EXTENSIONS OF HOM-LIE ALGEBRAS IN TERMS OF COHOMOLOGY

Abdoreza R. Armakan, Mohammed Reza Farhangdoost, Shiraz

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Abstract. We study (non-abelian) extensions of a given hom-Lie algebra and provide a geometrical interpretation of extensions, in particular, we characterize an extension of a hom-Lie algebra $\mathfrak{g}$ by another hom-Lie algebra $\mathfrak{h}$ and discuss the case where $\mathfrak{h}$ has no center. We also deal with the setting of covariant exterior derivatives, Chevalley derivative, Maurer-Cartan formula, curvature and the Bianchi identity for the possible extensions in differential geometry. Moreover, we find a cohomological obstruction to the existence of extensions of hom-Lie algebras, i.e., we show that in order to have an extendible hom-Lie algebra, there should exist a trivial member of the third cohomology.

Keywords: hom-Lie algebras; cohomology of hom-Lie algebras; extensions of hom-Lie algebras

MSC 2010: 17B99, 55U15

1. Introduction

The notion of hom-Lie algebras was first introduced by Hartwig, Larsson, and Silvestrov in [5], while they where studying deformations of the Witt and the Virasoro algebras. In a hom-Lie algebra, the Jacobi identity is twisted with an additional linear map, which is called the hom-Jacobi identity, see [9], [11]. In recent years, Makhlouf, Silvestrov, Sheng and other authors have studied different aspects of hom-Lie algebras, see [1], [3]–[5], [7], [8], [10], [12], [13]. The problem of group extensions in terms of cohomology is well known. Here, we would like to extend it to the hom-Lie algebras by a more geometric method. The cohomology of hom-Lie algebras is introduced in [1], [9], independently.

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In the first section, hom-Lie algebras and some of their useful related definitions are presented. In the second section we introduce hom-Lie algebra extensions with more geometric aspects and finally, in the third section, introducing the Chevalley cohomology for hom-Lie algebras, we find a cohomological obstruction to the existence of extensions.

Definition 1.1 ([11]). A hom-Lie algebra is a triple $(\mathfrak{g}, [,], \alpha)$, where $\mathfrak{g}$ is a vector space equipped with a skew-symmetric bilinear map $[,]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and a linear map $\alpha\colon \mathfrak{g} \to \mathfrak{g}$ such that

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0$$

for all $x, y, z \in \mathfrak{g}$, which is called the hom-Jacobi identity.

A hom-Lie algebra is called a multiplicative hom-Lie algebra if α is an algebraic morphism, i.e. for any $x, y \in \mathfrak{g}$,

$$\alpha([x, y]) = [\alpha(x), \alpha(y)].$$

A hom-Lie algebra is called regular if α is an automorphism.

A sub-vector space $\mathfrak{h} \subset \mathfrak{g}$ is a hom-Lie sub-algebra of $(\mathfrak{g}, [,], \alpha)$ if $\alpha(\mathfrak{h}) \subset \mathfrak{h}$ and $\mathfrak{h}$ is closed under the bracket operation, i.e.

$$[x_1, x_2]_g \in \mathfrak{h}$$

for all $x_1, x_2 \in \mathfrak{h}$. Let $(\mathfrak{g}, [,], \alpha)$ be a multiplicative hom-Lie algebra. For any nonnegative integer k let α^k denote the k-times composition of α by itself, i.e.

$$\alpha^k = \alpha \circ \ldots \circ \alpha \ (k \text{ times}),$$

where we define $\alpha^0 = \mathrm{Id}$ and $\alpha^1 = \alpha$. If $\mathfrak{g}$ is a regular hom-Lie algebra, let

$$\alpha^{-k} = \alpha^{-1} \circ \ldots \circ \alpha^{-1} \ (k \text{ times}).$$

Definition 1.2 ([9]). For any nonnegative integer k, a linear map $D\colon \mathfrak{g} \to \mathfrak{g}$ is called an α^k-derivation of the multiplicative hom-Lie algebra $(\mathfrak{g}, [,], \alpha)$, if

(i) $[D, \alpha] = 0$, i.e. $D \circ \alpha = \alpha \circ D$,
(ii) $D[x, y]_g = [D(x), \alpha^k(y)]_{\mathfrak{g}} + [\alpha^k(x), D(y)]_{\mathfrak{g}}$ for all $x, y \in \mathfrak{g}$.

Denote by $\mathrm{Der}_{\alpha^k}(\mathfrak{g})$ the set of all α^k-derivations of the multiplicative hom-Lie algebra $(\mathfrak{g}, [,], \alpha)$.

