Czechoslovak Mathematical Journal, 67 (142) (2017), 297–316

THE GENERAL RIGIDITY RESULT FOR BUNDLES OF A-COVELOCITIES AND A-JETS

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Received October 23, 2015. First published March 1, 2017.

Abstract. Let M be an m-dimensional manifold and $A = \mathbb{D}^r_k/I = \mathbb{R} \oplus N_A$ a Weil algebra of height r. We prove that any A-covelocity $T^A_x f \in T^{A*}_x M$, $x \in M$ is determined by its values over arbitrary $\max\{\operatorname{width} A, m\}$ regular and under the first jet projection linearly independent elements of $T^A_x M$. Further, we prove the rigidity of the so-called universally reparametrizable Weil algebras. Applying essentially those partial results we give the proof of the general rigidity result $T^{A*}M \simeq T^{r*}M$ without coordinate computations, which improves and generalizes the partial result obtained in Tomáš (2009) from $m \geqslant k$ to all cases of m.

We also introduce the space $J^A(M,N)$ of A-jets and prove its rigidity in the sense of its coincidence with the classical jet space $J^r(M,N)$.

Keywords: r-jet; bundle functor; Weil functor; Lie group; jet group; B-admissible A-velocity

MSC 2010: 58A20, 58A32, 53C24

1. Preliminaries

1.1. We give a contribution to the theory of Weil functors. We start from the concepts of the r-jet, the jet space $J^r(M,N)$ and the bundle functor, all defined in [8]. By M we denote an m-dimensional manifold and by N a smooth manifold. All manifolds are supposed to be of class C^∞. By $\mathcal{M}f_m$ we denote the category of m-dimensional manifolds with local diffeomorphisms, by $\mathcal{M}f$ the category of smooth manifolds with smooth maps and by $\mathcal{FM}$ the category of fibered manifolds with smooth fibered maps. Further, we denote by $\mathcal{FM}_m$ the category of fibered manifolds with m-dimensional bases with fibered maps covering local diffeomorphisms. By P^rM we denote the frame bundle on M. A bundle functor defined on the category $\mathcal{M}f_m$ is said to be a natural bundle, see [8]. There is an important bundle functor

DOI: 10.21136/CMJ.2017.0566-15

defined on the category $\mathcal{M}f_m \times \mathcal{M}f$, the so-called jet functor J^r. It assigns the space of r-jets $J^r(M,N)$ to any couple $(M,N) \in \mathrm{Obj}(\mathcal{M}f_m \times \mathcal{M}f)$ and the map $J^r(g,h)\colon J^r(M_1,N_1) \to J^r(M_2,N_2)$ defined by $j^r_x f \mapsto j^r_{f(x)} h \circ j^r_x f \circ (j^r_x g)^{-1}$ to any local diffeomorphism $g\colon M_1 \to M_2$ defined near x and any smooth map $h\colon N_1 \to N_2$.

There is a significant class of product-preserving bundle functors defined on the category $\mathcal{M}f$. The classical result (see [6], [5], [8], [11]) reads that the functors of this kind coincide with Weil functors. Denoting them by T^A we involve the associated Weil algebras to their notation. The restriction of a Weil functor to $\mathcal{M}f_m$ is said to be a Weil bundle.

The concept of a Weil functor goes back to 1953, when Weil defined and studied the spaces of the infinitely near points on manifolds, see [20]. This concept led to the contravariant definition of the Weil functor presented below. However, most authors studied Weil functors from the covariant point of view. The equivalence of both approaches was proved in [8]. Besides the authors of those fundamental results, there are many others studying not only the geometry of Weil functors but also their applications in various research areas like the differential invariant theory and the theory of natural operators, e.g., Kolář, Mikulski, Shurygin, Bushueva and many others (see e.g. [8], [7], [9], [12], [16], [15], [4]). However, there are also authors applying and connecting Weil functors to problems from the Lie group theory, e.g., Alonso, Muriel, Muňoz, Rodriguez, and others, see [1], [2], [13]. Further, there are papers applying Weil functors in the theory of jets (e.g., Kureš in [10], who also studies their applications in the theoretical mechanics).

