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H-ANTI-INVARIANT SUBMERSIONS FROM ALMOST QUATERNIONIC HERMITIAN MANIFOLDS

Kwang-Soon Park, Seoul

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Abstract. As a generalization of anti-invariant Riemannian submersions and Lagrangian Riemannian submersions, we introduce the notions of h-anti-invariant submersions and h-Lagrangian submersions from almost quaternionic Hermitian manifolds onto Riemannian manifolds. We obtain characterizations and investigate some properties: the integrability of distributions, the geometry of foliations, and the harmonicity of such maps. We also find a condition for such maps to be totally geodesic and give some examples of such maps. Finally, we obtain some types of decomposition theorems.

Keywords: Riemannian submersion; Lagrangian Riemannian submersion; decomposition theorem; totally geodesic

MSC 2010: 53C15, 53C26

1. Introduction

In 1960s, O'Neill in [17] and Gray in [10] introduced independently the notion of a Riemannian submersion, which is useful in many areas: physics ([6], [25], [5], [12], [13], [16]), medical imaging [15], robotic theory [1] (see [23]).

In 1976, Watson in [24] defined almost Hermitian submersions, which are Riemannian submersions from almost Hermitian manifolds onto almost Hermitian manifolds. Using this notion, he investigates a kind of structural problems among base manifold, fibers, total manifold. This notion was extended to almost contact manifolds in [7], locally conformal Kähler manifolds in [14], and quaternion Kähler manifolds in [11].

In 2010, Sahin in [22] introduced the notions of anti-invariant Riemannian submersions and Lagrangian Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds. Using this notions, he studies total manifolds. In particular, he investigates some kinds of decomposition theorems.

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We know that Riemannian submersions are related with physics and have applications in Yang-Mills theory ([6], [25]), Kaluza-Klein theory ([5], [12]), supergravity and superstring theories ([13], [16]). And quaternionic Kähler manifolds have applications in physics as the target spaces for nonlinear σ-models with supersymmetry, see [8].

The paper is organized as follows. In Section 2 we recall some notions, which are needed in the later sections. In Section 3 we introduce the notions of h-anti-invariant submersions and h-Lagrangian submersions from almost quaternionic Hermitian manifolds onto Riemannian manifolds, give examples, and investigate some properties: the integrability of distributions, the geometry of foliations, the condition for such maps to be totally geodesic, and the condition for such maps to be harmonic. In Section 4 under h-anti-invariant submersions and h-Lagrangian submersions, we consider some decomposition theorems.

2. Preliminaries

Let (M, g, J) be an almost Hermitian manifold, where M is a C^∞-manifold, g is a Riemannian metric on M, and J is a compatible almost complex structure on (M, g) (i.e., $J \in \operatorname{End}(TM)$, $J^2 = -\mathrm{id}$, $g(JX, JY) = g(X, Y)$ for $X, Y \in \Gamma(TM)$).

We call (M, g, J) a *Kähler manifold* if $\nabla J = 0$, where ∇ is the Levi-Civita connection of g.

Let (M, g_M) and (N, g_N) be Riemannian manifolds.

Let $F\colon (M, g_M) \to (N, g_N)$ be a C^∞-map.

The *second fundamental form* of F is given by

$$(\nabla F_*)(U, V) := \nabla^F_U F_* V - F_*(\nabla_U V) \quad \text{for } U, V \in \Gamma(TM),$$

where ∇^F is the pullback connection along F and ∇ is the Levi-Civita connection of g_M, see [3].

Then the map F is *harmonic* if and only if $\operatorname{trace}(\nabla F_*) = 0$, see [3].

We call F a *totally geodesic map* if $(\nabla F_*)(U, V) = 0$ for $U, V \in \Gamma(TM)$, see [3].

The map F is said to be a *C^∞-submersion* if F is surjective and the differential $(F_*)_p$ has maximal rank for any $p \in M$.

We call F a *Riemannian submersion* ([17], [9]) if F is a C^∞-submersion and

$$(F_*)_p\colon ((\ker(F_*)_p)^\perp, (g_M)_p) \to (T_{F(p)}N, (g_N)_{F(p)}) \tag{2.1}$$

is a linear isometry for any $p \in M$, where $(\ker(F_*)_p)^\perp$ is the orthogonal complement of the space $\ker(F_*)_p$ in the tangent space T_pM to M at p.

Let $F\colon (M, g_M) \to (N, g_N)$ be a Riemannian submersion.

For any vector field $U \in \Gamma(TM)$ we write

$$U = \mathcal{V}U + \mathcal{H}U, \tag{2.2}$$

where $\mathcal{V}U \in \Gamma(\ker F_*)$ and $\mathcal{H}U \in \Gamma((\ker F_*)^\perp)$.

Define the O'Neill tensors $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_U V = \mathcal{H}\nabla_{\mathcal{H}U}\mathcal{V}V + \mathcal{V}\nabla_{\mathcal{H}U}\mathcal{H}V, \tag{2.3}$$

$$\mathcal{T}_U V = \mathcal{H}\nabla_{\mathcal{V}U}\mathcal{V}V + \mathcal{V}\nabla_{\mathcal{V}U}\mathcal{H}V \tag{2.4}$$

for $U, V \in \Gamma(TM)$, where ∇ is the Levi-Civita connection of g_M ([17], [9]).

Let

$$\widehat{\nabla}_V W := \mathcal{V}\nabla_V W \quad \text{for } V, W \in \Gamma(\ker F_*). \tag{2.5}$$

Then we have

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y], \tag{2.6}$$

$$\mathcal{T}_U V = \mathcal{T}_V U \tag{2.7}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U, V \in \Gamma(\ker F_*)$.

Proposition 2.1 ([17], [9])**.** *Let F be a Riemannian submersion from a Riemannian manifold (M, g_M) onto a Riemannian manifold (N, g_N). Then we obtain*

$$g_M(\mathcal{T}_U V, W) = -g_M(V, \mathcal{T}_U W), \tag{2.8}$$

$$g_M(\mathcal{A}_U V, W) = -g_M(V, \mathcal{A}_U W), \tag{2.9}$$

$$(\nabla F_*)(U, V) = (\nabla F_*)(V, U), \tag{2.10}$$

$$(\nabla F_*)(X, Y) = 0 \tag{2.11}$$

for $U, V, W \in \Gamma(TM)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

We recall the notions of an anti-invariant Riemannian submersion and a Lagrangian Riemannian submersion.

Let F be a Riemannian submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). The map F is said to be an *anti-invariant Riemannian submersion*, see [22], if $J(\ker F_*) \subset (\ker F_*)^\perp$.