Definition 1.3 ([9]). For any $x \in \mathfrak{g}$ satisfying $\alpha(x) = x$, define $D_k(x)\colon \mathfrak{g} \to \mathfrak{g}$ by

$$D_k(x)(y) = [\alpha^k(y), x]_{\mathfrak{g}}$$

for all $y \in \mathfrak{g}$.

It is shown in [9] that $D_k(x)$ is an α^{k+1}-derivation, which is called an inner α^k-derivation. So

$$\mathrm{Inn}_{\alpha^k}(\mathfrak{g}) = \{[\alpha^{k-1}(.), x]_{\mathfrak{g}}\colon\ x \in \mathfrak{g},\ \alpha(x) = x\}.$$

It is also shown that

$$\mathrm{Der}(\mathfrak{g}) = \bigoplus_{k \geqslant 0} \mathrm{Der}_{\alpha^k}(\mathfrak{g})$$

is a Lie algebra.

2. Extensions of hom-Lie algebras

In this section we clarify what we mean by an extension of a hom-Lie algebra. Although it is shown in [9] that extensions of a given hom-Lie algebra is characterized by elements of its second cohomology group, we concentrate on some geometric aspects here.

Definition 2.1. Let $\mathfrak{g}$, $\mathfrak{h}$ be two hom-Lie algebras. We call $\mathfrak{e}$ an extension of the hom-Lie algebra $\mathfrak{g}$ by $\mathfrak{h}$, if there exists a short exact sequence

$$0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0$$

of hom-Lie algebras and their morphisms.

We want to study the possible extensions, so suppose there exists an extension

$$0 \longrightarrow \mathfrak{h} \xrightarrow{\ i\ } \mathfrak{e} \xrightarrow{\ p\ } \mathfrak{g} \longrightarrow 0\ ,$$

and let $s\colon \mathfrak{g} \to \mathfrak{e}$ be such that $p \circ s = \mathrm{Id}_{\mathfrak{g}}$. We define

$$\begin{gathered} \varphi\colon\ \mathfrak{g} \to \mathrm{Der}_{\alpha^k}(\mathfrak{h}), \\ \varphi_x(y) = [\alpha^k(s(x)), y], \end{gathered} \tag{2.1}$$

and

$$\begin{gathered} \varrho\colon \bigwedge^2 \mathfrak{g} \to \mathfrak{h}, \\ \varrho(x, y) = [s(x), s(y)] - s([x, y]). \end{gathered} \tag{2.2}$$

Recalling that $\mathrm{ad}_k(u)(v)$ for all u, v in $\mathfrak{g}$ is defined in [9] as an α^k-adjoint representation, we have the following lemma.

Lemma 2.2. *The maps φ and ϱ defined in* (2.2) *and* (2.1) *satisfy*

$$[\varphi_x, \varphi_y] - \varphi_{[x,y]} = \mathrm{ad}_k(\varrho(x, y)), \tag{2.3}$$

$$\sum_{\mathrm{cyclic}\{x,y,z\}} (\varphi_x \varrho(y, z) - \varrho([x, y], z)) = 0. \tag{2.4}$$

P r o o f. First note that $\varphi_x = \mathrm{ad}_k(s(x))$. So we have

$$\begin{aligned}[\varphi_x, \varphi_y] &= [\mathrm{ad}_k(s(x)), \mathrm{ad}_k(s(y))] - \mathrm{ad}_k(s([x, y]))\\ &= \mathrm{ad}_k([s(x), s(y)] - s([x, y])) = \mathrm{ad}_k(\varrho(x, y)).\end{aligned}$$

For the second equation we have

$$\begin{aligned}&\sum_{\mathrm{cyclic}\{x,y,z\}} (\varphi_x \varrho(y, z) - \varrho([x, y], z))\\ &= \varphi_x \varrho(y, z) - \varrho([x, y], z) + \varphi_y \varrho(z, x) - \varrho([y, z], x) + \varphi_z \varrho(x, y) - \varrho([z, x], y)\\ &= [\alpha^k(s(x)), [s(y), s(z)]] - [\alpha^k(s(x)), s([y, z])] - [s([x, y]), s(z)] + s[[x, y], z]\\ &\quad + [\alpha^k(s(y)), [s(z), s(x)]] - [\alpha^k(s(y)), s([z, x])] - [s([y, z]), s(x)] + s[[y, z], x]\\ &\quad + [\alpha^k(s(z)), [s(x), s(y)]] - [\alpha^k(s(z)), s([x, y])] - [s([z, x]), s(y)] + s[[z, x], y] = 0.\end{aligned}$$