From the algebraic point of view, Weil functors are studied by Bertram (see [3]), who generalizes their definition in the sense of substituting general fields and rings for reals to the definition of the associated Weil algebra. Weil functors are also applied in the research of geometrical categories, namely in the papers by Nishimura, see [14].

We recall briefly the elementary concepts concerning Weil functors from [8]. Let $\mathcal{E}(k)$ be the algebra of germs of smooth functions defined on $\mathbb{R}^k$ with the source at zero. A Weil algebra A can be defined either as the direct sum $\mathbb{R} \oplus N_A$ of reals with the so-called nilpotent ideal N_A of all of its nilpotent elements, or as a quotient $\mathcal{E}(k)/I$ by an ideal $I \subset \mathcal{E}(k)$ of a finite codimension. A can be also defined as a quotient $\mathbb{D}^r_k/J$ of the so-called jet algebra $\mathbb{D}^r_k$ by its ideal J. In other words, A can be considered as the algebra of polynomials of k indeterminates of order at most r with the truncated multiplication factorized by one of its ideals. Finally, we define $\operatorname{width} A = \dim(N_A/N_A^2)$ and $\operatorname{height} A$ as the minimal r for which $A = \mathbb{D}^r_k/J$. For the nilpotent ideal of $\mathbb{D}^r_k$, we simplify the notation from $N_{\mathbb{D}^r_k}$ to N^r_k.

We remark that if N is associated with the symbol of a Weil algebra, then it denotes its nilpotent ideal, e.g., N_A or N^r_k. On the other hand, N endowed with no index is a symbol reserved for a manifold. This convention will hold in the whole paper.

In the whole paper, we apply the covariant approach to the definition of Weil functors. This comes out from the I-factorization of germs in the following sense. Two germs $\operatorname{germ}_0 g\colon \mathbb{R}^k_0 \to M_x$ and $\operatorname{germ}_0 h\colon \mathbb{R}^k_0 \to M_x$ taking the same value x at the source $0 \in \mathbb{R}^k$ are said to be I-equivalent if and only if $\operatorname{germ}_x \gamma \circ \operatorname{germ}_0 g - \operatorname{germ}_x \gamma \circ \operatorname{germ}_0 h \in I$ for any smooth function $\gamma\colon M \to \mathbb{R}$ defined near x. Classes of such equivalence relation are denoted by $j^A g$ and the space of them by $J^A M$. For a smooth map $\varphi\colon M \to N$, there is a map $J^A\varphi\colon J^A M \to J^A N$ defined by the assignment $j^A g \mapsto j^A(\varphi \circ g)$. For $A = \mathbb{D}^r_k$ we have $J^A M = J^r_0(\mathbb{R}^k, M)$ and $J^A\varphi = J^r_0(\mathrm{id}_{\mathbb{R}^k}, \varphi)$.

We remark that a Weil functor can be also defined from the contravariant point of view as a functor T^A defined by $T^A M = \operatorname{Hom}(C^\infty(M,\mathbb{R}), A)$ on objects and by $T^A f(H)(\varphi) = H(\varphi \circ f)$ on a morphism $f\colon M \to N$, where $H \in T^A M$ and $\varphi \in C^\infty(N, \mathbb{R})$. There is the identification $\theta\colon J^A M \to T^A M$ defined by $\theta(j^A g)(\varphi) = j^A(\varphi \circ g)$ for any $\varphi \in C^\infty(M,\mathbb{R})$ defined near $g(0)$ and by $\theta(J^A f)(H)(\psi) = H(\psi \circ f)$ for any smooth $f\colon M \to N$, $H \in T^A M$ and an any function $\psi \in C^\infty(N,\mathbb{R})$. In what follows we will use only the notation T^A for Weil functors in spite of applying the covariant approach to them.