We call F a *Lagrangian Riemannian submersion*, see [22], if $J(\ker F_*) = (\ker F_*)^\perp$.

Let M be a $4m$-dimensional C^∞-manifold and let E be a rank 3 subbundle of $\mathrm{End}(TM)$ such that for any point $p \in M$ with a neighborhood U there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -\mathrm{id}, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2},$$

where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Then we call E an *almost quaternionic structure* on M and (M, E) an *almost quaternionic manifold*, see [2].

Moreover, let g be a Riemannian metric on M such that for any point $p \in M$ with a neighborhood U there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -\mathrm{id}, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}, \tag{2.12}$$

$$g(J_\alpha X, J_\alpha Y) = g(X, Y) \tag{2.13}$$

for $X, Y \in \Gamma(TM)$, where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Then we call (M, E, g) an *almost quaternionic Hermitian manifold*, see [11].

For convenience, the above basis $\{J_1, J_2, J_3\}$ satisfying (2.12) and (2.13) is said to be a *quaternionic Hermitian basis*.

Let (M, E, g) be an almost quaternionic Hermitian manifold.

We call (M, E, g) a *quaternionic Kähler manifold* if given a point $p \in M$ with a neighborhood U, there exist 1-forms $\omega_1, \omega_2, \omega_3$ on U such that for any $\alpha \in \{1, 2, 3\}$,

$$\nabla_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2}$$

for $X \in \Gamma(TM)$, where the indices are taken from $\{1, 2, 3\}$ modulo 3, see [11].

If there exists a global parallel quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of sections of E on M (i.e., $\nabla J_\alpha = 0$ for $\alpha \in \{1, 2, 3\}$, where ∇ is the Levi-Civita connection of g), then (M, E, g) is said to be a *hyperkähler manifold*. Furthermore, we call (J_1, J_2, J_3, g) a *hyperkähler structure* on M and g a *hyperkähler metric*, see [4].

Now, we recall the notions of almost h-slant submersions, almost h-semi-invariant submersions, and almost h-semi-slant submersions.

Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold.

A Riemannian submersion $F\colon (M, E, g_M) \to (N, g_N)$ is said to be an *almost h-slant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for $R \in \{I, J, K\}$ the angle $\theta_R(X)$ between RX and the space $\ker(F_*)_q$ is constant for nonzero $X \in \ker(F_*)_q$ and $q \in U$, see [19].

A Riemannian submersion $F\colon (M, E, g_M) \to (N, g_N)$ is called an *almost h-semi-invariant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$ there is a distribution $\mathcal{D}_1^R \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \quad R(\mathcal{D}_1^R) = \mathcal{D}_1^R, \quad R(\mathcal{D}_2^R) \subset (\ker F_*)^\perp,$$

where $\mathcal{D}_2^R$ is the orthogonal complement of $\mathcal{D}_1^R$ in $\ker F_*$, see [18].

A Riemannian submersion $F\colon (M, E, g_M) \to (N, g_N)$ is called an *almost h-semi-slant submersions* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$ there is a distribution $\mathcal{D}_1^R \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \quad R(\mathcal{D}_1^R) = \mathcal{D}_1^R,$$

and the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathcal{D}_2^R)_q$ is constant for nonzero $X \in (\mathcal{D}_2^R)_q$ and $q \in U$, where $\mathcal{D}_2^R$ is the orthogonal complement of $\mathcal{D}_1^R$ in $\ker F_*$, see [20].

Throughout this paper, we will use the above notation.

3. H-anti-invariant submersions

In this section, we introduce the notions of h-anti-invariant submersions and h-Lagrangian submersions from almost quaternionic Hermitian manifolds onto Riemannian manifolds and investigate their properties.

Definition 3.1. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Let $F\colon (M, E, g_M) \to (N, g_N)$ be a Riemannian submersion. We call the map F an *h-anti-invariant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that $R(\ker F_*) \subset (\ker F_*)^\perp$ for $R \in \{I, J, K\}$.

We call such a basis $\{I, J, K\}$ an *h-anti-invariant basis.*

Remark 3.2. As we see, an h-anti-invariant submersion is one of the particular cases of an almost h-slant submersion, an almost h-semi-invariant submersion, and an almost h-semi-slant submersion.

Remark 3.3. Let F be an h-anti-invariant submersion from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N). Then there does not exist a map F such that $\dim(\ker F_*) = \dim((\ker F_*)^\perp)$. If it did, then given a local quaternionic Hermitian basis $\{I, J, K\}$ of E with $R(\ker F_*) \subset (\ker F_*)^\perp$ for $R \in \{I, J, K\}$, we should have

$$R(\ker F_*) = (\ker F_*)^\perp \quad \text{for } R \in \{I, J, K\}$$

so that

$$K(\ker F_*) = IJ(\ker F_*) = I((\ker F_*)^\perp) = (\ker F_*),$$

contradiction!

Due to Remark 3.3, we need to define another type of such a map.

Definition 3.4. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Let $F\colon (M, E, g_M) \to (N, g_N)$ be a Riemannian submersion. We call the map F a *h-Lagrangian submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that $I(\ker F_*) = (\ker F_*)^\perp$, $J(\ker F_*) = \ker F_*$, and $K(\ker F_*) = (\ker F_*)^\perp$.

We call such a basis $\{I, J, K\}$ an *h-Lagrangian basis.*

Remark 3.5. (a) It is easy to check that $J(\ker F_*) = \ker F_*$ implies $J((\ker F_*)^\perp) = (\ker F_*)^\perp$.

(b) Let F be a Riemannian submersion from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N) such that $\dim(\ker F_*) = \dim((\ker F_*)^\perp)$. Then there does not exist a map F that for some local quaternionic Hermitian basis $\{I, J, K\}$ of E we have

$$I(\ker F_*) = \ker F_*, \quad J(\ker F_*) = \ker F_*, \quad K(\ker F_*) = (\ker F_*)^\perp.$$

If it did, then $K(\ker F_*) = IJ(\ker F_*) = I(\ker F_*) = \ker F_*$, contradiction!

