$. By (2.4), we have $x/\log x < y/\log y$, and by (2.9), we get

$$z < 19.5540(\log x)(\log y)\left(0.7271 + \log\left(\frac{2y}{\log y}\right)\right)^2. \tag{2.10}$$

Further, by (2.4), (2.5) and (2.10), we have

$$\begin{aligned} y &< \frac{z\log z}{\log x} < 19.5540(\log y)(\log z)\left(0.7271 + \log\left(\frac{2y}{\log y}\right)\right)^2 \\ &< 19.5540(\log y)^2\left(0.7271 + \log\left(\frac{2y}{\log y}\right)\right)^2, \end{aligned}$$

whence we conclude that $y < 480000$. Thus, by (2.4), the theorem is proved. □

3. Proof of Theorem 1.2

Lemma 3.1 ([2]). *If X, Y, n are positive integers such that $X > Y$, $\gcd(X, Y) = 1$ and $n > 6$, then $X^n - Y^n$ has a prime divisor p with $p > n$.*

Lemma 3.2 ([11], Theorem 8.1). *Every solution (u, v) of (1.2) can be expressed as*

$$u + v\sqrt{D} = \lambda_1\left(u_1 + \lambda_2 v_1\sqrt{D}\right)^s, \quad \lambda_1, \lambda_2 \in \{\pm 1\},\ s \in \mathbb{Z},\ s \geqslant 0,$$

where (u_1, v_1) is the least positive integer solution of (1.2).

Lemma 3.3 ([6], Theorem 1 and 2). *Let D, k be positive integers such that D is not a square, $k > 1$, $2 \nmid k$ and $\gcd(D, k) = 1$. Every solution (X, Y, Z) of the equation*

$$X^2 - DY^2 = k^Z, \quad X, Y, Z \in \mathbb{Z},\ \gcd(X, Y) = 1,\ Z > 0$$

can be expressed as

$$X + Y\sqrt{D} = \bigl(X_1 + Y_1\sqrt{D}\bigr)^t \bigl(u + v\sqrt{D}\bigr), \quad Z = Z_1 t,\ t \in \mathbb{N},$$

where X_1, Y_1, Z_1 are positive integers satisfying

$$X_1^2 - DY_1^2 = k^{Z_1}, \quad \gcd(X_1, Y_1) = 1,\ h(4D) \equiv 0 \pmod{Z_1},$$

(u, v) is a solution of (1.2).

P r o o f of Theorem 1.2. We now assume that (x, y, z) is a solution of (1.1) with $2 \mid z$. If x is a square, then from (2.1) and (2.3) we get $x = a^2$, where a is an odd integer with $a \geqslant 3$. Substituting it into (1.1), by (2.2), we have

$$z^{z/2} + a^y = b^{a^2}, \quad z^{z/2} - a^y = c^{a^2}, \quad y = bc,\ b, c \in \mathbb{N},\ \gcd(b, c) = 1, \tag{3.1}$$

whence we get

$$2a^y = b^{a^2} - c^{a^2}. \tag{3.2}$$

However, since $a^2 \geqslant 9$, by Lemma 3.1, $b^{a^2} - c^{a^2}$ has a prime divisor p with $p > a^2$ and (3.2) is false. It implies that x is not a square. Similarly, we can prove that y is not a square.

We see from (1.1) and (2.3) that the equation

$$X^2 - yY^2 = x^Z, \quad X, Y, Z \in \mathbb{Z},\ \gcd(X, Y) = 1,\ Z > 0 \tag{3.3}$$

has the solution

$$(X, Y, Z) = (z^{z/2}, y^{(x-1)/2}, y). \tag{3.4}$$

Recall that $x > 1$, $2 \nmid x$, $\gcd(x, y) = 1$ and y is not a square. Applying Lemma 3.3 to (3.3) and (3.4), we have

$$y = Z_1 t, \quad t \in \mathbb{N}, \tag{3.5}$$

$$z^{z/2} + y^{(x-1)/2}\sqrt{y} = (X_1 + Y_1\sqrt{y})^t (u + v\sqrt{y}), \tag{3.6}$$

where X_1, Y_1, Z_1 are positive integers satisfying

$$X_1^2 - yY_1^2 = x^{Z_1}, \quad \gcd(X_1, Y_1) = 1, \tag{3.7}$$

$$h(4y) \equiv 0 \pmod{Z_1}, \tag{3.8}$$

(u, v) is a solution of the Pell equation

$$u^2 - yv^2 = 1, \quad u, v \in \mathbb{Z}. \tag{3.9}$$

Since $z^{z/2} + y^{(x-1)/2}\sqrt{y} > 0$ and $X_1 + Y_1\sqrt{y} > 0$, by Lemma 3.2, we get from (3.6) that

$$u + v\sqrt{y} = (u_1 + \lambda v_1\sqrt{y})^s, \quad \lambda \in \{\pm 1\},\ s \in \mathbb{Z},\ s \geqslant 0, \tag{3.10}$$

where (u_1, v_1) is the least positive integer solution of (3.9). Substituting (3.10) into (3.6), we have

$$z^{z/2} + y^{(x-1)/2}\sqrt{y} = (X_1 + Y_1\sqrt{y})^t(u_1 + \lambda v_1\sqrt{y})^s. \tag{3.11}$$

Let $d = \gcd(s, t)$. If $d > 1$, since $2 \nmid t$ by (2.3) and (3.5), then d has an odd prime divisor p. Further, let

$$f + g\sqrt{y} = (X_1 + Y_1\sqrt{y})^{t/p}(u_1 + \lambda v_1\sqrt{y})^{s/p}. \tag{3.12}$$