□

Therefore, using φ and ϱ which satisfy (2.3) and (2.4), the hom-Lie algebra structure on $\mathfrak{e} = \mathfrak{h} \oplus s(\mathfrak{g})$ will be in the form

$$[y_1 + s(x_1), y_2 + s(x_2)] = [y_1, y_2] + \varphi_{x_1} y_2 - \varphi_{x_2} y_1 + \varrho(x_1, x_2) + s([x_1, x_2]).$$

Definition 2.3. For a linear space V, the space of p-linear skew symmetric maps $\mathfrak{g} \to V$ is denoted by $A^p(\mathfrak{g}, V)$. The Chevalley derivative is defined by

$$\begin{gathered} d\colon\ A^p(\mathfrak{g}, V) \to A^{p+1}(\mathfrak{g}, V),\\ d\phi(x_0, \ldots, x_p) = \sum_{i<j} (-1)^{i+j} \phi([x_i, x_j], x_0, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_p). \end{gathered} \tag{2.5}$$

Definition 2.4. For a linear space W and a hom-Lie algebra $\mathfrak{f}$, the super Lie bracket on

$$A * (W, \mathfrak{f}) = \bigoplus_{p\in\mathbb{N}} A^p(\mathfrak{g}, \mathfrak{f})$$

is defined by

$$[\zeta, \xi]_\wedge = \frac{1}{p!q!} \sum_\sigma [\zeta(x_{\sigma_1}, \ldots, x_{\sigma_p}), \xi(x_{\sigma_{p+1}}, \ldots,)]_{\mathfrak{f}}.$$

Noting these definitions, we see that the map ϱ given by (2.2) satisfies the Maurer-Cartan formula for curvatures on principal bundles in differential geometry, i.e.

$$\varrho = ds + \frac{1}{2}[s, s]_\wedge.$$

Analogously, (2.3) can be written in the form

$$\mathrm{ad}_k(\varrho) = d\varphi + \frac{1}{2}[\varphi, \varphi]_\wedge.$$

Thus, we can see s as a connection in the sense of the horizontal lift of vector fields on the base of a bundle. Moreover, φ is an induced connection. See [6] for more background information.

Let R be a ring. An R-module M is an additive Abelian group, in which a product is defined between elements of the ring and elements of the module that is distributive over the addition operation of each parameter and is compatible with the ring multiplication, see [2]. In fact for every $\mathrm{Der}_{\alpha^k}(\mathfrak{h})$-module V we put

$$\begin{gathered}\varphi_\wedge\colon\ A^p(\mathfrak{g}, V) \to A^{p+1}(\mathfrak{g}, V),\\ \varphi \wedge \xi(x_0, \ldots, x_p) = \sum_{i=0}^{p} (-1)^i \varphi_{x_i}(\xi(x_0, \ldots, \widehat{x}_i, \ldots, x_p)).\end{gathered}$$

So the exterior covariant derivative will be defined by

$$\begin{gathered}\delta_\varphi\colon\ A^p(\mathfrak{g}, V) \to A^{p+1}(\mathfrak{g}, V),\\ \delta_\varphi \xi = \varphi \wedge \xi + d\xi.\end{gathered}$$

So the formula (2.4) will become the Bianchi identity

$$\delta_\varphi \varrho = 0.$$

Moreover, we deduce that

$$\delta_\varphi \delta_\varphi(\psi) = [\varrho, \psi]_\wedge, \quad \psi \in A^p(\mathfrak{g}, \mathfrak{h}). \tag{2.6}$$

If we put $s' = s + b$ instead of s, where $b\colon\ \mathfrak{g} \to \mathfrak{h}$ is a linear map, we have

$$\varphi'_x(y) = [\alpha^k(s(x) + b(x)), y] = [\alpha^k(s(x)), y] + [\alpha^k(b(x)), y] = \varphi_x + \mathrm{ad}_k^{\mathfrak{h}}(b(x))$$

and

$$\begin{aligned}\varrho'(x, y) &= \varrho(x, y) + \varphi_x b(y) - \varphi_y b(x) - b([x, y]) + [b(x), b(y)]\\ &= \varrho(x, y) + \delta_\varphi b(x, y) + [b(x), b(y)],\end{aligned}$$

i.e.