Now, we give some examples. Note that given a Euclidean space $\mathbb{R}^{4m}$ with coordinates $(x_1, x_2, \dots, x_{4m})$, we can canonically choose complex structures I, J, K on $\mathbb{R}^{4m}$ as follows:

$$\begin{aligned}
&I\Big(\frac{\partial}{\partial x_{4k+1}}\Big) = \frac{\partial}{\partial x_{4k+2}}, \quad I\Big(\frac{\partial}{\partial x_{4k+2}}\Big) = -\frac{\partial}{\partial x_{4k+1}}, \quad I\Big(\frac{\partial}{\partial x_{4k+3}}\Big) = \frac{\partial}{\partial x_{4k+4}}, \\
&I\Big(\frac{\partial}{\partial x_{4k+4}}\Big) = -\frac{\partial}{\partial x_{4k+3}}, \quad J\Big(\frac{\partial}{\partial x_{4k+1}}\Big) = \frac{\partial}{\partial x_{4k+3}}, \quad J\Big(\frac{\partial}{\partial x_{4k+2}}\Big) = -\frac{\partial}{\partial x_{4k+4}}, \\
&J\Big(\frac{\partial}{\partial x_{4k+3}}\Big) = -\frac{\partial}{\partial x_{4k+1}}, \quad J\Big(\frac{\partial}{\partial x_{4k+4}}\Big) = \frac{\partial}{\partial x_{4k+2}}, \quad K\Big(\frac{\partial}{\partial x_{4k+1}}\Big) = \frac{\partial}{\partial x_{4k+4}}, \\
&K\Big(\frac{\partial}{\partial x_{4k+2}}\Big) = \frac{\partial}{\partial x_{4k+3}}, \quad K\Big(\frac{\partial}{\partial x_{4k+3}}\Big) = -\frac{\partial}{\partial x_{4k+2}}, \quad K\Big(\frac{\partial}{\partial x_{4k+4}}\Big) = -\frac{\partial}{\partial x_{4k+1}}
\end{aligned}$$

for $k \in \{0, 1, \dots, m-1\}$.

Then we easily check that $(I, J, K, \langle\ ,\ \rangle)$ is a hyperkähler structure on $\mathbb{R}^{4m}$, where $\langle\ ,\ \rangle$ denotes the Euclidean metric on $\mathbb{R}^{4m}$.

Example 3.6. Define a map $F\colon\ \mathbb{R}^{12} \to \mathbb{R}^9$ by

$$F(x_1, \dots, x_{12}) = (x_{10}, x_{11}, x_{12}, x_4, x_3, x_2, x_8, x_6, x_7).$$

Then the map F is an h-anti-invariant submersion such that

$$\begin{aligned}
\ker F_* &= \Big\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_9} \Big\rangle, \\
(\ker F_*)^\perp &= \Big\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} \Big\rangle, \\
I\Big(\frac{\partial}{\partial x_1}\Big) &= \frac{\partial}{\partial x_2}, \quad I\Big(\frac{\partial}{\partial x_5}\Big) = \frac{\partial}{\partial x_6}, \quad I\Big(\frac{\partial}{\partial x_9}\Big) = \frac{\partial}{\partial x_{10}}, \\
J\Big(\frac{\partial}{\partial x_1}\Big) &= \frac{\partial}{\partial x_3}, \quad J\Big(\frac{\partial}{\partial x_5}\Big) = \frac{\partial}{\partial x_7}, \quad J\Big(\frac{\partial}{\partial x_9}\Big) = \frac{\partial}{\partial x_{11}}, \\
K\Big(\frac{\partial}{\partial x_1}\Big) &= \frac{\partial}{\partial x_4}, \quad K\Big(\frac{\partial}{\partial x_5}\Big) = \frac{\partial}{\partial x_8}, \quad K\Big(\frac{\partial}{\partial x_9}\Big) = \frac{\partial}{\partial x_{12}}.
\end{aligned}$$

Example 3.7. Define a map $F\colon\ \mathbb{R}^4 \to \mathbb{R}^2$ by

$$F(x_1, \dots, x_4) = \Big(\frac{x_2 + x_3}{\sqrt{2}}, \frac{x_1 + x_4}{\sqrt{2}}\Big).$$

Then the map F is an h-Lagrangian submersion such that

$$\begin{aligned}
\ker F_* &= \Big\langle V_1 = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}, V_2 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_4} \Big\rangle, \\
(\ker F_*)^\perp &= \Big\langle X_1 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}, X_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4} \Big\rangle, \\
&I(V_1) = -X_2, \quad I(V_2) = X_1, \\
&J(V_1) = V_2, \quad J(V_2) = -V_1, \\
&K(V_1) = X_1, \quad K(V_2) = X_2.
\end{aligned}$$

Let F be an h-anti-invariant submersion (or an h-Lagrangian submersion) from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N). Given a point $p \in M$ with a neighborhood U, we have an h-anti-invariant basis (or an h-Lagrangian basis, respectively) $\{I, J, K\}$ of sections of E on U.

Then given $X \in \Gamma((\ker F_*)^\perp)$ and $R \in \{I, J, K\}$, we write

$$RX = B_R X + C_R X, \tag{3.1}$$

where $B_R X \in \Gamma(\ker F_*)$ and $C_R X \in \Gamma((\ker F_*)^\perp)$.

If $F\colon (M,E,g_M)\to(N,g_N)$ is an h-anti-invariant submersion, then we get

$$(\ker F_*)^\perp = R(\ker F_*)\oplus\mu_R \quad \text{for } R\in\{I,J,K\}. \tag{3.2}$$

Then it is easy to check that μ_R is R-invariant for $R\in\{I,J,K\}$.

Given $X\in\Gamma((\ker F_*)^\perp)$ and $R\in\{I,J,K\}$, we have

$$X = P_R X + Q_R X, \tag{3.3}$$

where $P_R X\in\Gamma(R(\ker F_*))$ and $Q_R X\in\Gamma(\mu_R)$.

Furthermore, given $R\in\{I,J,K\}$, we obtain

$$C_R X\in\Gamma(\mu_R) \quad \text{for } X\in\Gamma((\ker F_*)^\perp) \tag{3.4}$$

and

$$g_M(C_R X, RV) = 0 \quad \text{for } V\in\Gamma(\ker F_*). \tag{3.5}$$

Then it is easy to have

Lemma 3.8. *Let F be an h-anti-invariant submersion from a hyperkähler manifold (M,I,J,K,g_M) onto a Riemannian manifold (N,g_N) such that (I,J,K) is an h-anti-invariant basis. Then we get*

(1)
$$\mathcal{T}_V RW = B_R\mathcal{T}_V W$$
$$\mathcal{H}\nabla_V RW = C_R\mathcal{T}_V W + R\widehat{\nabla}_V W$$

for $V,W\in\Gamma(\ker F_*)$ and $R\in\{I,J,K\}$;

(2)
$$\mathcal{A}_X C_R Y + \mathcal{V}\nabla_X B_R Y = B_R\mathcal{H}\nabla_X Y$$
$$\mathcal{H}\nabla_X C_R Y + \mathcal{A}_X B_R Y = R\mathcal{A}_X Y + C_R\mathcal{H}\nabla_X Y$$

for $X,Y\in\Gamma((\ker F_*)^\perp)$ and $R\in\{I,J,K\}$;

(3)
$$\mathcal{A}_X RV = B_R\mathcal{A}_X V$$
$$\mathcal{H}\nabla_X RV = C_R\mathcal{A}_X V + R\mathcal{V}\nabla_X V$$

for $V\in\Gamma(\ker F_*)$, $X\in\Gamma((\ker F_*)^\perp)$, and $R\in\{I,J,K\}$.