We remark that for $A = \mathbb{D}^r_k$ the functor $T^A = T^{\mathbb{D}^r_k}$ coincides with the well-known functor T^r_k of (k,r)-velocities defined by $T^r_k M = J^r_0(\mathbb{R}^k, M)$ on objects and by $T^r_k f(j^r_0\varphi) = j^r_0(f \circ \varphi)$ on morphisms for any $j^r_0\varphi \in T^r_k M$, see [8].

As for natural transformations $\widetilde{t}_M\colon T^B M \to T^A M$, they bijectively correspond to homomorphisms $t\colon B \to A$. Further, homomorphisms $B \to A$ correspond bijectively to the so-called B-admissible A-velocities defined in [7] as follows. Let $A = \mathcal{E}(k)/I$ and $B = \mathcal{E}(p)/J$ be Weil algebras. For a smooth map $f\colon \mathbb{R}^k_0 \to \mathbb{R}^p_0$, an A-velocity $j^A f$ is said to be B-admissible if and only if $\operatorname{germ}_0 \varphi \in J \Rightarrow \operatorname{germ}_0(\varphi \circ f) \in I$.

Thus every B-admissible A-velocity $j^A f$ is assigned bijectively a natural transformation $\widetilde{t}_M\colon T^B M \to T^A M$ induced by a Weil algebra homomorphism $t = t^{j^A f}\colon B \to A$. The natural transformation $\widetilde{t}_M$ is defined as

$$\widetilde{t}_M(j^B\varphi) = \widetilde{t}^{j^A f}_M(j^B\varphi) = j^A(\varphi \circ f). \tag{1.1}$$

In other words, a Weil algebra homomorphism $t\colon B \to A$ together with its associated natural transformation $\widetilde{t}_M\colon T^B M \to T^A M$ corresponds to a reparametrization of the indeterminates $t_1, \ldots, t_p$ by suitable polynomials in $\tau_1, \ldots, \tau_k$, transforming the ideal J to I. In particular, isomorphisms of Weil algebras correspond to invertible reparametrizations of this kind transforming bijectively the ideals defining them, which in the case of automorphisms coincide.

Let G^r_k denote the general jet group, see [8]. We recall that its support is formed by $J^r_0(\mathbb{R}^k, \mathbb{R}^k)_0$ endowed with the multiplication defined by the composition of jets.

It is a well-known fact that every $j^r_0 g \in G^r_k$ determines an automorphism $t_{j^r_0 g}$ of $\mathbb{D}^r_k$ defined by the assignment

$$j^r_0\eta \mapsto j^r_0\eta \circ (j^r_0 g)^{-1} \tag{1.2}$$

for any $j^r_0\eta \in \mathbb{D}^r_k$. If it is clear from the context we can write only $j^r_0 g$ instead of $t_{j^r_0 g}$, applying this convention particularly in (1.3), (1.4) and (1.5) below. Further, every automorphism of this kind determines the natural equivalence $\widetilde{t}_{j^r_0 g, M}$ of $T^r_k M$. For a Weil algebra $A = \mathbb{D}^r_k/I$ and the canonical projection homomorphism $p\colon \mathbb{D}^r_k \to A$, Alonso in [1] defined the subgroups G_A and $G^A \subseteq G^r_k \simeq \operatorname{Aut}(\mathbb{D}^r_k)$ as

$$G_A = \{j^r_0 g \in G^r_k\colon\ p \circ j^r_0 g = p\}, \quad G^A = \{j^r_0 g \in G^r_k\colon\ \operatorname{Ker}(p \circ j^r_0 g) = \operatorname{Ker}(p)\}. \tag{1.3}$$

He also proved that G_A is a normal subgroup of G^A and the identification $G^A/G_A \simeq \operatorname{Aut} A$. Clearly, $j^r_0 g$ determines an automorphism of A and consequently a natural equivalence over $T^A M$ if and only if $j^r_0 g \in G^A$. Clearly, $G_{\mathbb{D}^r_k} = \{j^r_0 \operatorname{id}_{\mathbb{R}^k}\}$ and $G^{\mathbb{D}^r_k} = G^r_k$.