$$\varrho' = \varrho + \delta_\varphi b + \frac{1}{2}[b,b]_\wedge.$$

Thus, so far we have proved

Theorem 2.5. *Let $\mathfrak{g}$, $\mathfrak{h}$ be two hom-Lie algebras. Extensions of $\mathfrak{g}$ on $\mathfrak{h}$, i.e. the short exact sequences of the form*

$$0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0,$$

are in one to one correspondence with the data of the following form: a linear map $\varphi\colon \mathfrak{g} \to \mathrm{Der}_{\alpha^k}(\mathfrak{h})$ and a skew symmetric bilinear map $\varrho\colon \mathfrak{g}\times\mathfrak{g} \to \mathfrak{h}$, such that

$$[\varphi_x, \varphi_y] - \varphi_{[x,y]} = \mathrm{ad}_k(\varrho(x,y)), \tag{2.7}$$

and

$$\sum_{\mathrm{cyclic}\{x,y,z\}} (\varphi_x \varrho(y,z) - \varrho([x,y],z)) = 0, \tag{2.8}$$

or in other words $\delta_\varphi \varrho = 0$. The extension which corresponds to φ and ϱ is the vector space $\mathfrak{e} = \mathfrak{h} \oplus \mathfrak{g}$ whose hom-Lie algebra structure is given by

$$[y_1 + s(x_1), y_2 + s(x_2)] = ([y_1,y_2] + \varphi_{x_1} y_2 - \varphi_{x_2} y_1 + \varrho(x_1,x_2) + s([x_1,x_2]),$$

and its short exact sequence is

$$0 \longrightarrow \mathfrak{h} \xrightarrow{i_2} \mathfrak{h}\oplus\mathfrak{g} = \mathfrak{e} \xrightarrow{pr_1} \mathfrak{g} \longrightarrow 0\ .$$

Two data (φ, ϱ) and (φ', ϱ') are equivalent if there exists a linear map $b\colon \mathfrak{g} \to \mathfrak{h}$ such that

$$\varphi'_x = \varphi_x + \mathrm{ad}_k^{\mathfrak{h}}(b(x)),$$

and

$$\begin{aligned}\varrho'(x,y) &= \varrho(x,y) + \varphi_x b(y) - \varphi_y b(x) - b([x,y]) + [b(x), b(y)]\\ &= \varrho(x,y) + \delta_\varphi b(x,y) + [b(x), b(y)].\end{aligned}$$

So the corresponding equivalence is

$$\begin{aligned}\mathfrak{e} = \mathfrak{h}\oplus\mathfrak{g} &\to \mathfrak{h}\oplus\mathfrak{g} = \mathfrak{e}',\\ y + x &\mapsto y - b(x) + x.\end{aligned}$$

Moreover, the datum (φ, ϱ) represents a split extension if and only if (φ, ϱ) corresponds to a datum of the form $(\varphi', 0)$ (so that φ' is an isomorphism). In this case there exists a linear map $b\colon \mathfrak{g} \to \mathfrak{h}$ such that

$$\varrho = -\delta_\varphi b - \frac{1}{2}[b, b]_\wedge.$$

Example 2.6. Let $\pi\colon B \to M = B/G$ be a principal bundle with structure group G, i.e., B is a manifold with a free right action of a Lie group G and π is the projection on the orbit space $M = B/G$. Denote by $\mathfrak{g} = \mathfrak{X}(M)$ the hom-Lie algebra of the vector fields on M, by $\mathfrak{e} = \mathfrak{X}(B)^G$ the hom-Lie algebra of G-invariant vector fields on B and by $\mathfrak{X}_v(B)^G$ the ideal of the G-invariant vertical vector fields of $\mathfrak{e}$. We have a natural homomorphism $\pi_*\colon \mathfrak{e} \to \mathfrak{g}$ with the kernel $\mathfrak{h}$, i.e., $\mathfrak{e}$ is an extension of $\mathfrak{g}$ by means of $\mathfrak{h}$.

Note that we have a $C^\infty(M)$-module structure on $\mathfrak{g}, \mathfrak{e}, \mathfrak{h}$. In particular, $\mathfrak{h}$ is a hom-Lie algebra over $C^\infty(M)$. The extension

$$0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0$$

is also an extension of $C^\infty(M)$-modules. Now assume that the section $s\colon \mathfrak{g} \to \mathfrak{e}$ is a homomorphism of $C^\infty(M)$-modules. Then it can be viewed as a connection in the principal bundle π, and the $\mathfrak{h}$-valued 2-form ϱ as its curvature.

Corollary 2.7. *Let $\mathfrak{g}, \mathfrak{h}$ be two hom-Lie algebras such that $\mathfrak{h}$ has no center, i.e. $Z(\mathfrak{h}) = 0$. Then the extensions of $\mathfrak{g}$ by $\mathfrak{h}$ are in one to one correspondence with the isomorphisms of the form*

$$\bar\varphi\colon \mathfrak{g} \to \mathrm{out}(\mathfrak{h}) = \frac{\mathrm{Der}_{\alpha^k}(\mathfrak{h})}{\mathrm{ad}_k(\mathfrak{h})}.$$