Theorem 3.9. *Let F be an h-anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-anti-invariant basis. Then the following conditions are equivalent:*

(a) *the distribution $(\ker F_*)^\perp$ is integrable.*

(b)
$$g_M(\mathcal{A}_X B_I Y - \mathcal{A}_Y B_I X, IV) = g_M(C_I Y, I\mathcal{A}_X V) - g_M(C_I X, I\mathcal{A}_Y V)$$
for $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

(c)
$$g_M(\mathcal{A}_X B_J Y - \mathcal{A}_Y B_J X, JV) = g_M(C_J Y, J\mathcal{A}_X V) - g_M(C_J X, J\mathcal{A}_Y V)$$
for $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

(d)
$$g_M(\mathcal{A}_X B_K Y - \mathcal{A}_Y B_K X, KV) = g_M(C_K Y, K\mathcal{A}_X V) - g_M(C_K X, K\mathcal{A}_Y V)$$
for $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

Proof. Given $V \in \Gamma(\ker F_*)$, $X, Y \in \Gamma((\ker F_*)^\perp)$, and $R \in \{I, J, K\}$, using (3.5) we get

$$\begin{aligned}
g_M([X, Y], V) &= g_M(\nabla_X RY - \nabla_Y RX, RV) \\
&= g_M(\nabla_X B_R Y + \nabla_X C_R Y - \nabla_Y B_R X - \nabla_Y C_R X, RV) \\
&= g_M(\mathcal{A}_X B_R Y - \mathcal{A}_Y B_R X, RV) - g_M(C_R Y, \nabla_X RV) + g_M(C_R X, \nabla_Y RV) \\
&= g_M(\mathcal{A}_X B_R Y - \mathcal{A}_Y B_R X, RV) - g_M(C_R Y, R\mathcal{A}_X V) + g_M(C_R X, R\mathcal{A}_Y V).
\end{aligned}$$

Hence,
$$\text{(a)} \Leftrightarrow \text{(b)}, \quad \text{(a)} \Leftrightarrow \text{(c)}, \quad \text{(a)} \Leftrightarrow \text{(d)}.$$
Therefore, the result follows. □

Lemma 3.10. *Let F be an h-Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-Lagrangian basis. Then the following conditions are equivalent:*

(a) *The distribution $(\ker F_*)^\perp$ is integrable.*

(b) $\mathcal{A}_X IY = \mathcal{A}_Y IX$ *for $X, Y \in \Gamma((\ker F_*)^\perp)$.*

(c) $\mathcal{A}_X KY = \mathcal{A}_Y KX$ *for $X, Y \in \Gamma((\ker F_*)^\perp)$.*

(d) $\mathcal{A}_X JY = \mathcal{A}_Y JX$ *for $X, Y \in \Gamma((\ker F_*)^\perp)$.*

Proof. By the proof of Theorem 3.9, we get (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c).

Given $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$, since $J(\ker F_*) = \ker F_*$, we obtain

$$\begin{aligned}
g_M([X, Y], JV) &= -g_M(\nabla_X JY - \nabla_Y JX, V) \\
&= g_M(\mathcal{A}_Y JX - \mathcal{A}_X JY, V),
\end{aligned}$$

which implies (a) $\Leftrightarrow$ (d).

Therefore, the result follows. □

We consider equivalent conditions for distributions to be totally geodesic.

Theorem 3.11. *Let F be an h-anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-anti-invariant basis. Then the following conditions are equivalent:*

(a) *The distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation on M.*

(b)
$$g_M(\mathcal{A}_X B_I Y, IV) = g_M(C_I Y, I\mathcal{A}_X V)$$
for $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

(c)
$$g_M(\mathcal{A}_X B_J Y, JV) = g_M(C_J Y, J\mathcal{A}_X V)$$
for $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

(d)
$$g_M(\mathcal{A}_X B_K Y, KV) = g_M(C_K Y, K\mathcal{A}_X V)$$
for $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

P r o o f. Given $V \in \Gamma(\ker F_*)$, $X, Y \in \Gamma((\ker F_*)^\perp)$, and $R \in \{I, J, K\}$, using (3.5) we have

$$\begin{aligned} g_M(\nabla_X Y, V) &= g_M(\nabla_X B_R Y + \nabla_X C_R Y, RV) \\ &= g_M(\mathcal{A}_X B_R Y, RV) - g_M(C_R Y, \nabla_X RV) \\ &= g_M(\mathcal{A}_X B_R Y, RV) - g_M(C_R Y, R\mathcal{A}_X V), \end{aligned}$$

which implies (a) $\Leftrightarrow$ (b), (a) $\Leftrightarrow$ (c), (a) $\Leftrightarrow$ (d).

Therefore, the result follows. □

Lemma 3.12. *Let F be an h-Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-Lagrangian basis. Then the following conditions are equivalent:*

(a) *The distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation on M.*

(b) $\mathcal{A}_X IY = 0$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$.

(c) $\mathcal{A}_X KY = 0$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$.

(d) $\mathcal{A}_X JY = 0$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$.

P r o o f. By the proof of Theorem 3.11, we get (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c).

Given $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$, we obtain

$$g_M(\nabla_X Y, JV) = -g_M(\nabla_X JY, V) = -g_M(\mathcal{A}_X JY, V),$$

which implies (a) $\Leftrightarrow$ (d).