In the rest of the paper, we will involve the Weil algebra into the notation of p from (1.3), i.e., we will write p_A instead p. Consider the left action l of G^A and the effective left action $\bar{l}$ of $\operatorname{Aut} A$ on $T^A_x M$ defined as

$$l(j^r_0 g, j^A\varphi_x) = \bar{l}(\pi(j^r_0 g), j^A\varphi_x) = \pi(j^r_0 g)(j^A\varphi_x) = \widetilde{p}_{A,M}(j^r_0\widehat{\varphi}_x \circ (j^r_0 g)^{-1}), \tag{1.4}$$

where $\pi\colon\ G^A \to \operatorname{Aut} A$ is the projection Lie group homomorphism and $j^r_0\widehat{\varphi}_x \in j^A\varphi_x$ is arbitrary. For a Lie subgroup $H \subseteq \operatorname{Aut} A$ let us denote by $\mathcal{O}rb(H, j^A\varphi_x)$ the H-orbit of $j^A\varphi_x$ with respect to $\bar{l}$ restricted to $H \times T^A_x M$. Since every element of $\operatorname{Aut} A$ determines a natural transformation over $T^A M$ determined by (1.4), the values of $T^A_x f$ on the whole $\mathcal{O}rb(\operatorname{Aut} A, j^A\varphi_x)$ are determined by $T^A_x f(j^A\varphi_x)$. This follows from the identity

$$\begin{aligned} T^A_x f(l(j^r_0 g, j^A\varphi_x)) &= T^A_x f(\bar{l}(\pi(j^r_0 g), j^A\varphi_x)) = (T^A_x f \circ \pi(j^r_0 g))(j^A\varphi_x) \\ &= (\pi(j^r_0 g) \circ T^A_x f)(j^A\varphi_x) = \bar{l}(\pi(j^r_0 g), T^A_x f(j^A\varphi_x)) \\ &= l(j^r_0 g, T^A_x f(j^A\varphi_x)). \end{aligned} \tag{1.5}$$

It is easy to see that $j^A\varphi_{x,1} = j^A\varphi_{x,2}$ implies $(j^r_0\varphi_{x,1})^{-1} \circ j^r_0\varphi_{x,2} \in G_A$.

1.2. Besides Weil bundles there are significant natural bundles $T^{r*}M$, the bundles of the r-th order covelocities (see (1.3)). They are defined by $T^{r*}_x M = J^r_x(M, \mathbb{R})_0$ on objects and by $T^{r*}g(j^r_x f) = j^r_x f \circ (j^r_x g)^{-1}$ on morphisms, where $j^r_x f \in T^{r*}_x M$ is arbitrary. In [17] we have presented the bundles $T^{r*}M$ in the form of $P^r M[N^r_m, l]$, where $l\colon G^r_m \times N^r_m \to N^r_m$ denotes the left action of the jet group on the standard fiber determined by (1.4) and applied to the case of $A = \mathbb{D}^r_m$.

Moreover, we have generalized the space of higher-order covelocities $T^{r*}M$ to the space of A-covelocities $T^{A*}M$ consisting of the elements $T^A_x f\colon T^A_x M \to T^A_0\mathbb{R} \simeq N_A$. Further, for any local diffeomorphism g defined near x we have defined the so-called T^{A*}-map as

$$T^{A*}g(T^A_x f) = T^A_x f \circ (T^A_x g)^{-1}. \tag{1.6}$$

We recall the partial result from [17] obtained for such spaces and maps.

Let $A = \mathbb{D}^r_k/I$ and $m = \dim M$. Then the following holds:

For $m \geqslant k$, the spaces $T^{A*}M$ together with the maps defined by (1.6) form the natural bundle $P^rM[N^r_m, l]$ identified with $T^{r*}M$.