Proof. If (φ, ϱ) is a datum, the map $\bar\varphi\colon \mathfrak{g} \to \mathrm{Der}_{\alpha^k}(\mathfrak{h})/\,\mathrm{ad}_k(\mathfrak{h})$ defined by

$$\mathfrak{g} \xrightarrow{\varphi} \mathrm{Der}_{\alpha^k}(\mathfrak{h}) \xrightarrow{\pi} \frac{\mathrm{Der}_{\alpha^k}(\mathfrak{h})}{\mathrm{ad}_k(\mathfrak{h})}\,,$$
$$\bar\varphi = \pi \circ \varphi$$

is a hom-Lie algebra homomorphism, because

$$\begin{aligned}\bar\varphi_{[x,y]} &= \pi(\varphi_{[x,y]}) = \pi([\varphi_x, \varphi_y] - \mathrm{ad}_k(\varrho(x,y))) \\ &= \pi([\varphi_x, \varphi_y]) = [\pi \circ \varphi_x, \pi \circ \varphi_y] = [\bar\varphi_x, \bar\varphi_y].\end{aligned} \tag{2.9}$$

Conversely, suppose we have the map $\bar{\varphi}$. A linear lift $\varphi\colon \mathfrak{g} \to \mathrm{Der}_{\alpha^k}(\mathfrak{h})$ can be considered. Since $\bar{\varphi}$ is a hom-Lie algebra homomorphism and $\mathfrak{h}$ has no center, there exists a skew symmetric unique linear map $\varrho\colon \mathfrak{g}\times\mathfrak{g} \to \mathfrak{h}$ such that

$$[\varphi_x, \varphi_y] - \varphi_{[x,y]} = \mathrm{ad}_k(\varrho(x,y)).$$

So the equation (2.3) is fulfilled. Also it is easy to obtain (2.4). □

3. Cohomological obstruction to existence of extensions

In this section we present a proposition which shows that if there exists a hom-Lie algebra extension, there should be a trivial member of the third cohomology. We have to make some notes first.

Remark 3.1. The hom-Lie algebra $\mathfrak{h}$ is a $\mathrm{Der}_{\alpha^k}(\mathfrak{h})$-module with the multiplication rule

$$\mathrm{Der}(\mathfrak{h}) \times \mathfrak{h} \to \mathfrak{h}, \quad (h, x) \mapsto h(x),$$

and $Z(\mathfrak{h})$ is a submodule of $\mathfrak{h}$ with this multiplication, i.e., $h(x) \in Z(\mathfrak{h})$ for all $x \in Z(\mathfrak{h})$, $h \in \mathrm{Der}_{\alpha^k}(\mathfrak{h})$, since

$$[h(x), y] = h([x,y]) - [x, h(y)] = 0$$

for all $y \in \mathfrak{h}$. Thus $h(x) \in Z(\mathfrak{h})$. Also for all $\bar{h} \in \mathrm{Der}_{\alpha^k}(\mathfrak{h})/\,\mathrm{ad}_k(\mathfrak{h})$, there exists $h \in \mathrm{Der}_{\alpha^k}(\mathfrak{h})$ such that $\bar{h} = [h]$ and one can see $Z(\mathfrak{h})$ as a module on $\mathrm{Der}_{\alpha^k}(\mathfrak{h})/\,\mathrm{ad}_k(\mathfrak{h})$; it is sufficient to define the multiplication for all $x \in Z(\mathfrak{h})$ and $\bar{h} \in \mathrm{Der}_{\alpha^k}(\mathfrak{h})/\,\mathrm{ad}_k(\mathfrak{h})$ in the following way:

$$\bar{h}\cdot x = h(x).$$

Note that this notion is well defined since for $\bar{h} = [h']$ we have $h' = h + \mathrm{ad}_k(a)$, so

$$h'(x) = h(x) + \mathrm{ad}_k(a)(x) = h(x) + [a, x] = h(x),$$

since x is in the center of $\mathfrak{h}$ and $a \in \mathfrak{h}$. Now, using the module structure of $Z(\mathfrak{h})$ on $\mathrm{Der}_{\alpha^k}(\mathfrak{h})/\,\mathrm{ad}_k(\mathfrak{h})$, we can give $\mathfrak{g}$ a module structure by the map $\bar{\varphi}$, i.e., for $c \in \mathfrak{g}$ and $x \in Z(\mathfrak{h})$ we put

$$c\cdot x = \bar{\varphi}(c)\cdot x.$$

Remark 3.2. For hom-Lie algebra homomorphism $\bar{\varphi}\colon \mathfrak{g} \to \mathrm{Der}_{\alpha^k}(\mathfrak{h})/\,\mathrm{ad}_k(\mathfrak{h})$, if V is a vector space which has a $\mathrm{Der}_{\alpha^k}(\mathfrak{h})/\,\mathrm{ad}_k(\mathfrak{h})$-module structure, one can consider the space of all k linear forms on $\mathfrak{g}$ with values in V which is denoted by $\overset{k}{\bigwedge}(\mathfrak{g}, V)$.