Therefore, the result follows. □

Theorem 3.13. *Let F be an h-anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-anti-invariant basis. Then the following conditions are equivalent:*

(a) *The distribution $\ker F_*$ defines a totally geodesic foliation on M.*

(b)
$$\mathcal{T}_V B_I X + \mathcal{A}_{C_I X} V \in \Gamma(\mu_I)$$

for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

(c)
$$\mathcal{T}_V B_J X + \mathcal{A}_{C_J X} V \in \Gamma(\mu_J)$$

for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

(d)
$$\mathcal{T}_V B_K X + \mathcal{A}_{C_K X} V \in \Gamma(\mu_K)$$

for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

Proof. Given $V, W \in \Gamma(\ker F_*)$, $X \in \Gamma((\ker F_*)^\perp)$, and $R \in \{I, J, K\}$, using (3.5) we get

$$\begin{aligned} g_M(\nabla_V W, X) &= g_M(\nabla_V RW, RX) \\ &= -g_M(RW, \nabla_V B_R X + \nabla_V C_R X) \\ &= -g_M(RW, \mathcal{T}_V B_R X) - g_M(RW, \nabla_V C_R X). \end{aligned}$$

However,

$$\begin{aligned} g_M(RW, \nabla_V C_R X) &= g_N(F_* RW, F_* \nabla_V C_R X) \quad (\text{since } RW \in \Gamma((\ker F_*)^\perp)) \\ &= -g_N(F_* RW, (\nabla F_*)(V, C_R X)) \\ &= -g_N(F_* RW, (\nabla F_*)(C_R X, V)) \quad (\text{by (2.10)}) \\ &= g_M(RW, \nabla_{C_R X} V) \\ &= g_M(RW, \mathcal{A}_{C_R X} V). \end{aligned}$$

Hence,

$$g_M(\nabla_V W, X) = -g_M(RW, \mathcal{T}_V B_R X + \mathcal{A}_{C_R X} V),$$

which implies (a) $\Leftrightarrow$ (b), (a) $\Leftrightarrow$ (c), (a) $\Leftrightarrow$ (d).

Therefore, we obtain the result. □

Lemma 3.14. *Let F be an h-Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-Lagrangian basis. Then the following conditions are equivalent:*

(a) *The distribution $\ker F_*$ defines a totally geodesic foliation on M.*

(b) $\mathcal{T}_V IX = 0$ *for* $X \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$.

(c) $\mathcal{T}_V KX = 0$ *for* $X \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$.

(d) $\mathcal{T}_V JX = 0$ *for* $X \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$.

Proof. By the proof of Theorem 3.13, we have (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c).

Given $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$, we get

$$\begin{aligned} g_M(\nabla_V W, JX) &= -g_M(W, \nabla_V JX) \\ &= -g_M(W, \mathcal{T}_V JX) \quad (\text{since } JX \in \Gamma((\ker F_*)^\perp)), \end{aligned}$$

which implies (a) $\Leftrightarrow$ (d).

Therefore, the result follows. □

Now, we consider equivalent conditions for such maps to be either totally geodesic or harmonic.

Theorem 3.15. *Let F be an h-anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-anti-invariant basis. Then the following conditions are equivalent:*

(a) *The map F is a totally geodesic map.*

(b)
$$\mathcal{A}_X IV = 0, \quad Q_I \mathcal{H} \nabla_X IV = 0, \quad \mathcal{T}_V IW = 0, \quad Q_I \mathcal{H} \nabla_V IW = 0$$

for $V, W \in \Gamma(\ker F_*)$ *and* $X \in \Gamma((\ker F_*)^\perp)$.

(c)
$$\mathcal{A}_X JV = 0, \quad Q_J \mathcal{H} \nabla_X JV = 0, \quad \mathcal{T}_V JW = 0, \quad Q_J \mathcal{H} \nabla_V JW = 0$$

for $V, W \in \Gamma(\ker F_*)$ *and* $X \in \Gamma((\ker F_*)^\perp)$.

(d)
$$\mathcal{A}_X KV = 0, \quad Q_K \mathcal{H} \nabla_X KV = 0, \quad \mathcal{T}_V KW = 0, \quad Q_K \mathcal{H} \nabla_V KW = 0$$

for $V, W \in \Gamma(\ker F_*)$ *and* $X \in \Gamma((\ker F_*)^\perp)$.

Proof. By (2.11) we have $(\nabla F_*)(X, Y) = 0$ for $X, Y \in \Gamma((\ker F_*)^\perp)$.

Given $V, W \in \Gamma(\ker F_*)$, $X \in \Gamma((\ker F_*)^\perp)$, and $R \in \{I, J, K\}$, by using (3.2) and (3.3) we obtain

$$\begin{aligned} (\nabla F_*)(X, V) &= -F_*(\nabla_X V) = F_*(R \nabla_X RV) \\ &= F_*(R(\mathcal{A}_X RV + \mathcal{H} \nabla_X RV)) = 0 \end{aligned}$$

$\Leftrightarrow R(\mathcal{A}_X RV + Q_R \mathcal{H}\nabla_X RV) = 0 \Leftrightarrow \mathcal{A}_X RV = 0,\ Q_R \mathcal{H}\nabla_X RV = 0$ and

$$\begin{aligned}(\nabla F_*)(V, W) &= -F_*(\nabla_V W) = F_*(R\nabla_V RW)\\ &= F_*(R(\mathcal{T}_V RW + \mathcal{H}\nabla_V RW)) = 0\end{aligned}$$

$\Leftrightarrow R(\mathcal{T}_V RW + Q_R \mathcal{H}\nabla_V RW) = 0 \Leftrightarrow \mathcal{T}_V RW = 0,\ Q_R \mathcal{H}\nabla_V RW = 0.$

Hence,

$$\text{(a)} \Leftrightarrow \text{(b)}, \quad \text{(a)} \Leftrightarrow \text{(c)}, \quad \text{(a)} \Leftrightarrow \text{(d)}.$$

Therefore, the result follows. □

Lemma 3.16. *Let F be an h-Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-Lagrangian basis. Then the following conditions are equivalent:*

(a) *The map F is a totally geodesic map.*

(b) *$\mathcal{A}_X IV = 0$ and $\mathcal{T}_V IW = 0$ for $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.*

(c) *$\mathcal{A}_X KV = 0$ and $\mathcal{T}_V KW = 0$ for $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.*

(d) *$\mathcal{A}_X JV = 0$ and $\mathcal{T}_V JW = 0$ for $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.*

Proof. By the proof of Theorem 3.15, we have (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c).

Given $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$, we get

$$\begin{aligned}(\nabla F_*)(X, V) &= -F_*(\nabla_X V) = F_*(J\nabla_X JV)\\ &= F_*(J(\mathcal{A}_X JV + \mathcal{V}\nabla_X JV)) = F_* J\mathcal{A}_X JV = 0\end{aligned}$$

$\Leftrightarrow \mathcal{A}_X JV = 0$ and

$$\begin{aligned}(\nabla F_*)(V, W) &= -F_*(\nabla_V W) = F_*(J\nabla_V JW)\\ &= F_*(J(\mathcal{T}_V JW + \mathcal{V}\nabla_V JW))\\ &= F_* J\mathcal{T}_V JW = 0\end{aligned}$$

$\Leftrightarrow \mathcal{T}_V JW = 0$, which implies (a) $\Leftrightarrow$ (d).