For any $M \in \mathrm{Obj}(\mathcal{M}f_m)$ and $N \in \mathrm{Obj}(\mathcal{M}f)$ we define the space $J^A(M,N)$ by $J^A(M,N) = \{T^A_x f; f\colon M \to N\}$. For a local diffeomorphism $g\colon M_1 \to M_2$ and a smooth map $h\colon N_1 \to N_2$ we define the map $J^A(g,h)\colon J^A(M_1,N_1) \to J^A(M_2,N_2)$ by

$$J^A(g,h)(T^A_x f) = T^A_{f(x)}h \circ T^A_x f \circ (T^A_x g)^{-1}. \tag{1.7}$$

In [19] we have proved a partial result valid for $m \geqslant k$, which states the identification of $J^A(M,N)$ with $J^r(M,N)$. In the conference paper [18] we have also sketched a proof concerning the cases of $m < k$, following the coordinate proof of the mentioned partial result from [17].

In the present paper, the main result will be proved and generalized in a new way, applying essentially Proposition 2.2 below, formulated and proved in a rigorous and explicit way, and Proposition 4.1, which states the rigidity of the so-called universally reparametrizable Weil algebras. In this proof we essentially reduce the computations with coordinates.

At the end of this section we observe that the elements $T^A_x f \in T^{A*}_x M$ form the nilpotent ideal $N_{B_{x,M}}$ of a Weil algebra $B_{x,M} = \mathbb{D}^r_m/J_{x,M}$ obtained from the algebraic operations defined on $T^{A*}_x M$ as follows. The vector space operations are defined by $(T^A_x f_1 + T^A_x f_2)(j^A\varphi_x) = T^A_x f_1(j^A\varphi_x) + T^A_x f_2(j^A\varphi_x)$ and $(c\cdot T^A_x f)(j^A\varphi_x) = c\cdot T^A_x f(j^A\varphi_x)$ for any $j^A\varphi_x \in T^A_x M$ and $c \in \mathbb{R}$. Analogously we define the multiplication on $T^{A*}_x M$ by

$$\begin{aligned}(T^A_x f_1 \cdot T^A_x f_2)(j^A\varphi_x) &:= \mu_A(T^A_x f_1(j^A\varphi_x), T^A_x f_2(j^A\varphi_x))\\ &= T^A_{(0,0)}\mu \circ (T^A_x f_1(j^A\varphi_x), T^A_x f_2(j^A\varphi_x)),\end{aligned} \tag{1.8}$$

where μ denotes the standard multiplication on $\mathbb{R}$ and $\mu_A = T^A\mu$ the multiplication on $A = T^A\mathbb{R}$. In other words, the product $T^A_x f_1 \cdot T^A_x f_2$ is defined by

$$T^A_x f_1 \cdot T^A_x f_2 = T^A_{(0,0)}\mu \circ (T^A_x f_1, T^A_x f_2) = T^A_x(\mu \circ (f_1, f_2)). \tag{1.9}$$

Clearly, every local diffeomorphism $g\colon M \to N$ defined near x determines the isomorphism $T^{A*}_x g\colon T^{A*}_x M \to T^{A*}_{g(x)} N$ determined by the map $T^{A*}g$, which is defined by the assignment $T^A_x f \mapsto T^A_x f \circ (T^A_x g)^{-1}$ for an arbitrary $T^A_x f \in T^{A*}_x M$. Therefore $T^{A*}_x M$ can be identified with the nilpotent ideal of $B_{x,M}$ and consequently with the nilpotent ideal of the Weil algebra $\mathbb{R} \oplus T^{A*}_0 \mathbb{R}^m = B = \mathbb{D}^r_m / J$.

We present the already mentioned result $T^{r*}M \simeq P^r M[N^r_m, l]$ from [17] in a suitable form for direct application in the proof of the main result.