We can construct $\delta_{\bar{\varphi}}$ like δ_{φ}. First the exterior multiplication in $\bar{\varphi}$ is defined. For all $\psi \in \bigwedge^{k}(\mathfrak{g}, V)$, $\bar{\varphi} \wedge \psi$ is in $\bigwedge^{k+1}(\mathfrak{g}, V)$ and acts in the following way:

$$(\bar{\varphi} \wedge \psi)(x_0, \ldots, x_k) = \sum_{i=0}^{k} (-1)^i \bar{\varphi}(x_i).\psi(x_0, \ldots, \widehat{x}_i, \ldots, x_k),$$

and

$$\delta_{\bar{\varphi}}\colon \bigwedge^{k}(\mathfrak{g}, V) \to \bigwedge^{k+1}(\mathfrak{g}, V),\ \delta_{\bar{\varphi}}(\psi) = \bar{\varphi} \wedge \psi + d\psi.$$

In the special case where $V = Z(\mathfrak{h})$, $Z(\mathfrak{h})$ is a $\mathrm{Der}_{\alpha^k}(\mathfrak{h})/\operatorname{ad}_k(\mathfrak{h})$-module and if we consider

$$\varphi\colon\ \mathfrak{g} \to \mathrm{Der}_{\alpha^k}(\mathfrak{h})$$

to be such that $\bar{\varphi} = \pi \circ \varphi$, where $\pi\colon\ \mathrm{Der}_{\alpha^k}(\mathfrak{h}) \to \mathrm{Der}_{\alpha^k}(\mathfrak{h})/\operatorname{ad}_k(\mathfrak{h})$, then $Z(\mathfrak{h})$ is also a $\mathrm{Der}_{\alpha^k}(\mathfrak{h})$-module and

$$\delta_{\varphi}\colon\ \bigwedge^{k}(\mathfrak{g}, Z(\mathfrak{h})) \to \bigwedge^{k+1}(\mathfrak{g}, Z(\mathfrak{h}))$$

is defined too and in this case we have $\delta_{\varphi} = \delta_{\bar{\varphi}}$, since $\varphi \wedge \psi = \bar{\varphi} \wedge \psi$. Note that the multiplication rule between $\mathrm{Der}_{\alpha^k}(\mathfrak{h})/\operatorname{ad}_k(\mathfrak{h})$ and $Z(\mathfrak{h})$ is

$$[h] \cdot x = h(x), \quad h \in \mathrm{Der}_{\alpha^k}(\mathfrak{h}).$$

Therefore, since $\bar{\varphi}(x_i) = [\varphi(x_i)]$ we have

$$\begin{aligned}
(\bar{\varphi} \wedge \psi)(x_0, \ldots, x_k) &= \sum_{i=0}^{k} (-1)^i \bar{\varphi}(x_i) \cdot \psi(x_0, \ldots, \widehat{x}_i, \ldots, x_k) \\
&= \sum_{i=0}^{k} (-1)^i [\varphi(x_i)] \cdot \psi(x_0, \ldots, \widehat{x}_i, \ldots, x_k) \\
&= \sum_{i=0}^{k} (-1)^i (\varphi(x_i))(\psi(x_0, \ldots, \widehat{x}_i, \ldots, x_k)) \\
&= (\varphi \wedge \psi)((x_0, \ldots, x_k)).
\end{aligned}$$

Since $Z(\mathfrak{h})$ is the center of $\mathfrak{h}$, the operator $\delta_{\bar{\varphi}}$ or δ_{φ} satisfies $\delta_{\varphi} \circ \delta_{\varphi} = 0$, i.e.

$$\begin{aligned}
\delta_{\varphi} \circ \delta_{\varphi}(\psi)(x_1, \ldots, x_{k+1}) &= [\varrho, \psi]_{\wedge}(x_1, \ldots, x_{k+1}) \\
&= \frac{1}{2!k!} \sum_{\sigma \in S_{k+2}} \operatorname{sign}(\sigma)[\varrho(x_{\sigma_1}, x_{\sigma_2}), \psi(x_{\sigma_3}, \ldots, x_{\sigma_{k+2}})] = 0
\end{aligned}$$

for $\psi \in \bigwedge^{k+1}(\mathfrak{g}, Z(\mathfrak{h}))$ and $(x_1, \ldots, x_{k+1}) \in \mathfrak{g}$.