Therefore, we obtain the result. □

Theorem 3.17. *Let F be an h-anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-anti-invariant basis. Then the following conditions are equivalent:*

(a) *The map F is harmonic.*

(b) *$Q_I(\operatorname{trace}(\mathcal{T})) = 0$ on $\ker F_*$ and $\operatorname{trace}(I\mathcal{T}_V) = 0$ on $\ker F_*$ for $V \in \Gamma(\ker F_*)$.*

(c) $Q_J(\mathrm{trace}\,(\mathcal{T})) = 0$ *on* $\ker F_*$ *and* $\mathrm{trace}\,(J\mathcal{T}_V) = 0$ *on* $\ker F_*$ *for* $V \in \Gamma(\ker F_*)$.

(d) $Q_K(\mathrm{trace}\,(\mathcal{T})) = 0$ *on* $\ker F_*$ *and* $\mathrm{trace}\,(K\mathcal{T}_V) = 0$ *on* $\ker F_*$ *for* $V \in \Gamma(\ker F_*)$.

Proof. By (2.11) we know that the map F is harmonic if and only if $\sum_{i=1}^{m} \mathcal{T}_{e_i} e_i = 0$ for any local orthonormal frame $\{e_1, e_2, \ldots, e_m\}$ of $\ker F_*$.

Given $V, W \in \Gamma(\ker F_*)$, $R \in \{I, J, K\}$, and a local orthonormal frame $\{e_1, e_2, \ldots, e_m\}$ of $\ker F_*$, using (3.2) and (3.3) we obtain

$$\begin{aligned}\mathcal{T}_V RW &= \mathcal{V}\nabla_V RW = \mathcal{V}R\nabla_V W\\ &= \mathcal{V}R(\mathcal{T}_V W + \mathcal{V}\nabla_V W) = \mathcal{V}RP_R\mathcal{T}_V W\end{aligned}$$

so that using (2.7) and (2.8) we get

$$\begin{aligned} g_M\Big(\sum_{i=1}^{m} \mathcal{T}_{e_i} e_i, RV\Big) &= \sum_{i=1}^{m} g_M(\mathcal{T}_{e_i} e_i, RV) = \sum_{i=1}^{m} g_M(P_R\mathcal{T}_{e_i} e_i, RV)\\ &= -\sum_{i=1}^{m} g_M(RP_R\mathcal{T}_{e_i} e_i, V) = -\sum_{i=1}^{m} g_M(\mathcal{V}RP_R\mathcal{T}_{e_i} e_i, V)\\ &= -\sum_{i=1}^{m} g_M(\mathcal{T}_{e_i} Re_i, V) = \sum_{i=1}^{m} g_M(Re_i, \mathcal{T}_{e_i} V)\\ &= \sum_{i=1}^{m} g_M(Re_i, \mathcal{T}_V e_i) = -\sum_{i=1}^{m} g_M(e_i, R\mathcal{T}_V e_i) = 0\end{aligned}$$

$\Leftrightarrow \mathrm{trace}\,(R\mathcal{T}_V) = 0$ for $V \in \Gamma(\ker F_*)$.

Hence,

$$\text{(a)} \Leftrightarrow \text{(b)}, \quad \text{(a)} \Leftrightarrow \text{(c)}, \quad \text{(a)} \Leftrightarrow \text{(d)}.$$

Therefore, the result follows. □

Lemma 3.18. *Let F be an h-Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-Lagrangian basis. Then the map F is harmonic.*

Proof. Since $J(\ker F_*) = \ker F_*$, we can choose a local orthonormal frame $\{e_1, Je_1, \ldots, e_k, Je_k\}$ of $\ker F_*$.

Given $V, W \in \Gamma(\ker F_*)$, we have

$$\begin{aligned}\mathcal{T}_V JW &= \mathcal{H}\nabla_V JW = \mathcal{H}J\nabla_V W\\ &= \mathcal{H}J(\mathcal{T}_V W + \mathcal{V}\nabla_V W) = J\mathcal{T}_V W\end{aligned}$$

so that

$$\begin{aligned}\sum_{i=1}^{k}(\mathcal{T}_{e_i}e_i+\mathcal{T}_{Je_i}Je_i)&=\sum_{i=1}^{k}(\mathcal{T}_{e_i}e_i+J\mathcal{T}_{Je_i}e_i)=\sum_{i=1}^{k}(\mathcal{T}_{e_i}e_i+J\mathcal{T}_{e_i}Je_i)\\&=\sum_{i=1}^{k}(\mathcal{T}_{e_i}e_i+J^2\mathcal{T}_{e_i}e_i)=\sum_{i=1}^{k}(\mathcal{T}_{e_i}e_i-\mathcal{T}_{e_i}e_i)=0.\end{aligned}$$

Therefore, the result follows. □

4. Decomposition theorems

First of all, we recall some notions. Let (M,g) be a Riemannian manifold and L a foliation of M. Let ξ be the tangent bundle of L considered as a subbundle of the tangent bundle TM of M.

We call L a *totally umbilic foliation*, see [21], of M if

$$h(X,Y)=g(X,Y)H\quad\text{for } X,Y\in\Gamma(\xi),\tag{4.1}$$

where h is the second fundamental form of L in M and H is the mean curvature vector field of L in M.

The foliation L is said to be a *spheric foliation*, see [21], if it is a totally umbilic foliation and

$$\nabla_X H\in\Gamma(\xi)\quad\text{for } X\in\Gamma(\xi),\tag{4.2}$$

where ∇ is the Levi-Civita connection of g.

We call L a *totally geodesic foliation*, see [21], of M if

$$\nabla_X Y\in\Gamma(\xi)\quad\text{for } X,Y\in\Gamma(\xi).\tag{4.3}$$

Let (M_1,g_1) and (M_2,g_2) be Riemannian manifolds, $f_i\colon M_1\times M_2\to\mathbb{R}$ a positive C^∞-function, and $\pi_i\colon M_1\times M_2\to M_i$ the canonical projection for $i=1,2$.

We call $M_1\times_{(f_1,f_2)}M_2$ a *double-twisted product manifold*, see [21], of (M_1,g_1) and (M_2,g_2) if it is the product manifold $M:=M_1\times M_2$ with the Riemannian metric g such that

$$g(X,Y)=f_1^2\,g_1(\pi_{1*}X,\pi_{1*}Y)+f_2^2\,g_2(\pi_{2*}X,\pi_{2*}Y)\quad\text{for } X,Y\in\Gamma(TM).\tag{4.4}$$

We call $M_1\times_{(f_1,f_2)}M_2$ *nontrivial* if neither f_1 nor f_2 are constant functions.

The Riemannian manifold $M_1 \times_f M_2$ is said to be a *twisted product manifold*, see [21], of (M_1, g_1) and (M_2, g_2) if $M_1 \times_f M_2 = M_1 \times_{(1,f)} M_2$.