By Lemmas 3.2 and 3.3, we see from (3.5), (3.7) and (3.12) that f, g are integers satisfying

$$f^2 - yg^2 = x^{y/p}, \quad \gcd(f, g) = 1. \tag{3.13}$$

Substituting (3.12) into (3.11), we have

$$z^{z/2} + y^{(x-1)/2}\sqrt{y} = (f + g\sqrt{y})^p, \tag{3.14}$$

whence we get

$$y^{(x-1)/2} = g\sum_{i=0}^{(p-1)/2}\binom{p}{2i+1}f^{p-2i-1}(yg^2)^i. \tag{3.15}$$

When $p = 3$, by (3.5), we have $3 \mid y$ and

$$y = 3l, \quad l \in \mathbb{N}. \tag{3.16}$$

Further, by (3.15) and (3.16), we get

$$3^{(x-3)/2} l^{(x-1)/2} = g(f^2 + lg^2). \tag{3.17}$$

Since $\gcd(f, yg) = 1$ by (3.13), we have $\gcd(f, lg) = \gcd(f^2 + lg^2, l) = 1$. Hence, by (3.17), we get

$$f^2 + lg^2 \leqslant 3^{(x-3)/2}, \quad l^{(x-1)/2} \leqslant g. \tag{3.18}$$

We find from (3.18) that $l = 1$. Substituting it into (3.17), we have

$$3^{(x-3)/2} = g(f^2 + g^2). \tag{3.19}$$

But, since $f^2 + g^2 > 1$ and $3 \nmid f^2 + g^2$, (3.19) is false.

When $p > 3$, since $p \mid y$ and $\gcd(f, y) = 1$, we have

$$\binom{p}{2i+1} f^{p-2i-1}(yg^2)^i \equiv 0 \pmod{p^2}, \quad i = 1, \ldots, \frac{p-1}{2}, \tag{3.20}$$

$$p \,\|\, \sum_{i=0}^{(p-1)/2} \binom{p}{2i+1} f^{p-2i-1}(yg^2)^i \tag{3.21}$$

and

$$\gcd\left(y, \frac{1}{p} \sum_{i=0}^{(p-1)/2} \binom{p}{2i+1} f^{p-2i-1}(yg^2)^i\right) = 1. \tag{3.22}$$

Hence, we see from (3.15) and (3.22) that

$$p = \sum_{i=0}^{(p-1)/2} \binom{p}{2i+1} f^{p-2i-1}(yg^2)^i > p, \tag{3.23}$$

a contradiction. Therefore, we obtain

$$\gcd(s, t) = 1. \tag{3.24}$$

Let

$$X + Y\sqrt{y} = (X_1 + Y_1\sqrt{y})^t. \tag{3.25}$$

Since $2 \nmid t$, by (3.10) and (3.25), we have

$$
\begin{aligned}
(3.26)\quad u &= \sum_{i=0}^{[s/2]} \binom{s}{2i} u_1^{s-2i} (y v_1^2)^i, \qquad v = \lambda v_1 \sum_{i=0}^{[(s-1)/2]} \binom{s}{2i+1} u_1^{s-2i-1} (y v_1^2)^i, \\
X &= \sum_{i=0}^{(t-1)/2} \binom{t}{2i} X_1^{t-2i} (y Y_1^2)^i, \quad Y = Y_1 \sum_{i=0}^{(t-1)/2} \binom{t}{2i+1} X_1^{t-2i-1} (y Y_1^2)^i,
\end{aligned}
$$

where $[s/2]$ and $[(s-1)/2]$ are integer parts of $s/2$ and $(s-1)/2$, respectively. Substituting (3.25) into (3.6), we have

$$z^{z/2} + y^{(x-1)/2}\sqrt{y} = (X + Y\sqrt{y})(u + v\sqrt{y}),$$

whence we get

$$y^{(x-1)/2} = Xv + Yu. \tag{3.27}$$

By (3.26), we have

$$
\begin{aligned}
u &\equiv u_1^s \pmod{y}, \quad v \equiv \lambda s u_1^{s-1} v_1 \pmod{y}, \\
X &\equiv X_1^t \pmod{y}, \quad Y \equiv t X_1^{t-1} Y_1 \pmod{y}.
\end{aligned} \tag{3.28}
$$

Since $x > 1$, by (3.27) and (3.28), we get

$$D \equiv y^{(x-1)/2} \equiv Xv + Yu \equiv X_1^t(\lambda s u_1^{s-1} v_1) + t X_1^{t-1} Y_1 (u_1^s) \pmod{y}. \tag{3.29}$$

Further, by (3.7) and (3.9), we have $\gcd(X_1, y) = \gcd(u_1, y) = 1$. We see from (3.29) that

$$\lambda s X_1 v_1 + t Y_1 u_1 \equiv 0 \pmod{y}. \tag{3.30}$$

Furthermore, since $t \mid y$ by (3.5), we obtain from (3.24) and (3.30) that

$$v_1 \equiv 0 \pmod{t}. \tag{3.31}$$

Therefore, the combination of (3.5), (3.8) and (3.31) yields

$$v_1 h(4y) \equiv 0 \pmod{y}.$$

It implies that y is a singular number.

By the symmetry of x and y in (1.1), using the same method as above, we can prove that x is a singular number too. Thus, the theorem is proved. □

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