Lemma 1.1. *The multiplication in $T^{\mathbb{D}^r_k *}_0 \mathbb{R}^m$ defined by* (1.9) *coincides with the restriction of the standard multiplication in $\mathbb{D}^r_m$ to N^r_m.*

P r o o f. The elements $j^r_0 f$ and $j^r_0 g \in \mathbb{D}^r_m$ together with their product $j^r_0 h = j^r_0 f \times j^r_0 g$, all of them considered in the polynomial form, satisfy in coordinates the identity $h_\alpha = f_\beta g_{\alpha-\beta}\, \alpha!/\beta!(\alpha-\beta)!$, applying the standard notation with multiindices. On the other hand, (1.8) yields

$$\begin{aligned} ((T^r_k)_0 f \cdot (T^r_k)_0 g)(j^r_0\varphi) &= (T^r_k)_0 h(j^r_0\varphi) \\ &= f_{l_1\dots l_s}\varphi^{l_1}_{\mu_1}\dots\varphi^{l_s}_{\mu_s} \cdot g_{j_1\dots j_t}\varphi^{j_1}_{\nu_1}\dots\varphi^{j_t}_{\nu_t}\tau^{\mu\nu} \mod \langle \tau_1,\dots,\tau_k\rangle^{r+1}, \end{aligned} \tag{1.10}$$

where $j^r_0\varphi \in (T^r_k)_0\mathbb{R}^m$ is arbitrary and $\tau_1,\dots,\tau_k$ denote the indeterminates of $\mathbb{D}^r_k$. We apply the standard derivative indices to the notation of the coordinates of $(T^r_k)_0 f$ and $(T^r_k)_0 g$ while in case of $j^r_0\varphi$ we apply the notation with multiindices. Put $j^r_0\varphi = (j^r_0 \operatorname{pr}_1)^m$ for the the first canonical projection $\operatorname{pr}_1\colon \mathbb{R}^k \to \mathbb{R}$. In coordinates, we have $\varphi^l_j = \delta^1_j$ where $l = 1,\dots,m$. Applying the product $(T^r_k)_0 h$ from (1.10) to our $j^r_0\varphi$ and standard combinatorics, we obtain the coordinates of $(T^r_k)_0 h$ expressed in multiindices, which coincide with those of the coordinate identity in the very beginning of the proof. □

Further, there is the projection homomorphism $p_B\colon \mathbb{R} \oplus J^r_0(\mathbb{R}^m, \mathbb{R})_0 \simeq \mathbb{D}^r_m \to B$ defined by

$$p_B((T^r_k)_0 f) = T^A_0 f. \tag{1.11}$$

The verification of $p_B((T^r_k)_0 f \cdot (T^r_k)_0 g) = p_B((T^r_k)_0 f) \cdot p_B((T^r_k)_0 g) = T^A_0 f \cdot T^A_0 g$ is a direct consequence of (1.9). Indeed, $p_B((T^r_k)_0 f \cdot (T^r_k)_0 g) = p_B((T^r_k)_0 (f \cdot g)) = T^A_0 (f \cdot g) = T^A_0 f \cdot T^A_0 g$.

Moreover, any surjective homomorphism $p_{\bar{A},\widehat{A}}\colon\ \bar{A}\to\widehat{A}$ of Weil algebras determines the Weil algebra homomorphism $p_{\overline{B},\widehat{B}}\colon\ \overline{B}=T_0^{\bar{A}*}\mathbb{R}^m\to T_0^{\widehat{A}*}\mathbb{R}^m=\widehat{B}$ defined by

$$(p_{\overline{B},\widehat{B}}(T_0^{\bar{A}}f))(p_{\bar{A},\widehat{A}}(j^{\bar{A}}\varphi))=p_{\bar{A},\widehat{A}}(T_0^{\bar{A}}f(j^{\bar{A}}\varphi)) \tag{1.12}$$

for any $j^{\bar{A}}\varphi\in T_0^{\bar{A}}\mathbb{R}^m$, which can be easily and directly verified.