We call $M_1 \times_f M_2$ *nontrivial* if f is not a constant function.

The twisted product manifold $M_1 \times_f M_2$ is said to be a *warped product manifold*, see [21], of (M_1, g_1) and (M_2, g_2) if f depends only on the points of M_1. (i.e., $f \in C^\infty(M_1, \mathbb{R})$)

Let M_1 and M_2 be connected C^∞-manifolds and M the product manifold $M_1 \times M_2$. Let $\pi_i \colon M \to M_i$ be the canonical projection for $i = 1, 2$. Let $\xi_i := \ker \pi_{3-i*}$ and let $P_i \colon TM \to \xi_i$ be the vector bundle projection such that $TM = \xi_1 \oplus \xi_2$. And let L_i be the canonical foliation of M by the integral manifolds of ξ_i for $i = 1, 2$.

Proposition 4.1 ([21]). *Let g be a Riemannian metric on the product manifold $M_1 \times M_2$ and assume that the canonical foliations L_1 and L_2 intersect perpendicularly everywhere. Then g is a metric of*

(a) *a double-twisted product manifold $M_1 \times_{(f_1, f_2)} M_2$ if and only if L_1 and L_2 are totally umbilic foliations,*

(b) *a twisted product manifold $M_1 \times_f M_2$ if and only if L_1 is a totally geodesic foliation and L_2 is a totally umbilic foliation,*

(c) *a warped product manifold $M_1 \times_f M_2$ if and only if L_1 is a totally geodesic foliation and L_2 is a spheric foliation,*

(d) *a (usual) Riemannian product manifold $M_1 \times M_2$ if and only if L_1 and L_2 are totally geodesic foliations.*

Let F be a Riemannian submersion from a Riemannian manifold (M, g_M) onto a Riemannian manifold (N, g_N) such that the distributions $\ker F_*$ and $(\ker F_*)^\perp$ are integrable. Then we denote by $M_{\ker F_*}$ and $M_{(\ker F_*)^\perp}$ the integral manifolds of $\ker F_*$ and $(\ker F_*)^\perp$, respectively. We also denote by H and $H^\perp$ the mean curvature vector fields of $\ker F_*$ and $(\ker F_*)^\perp$, respectively, i.e., $H = m^{-1} \sum_{i=1}^{m} \mathcal{T}_{e_i} e_i$ and $H^\perp = n^{-1} \sum_{i=1}^{n} \mathcal{A}_{v_i} v_i$ for a local orthonormal frame $\{e_1, \ldots, e_m\}$ of $\ker F_*$ and a local orthonormal frame $\{v_1, \ldots, v_n\}$ of $(\ker F_*)^\perp$.

Using Proposition 4.1, Theorem 3.11, and Theorem 3.13, we get

Theorem 4.2. *Let F be an h-anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-anti-invariant basis. Then the following conditions are equivalent:*

(a) *(M, g_M) is locally a Riemannian product manifold of the form $M_{(\ker F_*)^\perp} \times M_{\ker F_*}$.*

(b)

$$g_M(\mathcal{A}_X B_I Y, IV) = g_M(C_I Y, I\mathcal{A}_X V) \quad \text{and} \quad \mathcal{T}_V B_I X + \mathcal{A}_{C_I X} V \in \Gamma(\mu_I)$$

for $V \in \Gamma(\ker F_*)$ *and* $X, Y \in \Gamma((\ker F_*)^\perp)$.

(c)

$$g_M(\mathcal{A}_X B_J Y, JV) = g_M(C_J Y, J\mathcal{A}_X V) \quad \text{and} \quad \mathcal{T}_V B_J X + \mathcal{A}_{C_J X} V \in \Gamma(\mu_J)$$

for $V \in \Gamma(\ker F_*)$ *and* $X, Y \in \Gamma((\ker F_*)^\perp)$.

(d)

$$g_M(\mathcal{A}_X B_K Y, KV) = g_M(C_K Y, K\mathcal{A}_X V) \quad \text{and} \quad \mathcal{T}_V B_K X + \mathcal{A}_{C_K X} V \in \Gamma(\mu_K)$$

for $V \in \Gamma(\ker F_*)$ *and* $X, Y \in \Gamma((\ker F_*)^\perp)$.

Using Proposition 4.1, Lemma 3.12, and Lemma 3.14, we obtain

Lemma 4.3. *Let* F *be an h-Lagrangian submersion from a hyperkähler manifold* (M, I, J, K, g_M) *onto a Riemannian manifold* (N, g_N) *such that* (I, J, K) *is an h-Lagrangian basis. Then the following conditions are equivalent:*

(a) (M, g_M) *is locally a Riemannian product manifold of the form* $M_{(\ker F_*)^\perp} \times M_{\ker F_*}$.

(b)

$$\mathcal{A}_X IY = 0 \quad \text{and} \quad \mathcal{T}_V IX = 0$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$.

(c)

$$\mathcal{A}_X KY = 0 \quad \text{and} \quad \mathcal{T}_V KX = 0$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$.

(d)

$$\mathcal{A}_X JY = 0 \quad \text{and} \quad \mathcal{T}_V JX = 0$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$.

Now, we deal with the geometry of distributions $\ker F_*$ and $(\ker F_*)^\perp$.

Theorem 4.4. *Let* F *be a Riemannian submersion from a Riemannian manifold* (M, g_M) *onto a Riemannian manifold* (N, g_N). *Assume that the distribution* $(\ker F_*)^\perp$ *defines a totally umbilic foliation on* M. *Then the distribution* $(\ker F_*)^\perp$ *also defines a totally geodesic foliation on* M.

Proof. Given $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$, we get

$$g_M(\nabla_X Y, V) = g_M(\mathcal{A}_X Y, V) = g_M(X, Y) g_M(H^\perp, V) \tag{4.5}$$

and

$$g_M(\nabla_X Y, V) = -g_M(Y, \nabla_X V) = -g_M(Y, \mathcal{A}_X V). \tag{4.6}$$

Comparing (4.5) and (4.6), we obtain $\mathcal{A}_X V = -g_M(H^\perp, V)X$.

Hence,

$$g_M(\mathcal{A}_X V, X) = -g_M(H^\perp, V)||X||^2. \tag{4.7}$$

But

$$\begin{aligned} g_M(\mathcal{A}_X V, X) &= g_M(\nabla_X V, X) = -g_M(V, \nabla_X X) \\ &= -g_M(V, \mathcal{A}_X X) = 0 \quad \text{(by (2.6))} \end{aligned}$$

so that from (4.7), we have $H^\perp = 0$.