Theorem 3.3. *Let $\mathfrak{g}, \mathfrak{h}$ be two hom-Lie algebras and $\bar{\varphi}\colon \mathfrak{g} \to \operatorname{Der}_{\alpha^k}(\mathfrak{h})/\operatorname{ad}_k(\mathfrak{h})$ a hom-Lie algebra homomorphism. Then the following conditions are equivalent:*

(i) *For any linear lift $\varphi\colon \mathfrak{g} \to \operatorname{Der}_{\alpha^k}(\mathfrak{h})$ of $\bar{\varphi}$, one can find a linear map*

$$\varrho\colon \bigwedge^2 \mathfrak{g} \to \mathfrak{h},$$

such that

$$[\varphi_x, \varphi_y] - \varphi_{[x,y]} = \operatorname{ad}_k(\varrho(x, y)).$$

In this case the $\delta_{\bar{\varphi}}$-cohomology classes λ will be trivial in $H^3(\mathfrak{g}, Z(\mathfrak{h}))$ where

$$\lambda = \lambda(\varphi, \varrho) := \delta_\varphi(\varrho)\colon \bigwedge^3 \mathfrak{g} \to Z(\mathfrak{h}).$$

(ii) *There exists an extension $0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0$ which induces the homomorphism $\bar{\varphi}$. In this case all the extensions $0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0$ inducing $\bar{\varphi}$ will be parametrized by $H^2(\mathfrak{g}, Z(\mathfrak{h}))$, where $H^2(\mathfrak{g}, Z(\mathfrak{h}))$ is the second cohomology space of $\mathfrak{g}$ with values in $Z(\mathfrak{h})$ which here is viewed as a $\mathfrak{g}$-module by $\bar{\varphi}$.*

Proof. Using the calculations in the proof of Corollary 2.7 we obtain

$$\operatorname{ad}_k(\lambda(x, y, z)) = \operatorname{ad}_k(\delta_\varphi \varrho(x, y, z)).$$

Therefore, $\lambda(x, y, z) \in Z(\mathfrak{h})$. The hom-Lie algebra $\operatorname{out}(\mathfrak{h}) = \operatorname{Der}_{\alpha^k}(\mathfrak{h})/\operatorname{ad}_k(\mathfrak{h})$ acts on $Z(\mathfrak{h})$, so $Z(\mathfrak{h})$ is a $\mathfrak{g}$-module by $\bar{\varphi}$ and $\delta_{\bar{\varphi}}$ is the cohomology differential. Using (2.6) we have

$$\delta_{\bar{\varphi}} = \delta_\varphi \delta_\varphi \varrho = [\varrho, \varrho]_\wedge = 0,$$

therefore,

$$[\lambda] \in H^3(\mathfrak{g}, Z(\mathfrak{h})).$$

We must show that the cohomology class $[\lambda]$ is independent of the choice of φ. If we have (φ, ϱ) like above and choose another linear lift $\varphi'\colon \mathfrak{g} \to \operatorname{Der}_{\alpha^k}(\mathfrak{h})$, then for a $b\colon \mathfrak{g} \to \mathfrak{h}$ we have $\varphi'(x) = \varphi(x) + \operatorname{ad}_k(b(x))$. We set

$$\varrho'\colon \bigwedge^2 \mathfrak{g} \to \mathfrak{h}, \quad \varrho'(x, y) = \varrho(x, y) + (\delta_\varphi b)(x, y) + [b(x), b(y)].$$

By calculations similar to Lemma 2.2 we obtain

$$[\varphi'_x, \varphi'_y] - \varphi'_{[x,y]} = \operatorname{ad}_k(\varrho'(x, y)),$$

and by the last part of Theorem 2.5,

$$\lambda(\varphi, \varrho) = \delta_\varphi \varrho = \delta_{\varphi'} \varrho' = \lambda(\varphi', \varrho'),$$

so the cochain λ remains unchanged. For a constant φ let ϱ, ϱ' be defined by

$$\varrho, \varrho' \colon \bigwedge^2 \mathfrak{g} \to \mathfrak{h}, \quad [\varphi_x, \varphi_y] - \varphi_{[x,y]} = \mathrm{ad}_k(\varrho(x, y)) = \mathrm{ad}_k(\varrho'(x, y)).$$

Therefore,

$$\varrho - \varrho' := \nu \colon \bigwedge^2 \mathfrak{g} \to Z(\mathfrak{h}).$$