2. A-covelocities determined by their values over $K=\max\{\operatorname{width}A,\dim M\}$ elements of $\operatorname{reg}T_x^AM$

In this section we prove that for any m-dimensional manifold M, every A-covelocity $T_x^Af\in T_x^{A*}M$ is determined by its values over arbitrary $\max\{\operatorname{width}A,m\}$ regular and under the first jet projection linearly independent elements of T_x^AM. We precise the partial result from the conference paper [18], in which we sketched the proof of the fact that any A-covelocity is determined by its values over a linearly independent series of 1-jets of regular elements from T_x^AM. We keep the basic idea but we add an explicit formulation and rigorous construction of this determination applicable particularly in the proof of the main rigidity result. We also discuss properly the case of $m>k$ mentioned in that paper.

Without loss of generality, $A=\mathbb{D}_k^r/I$ will be supposed in the so-called normal form. The correctness of such assumption will be verified by the next lemma. The ideal I is said to be normal if $I=J\vee\langle\tau_{j_1},\dots,\tau_{j_{k-l}}\rangle$ for another ideal J satisfying $J\subseteq\langle\tau_{i_1},\dots,\tau_{i_l}\rangle^2$, where $\{j_1,\dots,j_{k-l}\}\cup\{i_1,\dots,i_l\}=\{1,\dots,k\}$. Further, A is said to be in the normal form if I is normal. Clearly, the definition of normality can be reduced to the condition $I\subseteq N_A^2$ in the case of $\operatorname{width}A=k$. It follows from the comment after (1.1) that every Weil algebra is isomorphic to a normal one.

Lemma 2.1. *Let $t\colon\ A\to\widehat{A}$ be an isomorphism of Weil algebras. Then any $T_x^Af\in T_x^{A*}M$ is determined by its values over elements $j^A\varphi_{1,x},\dots,j^A\varphi_{h,x}\in T_x^AM$ if and only if $T_x^{\widehat{A}}f$ is determined by its values over $\widetilde{t}_M(j^A\varphi_{1,x}),\dots,\widetilde{t}_M(j^A\varphi_{h,x})$.*

P r o o f. Fix $j^A\varphi_x\in T_x^AM$, put $j^A\eta_i=T_x^Af(j^A\varphi_{i,x})$ and $j^A\eta=T_x^Af(j^A\varphi_x)$. Then we represent the values of individual T_x^Af over $j^A\varphi_x$ by the map $\omega\colon\ N_A^h\to N_A$ assigning some $j^A\eta$ to each h-tuple $(j^A\eta_1,\dots,j^A\eta_h)$. The same can be done with the space $T_x^{\widehat{A}}M$, the elements $j^{\widehat{A}}\widehat{\varphi}_{i,x}=\widetilde{t}_M(j^A\varphi_{i,x})$, $j^{\widehat{A}}\widehat{\varphi}_x=\widetilde{t}_M(j^A\varphi_x)$, $j^{\widehat{A}}\widehat{\eta}_i=\widetilde{t}_{\mathbb{R}}(j^A\eta_i)$ and $j^{\widehat{A}}\widehat{\eta}=\widetilde{t}_{\mathbb{R}}(j^A\eta)$, obtaining the map $\widehat{\omega}\colon\ N_{\widehat{A}}^h\to N_{\widehat{A}}$. To complete the proof it suffices to put $\widehat{\omega}=\widetilde{t}_{\mathbb{R}}\circ\omega\circ(\widetilde{t}_M^{-1}\times\ldots\times\widetilde{t}_M^{-1})$, which follows from the commutativity of morphisms with natural transformations. $\square$

Let $\widetilde{B}$ be some Weil algebra of height r, $q < r$ and $\widetilde{B}_q$ denote the subordinate Weil algebra obtained by truncating $\widetilde{B}$ to the order q. By $\pi_{q,\widetilde{B}}\colon \widetilde{B} \to \widetilde{B}_q$ we denote the projection homomorphism. Further, there is a homomorphism $\pi^q_{s,\widetilde{B}}\colon \widetilde{B}_q \to \widetilde{B}_s$ determined by $\pi_{q,\widetilde{B}}$ and $\pi_{s,\widetilde{B}}$, which induces the natural transformation $(\widetilde{\pi}^q_{s,\widetilde{B}})_M\colon T^{\widetilde{B}_q}M \to T^{\widetilde{B}_s}M$.