Therefore, the result follows. □

Remark 4.5. From the equation $\mathcal{A}_X Y = -\mathcal{A}_Y X$ for $X, Y \in \Gamma((\ker F_*)^\perp)$, we can obtain Theorem 4.4. But the equation $\mathcal{T}_V W = \mathcal{T}_W V$ for $V, W \in \Gamma(\ker F_*)$, yields no theorems like Theorem 4.4 on $\ker F_*$.

Theorem 4.6. *Let F be an h-anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-anti-invariant basis. Then the following conditions are equivalent:*

(a) *the distribution $\ker F_*$ defines a totally umbilic foliation on M.*

(b)
$$\mathcal{T}_V B_I X + \mathcal{H}\nabla_V C_I X = -g_M(H, X)IV$$

for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

(c)
$$\mathcal{T}_V B_J X + \mathcal{H}\nabla_V C_J X = -g_M(H, X)JV$$

for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

(d)
$$\mathcal{T}_V B_K X + \mathcal{H}\nabla_V C_K X = -g_M(H, X)KV$$

for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

Proof. Given $V, W \in \Gamma(\ker F_*)$, $X \in \Gamma((\ker F_*)^\perp)$, and $R \in \{I, J, K\}$, we obtain

$$\begin{aligned} g_M(\mathcal{T}_V W, X) &= g_M(\nabla_V RW, RX) \\ &= -g_M(RW, \nabla_V B_R X + \nabla_V C_R X) \\ &= -g_M(RW, \mathcal{T}_V B_R X + \mathcal{H}\nabla_V C_R X) \end{aligned}$$

so that it is easy to check that

$$\mathcal{T}_V W = g_M(V, W)H \Leftrightarrow \mathcal{T}_V B_R X + \mathcal{H}\nabla_V C_R X = -g_M(H, X)RV.$$

Hence,

$$(a) \Leftrightarrow (b), \quad (a) \Leftrightarrow (c), \quad (a) \Leftrightarrow (d).$$

Therefore, we get the result. □

Lemma 4.7. *Let F be an h-Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-Lagrangian basis. Then the following conditions are equivalent:*

(a) *The distribution $\ker F_*$ defines a totally umbilic foliation on M.*

(b) $\mathcal{T}_V IX = -g_M(H, X)IV$ *for* $X \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$.

(c) $\mathcal{T}_V KX = -g_M(H, X)KV$ *for* $X \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$.

(d) $\mathcal{T}_V JX = -g_M(H, X)JV$ *for* $X \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$.

Proof. By the proof of Theorem 4.6, we have (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c).
Given $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$, we get

$$\begin{aligned} g_M(\mathcal{T}_V W, X) &= g_M(\nabla_V JW, JX) \\ &= -g_M(JW, \nabla_V JX) \\ &= -g_M(JW, \mathcal{T}_V JX) \end{aligned}$$

so that we easily check that

$$\mathcal{T}_V W = g_M(V, W)H \Leftrightarrow \mathcal{T}_V JX = -g_M(H, X)JV.$$

Hence, (a) $\Leftrightarrow$ (d).
Therefore, the result follows. □

Using Proposition 4.1, Theorem 3.11, and Theorem 4.6, we get

Theorem 4.8. *Let F be an h-anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-anti-invariant basis. Then the following conditions are equivalent:*

(a) (M, g_M) *is locally a twisted product manifold of the form* $M_{(\ker F_*)^\perp} \times M_{\ker F_*}$.

(b)
$$g_M(\mathcal{A}_X B_I Y, IV) = g_M(C_I Y, I\mathcal{A}_X V)$$

and

$$\mathcal{T}_V B_I X + \mathcal{H}\nabla_V C_I X = -g_M(H, X)IV$$

for $V \in \Gamma(\ker F_*)$ *and* $X, Y \in \Gamma((\ker F_*)^\perp)$.

(c)
$$g_M(\mathcal{A}_X B_J Y, JV) = g_M(C_J Y, J\mathcal{A}_X V)$$

and

$$\mathcal{T}_V B_J X + \mathcal{H}\nabla_V C_J X = -g_M(H, X)JV$$

for $V \in \Gamma(\ker F_*)$ *and* $X, Y \in \Gamma((\ker F_*)^\perp)$.

(d)
$$g_M(\mathcal{A}_X B_K Y, KV) = g_M(C_K Y, K\mathcal{A}_X V)$$

and

$$\mathcal{T}_V B_K X + \mathcal{H}\nabla_V C_K X = -g_M(H, X)KV$$

for $V \in \Gamma(\ker F_*)$ *and* $X, Y \in \Gamma((\ker F_*)^\perp)$.

Using Proposition 4.1, Lemma 3.12, and Lemma 4.7, we have

Lemma 4.9. *Let F be an h-Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-Lagrangian basis. Then the following conditions are equivalent:*

(a) (M, g_M) *is locally a twisted product manifold of the form* $M_{(\ker F_*)^\perp} \times M_{\ker F_*}$.

(b)
$$\mathcal{A}_X IY = 0 \quad \text{and} \quad \mathcal{T}_V IX = -g_M(H, X)IV$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$.

(c)
$$\mathcal{A}_X KY = 0 \quad \text{and} \quad \mathcal{T}_V KX = -g_M(H, X)KV$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$.

(d)
$$\mathcal{A}_X JY = 0 \quad \text{and} \quad \mathcal{T}_V JX = -g_M(H, X)JV$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$.

Now, we consider the non-existence of some types of Riemannian submersions. Using Proposition 4.1 and Theorem 4.4, we get

Theorem 4.10. *Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Then there exists no h-anti-invariant submersion from $M = (M, E, g_M)$ onto (N, g_N) such that M is locally a nontrivial double-twisted product manifold of the form $M_{(\ker F_*)^\perp} \times M_{\ker F_*}$.*

Lemma 4.11. *Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Then there exists no h-Lagrangian submersion from $M = (M, E, g_M)$ onto (N, g_N) such that M is locally a nontrivial double-twisted product manifold of the form $M_{(\ker F_*)^\perp} \times M_{\ker F_*}$.*

Theorem 4.12. *Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Then there exists no h-anti-invariant submersion from $M = (M, E, g_M)$ onto (N, g_N) such that M is locally a nontrivial twisted product manifold of the form $M_{\ker F_*} \times M_{(\ker F_*)^\perp}$.*

Lemma 4.13. *Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Then there exists no h-Lagrangian submersion from $M = (M, E, g_M)$ onto (N, g_N) such that M is locally a nontrivial twisted product manifold of the form $M_{\ker F_*} \times M_{(\ker F_*)^\perp}$.*

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Author's address: Kwang-Soon Park, Division of General Mathematics, Room 4-107, Changgong Hall, University of Seoul, Seoul 02504, Republic of Korea, e-mail: parkksn@gmail.com.