It is obvious that $\lambda(\varphi, \varrho) - \lambda(\varphi, \varrho') = \delta_\varphi \varrho = \delta_\varphi \varrho' = \delta_{\bar{\varphi}} \nu$. Now if there exists an extension inducing $\bar{\varphi}$, ϱ can be found like in Theorem 2.5 for each lift φ such that $\lambda(\varphi, \varrho) = 0$. On the other hand, for a given (φ, ϱ) as described in Theorem 2.5 such that

$$[\lambda(\varphi, \varrho)] = 0 \in H^3(\mathfrak{g}, Z(\mathfrak{h})),$$

there exists $\nu \colon \bigwedge^2 \mathfrak{g} \to Z(\mathfrak{h})$ such that $\delta_{\bar{\varphi}} \nu = \lambda$, therefore

$$\mathrm{ad}_k((\varrho - \nu)(x, y)) = \mathrm{ad}_k(\varrho(x, y)), \delta_\varphi(\varrho - \nu) = 0.$$

Thus $(\varphi, \varrho - \nu)$ satisfies the conditions of Theorem 2.5, so it describes an extension inducing $\bar{\varphi}$. Now consider the linear lift φ and a map $\varrho \colon \bigwedge^2 \mathfrak{g} \to \mathfrak{h}$ satisfying (2.7) and (2.8) and note all ϱ's which satisfy this condition. We have

$$\varrho - \varrho' := \nu \colon \bigwedge^2 \mathfrak{g} \to Z(\mathfrak{h}),$$

and

$$\delta_{\bar{\varphi}} \nu = \delta_\varphi \varrho - \delta_\varphi \varrho' = 0 - 0 = 0,$$

so ν is a 2-cocycle.

Moreover, analogous to Theorem 2.5, using a $b \colon \mathfrak{g} \to \mathfrak{h}$ which preserves φ, i.e. $b \colon \mathfrak{g} \to Z(\mathfrak{h})$, one can use the corresponding data. Also ϱ' can be found using (2.9)

$$\varrho' = \varrho + \delta_\varphi b + \frac{1}{2}[b, b]_\wedge = \varrho + \delta_{\bar{\varphi}} b.$$

Thus, it is just the cohomology class of ν that matters. □

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References

[1] *F. Ammar, Z. Ejbehi, A. Makhlouf*: Cohomology and deformations of Hom-algebras. J. Lie Theory *21* (2011), 813–836. zbl MR

[2] *F. W. Anderson, K. R. Fuller*: Rings and Categories of Modules. Graduate Texts in Mathematics 13, Springer, New York, 1992. zbl MR doi

[3] *S. Benayadi, A. Makhlouf*: Hom-Lie algebras with symmetric invariant nondegenerate bilinear forms. J. Geom. Phys. *76* (2014), 38–60. zbl MR doi

[4] *J. M. Casas, M. A. Insua, N. Pacheco*: On universal central extensions of Hom-Lie algebras. Hacet. J. Math. Stat. *44* (2015), 277–288. zbl MR

[5] *J. T. Hartwig, D. Larsson, S. D. Silvestrov*: Deformations of Lie algebras using σ-derivations. J. Algebra *295* (2006), 314–361. zbl MR doi

[6] *I. Kolář, P. W. Michor, J. Slovák*: Natural Operations in Differential Geometry. Springer, Berlin (corrected electronic version), 1993. zbl MR doi

[7] *A. Makhlouf, S. D. Silvestrov*: Hom-algebra structures. J. Gen. Lie Theory Appl. *2* (2008), 51–64. zbl MR doi

[8] *A. Makhlouf, S. Silvestrov*: Notes on 1-parameter formal deformations of Hom-associative and Hom-Lie algebras. Forum Math. *22* (2010), 715–739. zbl MR doi

[9] *Y. Sheng*: Representations of hom-Lie algebras. Algebr. Represent. Theory *15* (2012), 1081–1098. zbl MR doi

[10] *Y. Sheng, D. Chen*: Hom-Lie 2-algebras. J. Algebra *376* (2013), 174–195. zbl MR doi

[11] *Y. Sheng, Z. Xiong*: On Hom-Lie algebras. Linear Multilinear Algebra *63* (2015), 2379–2395. zbl MR doi

[12] *D. Yau*: Enveloping algebras of Hom-Lie algebras. J. Gen. Lie Theory Appl. *2* (2008), 95–108. zbl MR doi

[13] *D. Yau*: The Hom-Yang-Baxter equation, Hom-Lie algebras, and quasi-triangular bialgebras. J. Phys. A, Math. Theor. *42* (2009), Article ID 165202, 12 pages. zbl MR doi

Authors' address: Abdoreza R. Armakan, Mohammad Reza Farhangdoost (corresponding author), Department of Mathematics, College of Sciences, Shiraz university, Adabiat square, Shiraz, Fars, Iran, P.O. Box 71457-44776, e-mail: `r.armakan@shirazu.ac.ir`, `farhang@shirazu.ac.ir`.