Consider the Weil algebra $B = \mathbb{R} \oplus N_B = \mathbb{D}^r_m/J$ with $N_B = T_0^{A*}\mathbb{R}^m$, see (1.11). Recall that for $A = \mathbb{D}^r_k$ the algebra B is identified with $\mathbb{D}^r_m$. Further, assume $\operatorname{width} A = k$ in almost all of this section with the exception of Remark 2.2 and Corollary 2.1 at the very end.

In what follows, we need the cyclic permutations $\sigma_1, \dots, \sigma_m \in S_m$ considered in the form of the permutation matrices denoted by the same symbols. By S_m we denote the set of all permutations on $\{1, \dots, m\}$ and by $C_m \subseteq S_m$ the set of all cycles of length m. Further, for any permutation matrix σ, a real c and $l \in \{1, \dots, m\}$ let us define the matrix $\sigma(c, l)$ as

$$\sigma(c,l) = d(1, c^l, c^{2l}, \dots, c^{(m-1)l}) \cdot \sigma, \tag{2.1}$$

where $d(a_1, \dots, a_m)$ denotes in general the matrix with elements $a_1, \dots, a_m$ on its diagonal and zeros on the other positions. Clearly, $\sigma(1, l) = \sigma$.

Lemma 2.2. *Let $c \notin \{0, 1, -1\}$. Then the matrices $\sigma(c, l)$ for $\sigma \in C_m$ and $l = 1, \dots, m$ form a linear basis of the Lie algebra $gl(m, \mathbb{R})$.*

P r o o f. The assertion is obtained from the assignment of an m^2-dimensional row vector to every matrix $\sigma(c, l)$ read by rows and from the construction of the m^2-th order matrix with rows formed by $\sigma_1(c, 1), \dots, \sigma_1(c, m), \dots, \sigma_m(c, 1), \dots, \sigma_m(c, m)$ considered in the vector form. By a suitable permutation of columns of such matrix we obtain a block-diagonal matrix. Expanding its determinant by its m-th order diagonal blocks we obtain its value in the form of the product of m Van der Monde determinants. This proves our claim. □

In what follows, A is supposed to be normal. Let $B = \mathbb{D}^r_m/J = \mathbb{R} \oplus T_0^{A*}\mathbb{R}^m$ be the Weil algebra from (1.11). Further, let

$$\begin{aligned} T_0^A f &= p_B \circ (T^r_k)_0 f = p_B\left(\sum_{i=1}^r (T^r_k)_0 f_{(i)}\right) = \sum_{i=1}^r T_0^A f_{(i)} \\ &= p_B((T^r_k)_0 f_{(r)} + (T^r_k)_0 f^{(r-1)}) = T_0^A f_{(r)} + T_0^A f^{(r-1)} \end{aligned} \tag{2.2}$$

be some decomposition of $T_0^A f \in N_B$ such that $(T^r_k)_0 f_{(i)}$ corresponds to a homogenous polynomial of order i and $(T^r_k)_0 f^{(i)}$ denotes the sum $\sum\limits_{j=1}^{i} (T^r_k)_0 f^{(j)}$.

Let us denote by $\iota_k^r\colon \mathbb{D}_k^1 \to \mathbb{D}_k^r$ the canonical linear insertion of linear polynomials into the linear support of $\mathbb{D}_k^r$ and by $\iota_k^{r,m}\colon (\mathbb{D}_k^1)^m \to (\mathbb{D}_k^r)^m$ the associated product linear map. Further, consider the canonical linear insertion $i_A\colon \mathbb{D}_k^1 \to A$ defined by $i_A = p_A \circ \iota_k^r$. Recalling our assumptions of $\operatorname{width} A = k$ and the normality of A we deduce that this is a section with respect to $\pi_{1,A}\colon A \to A_1 \simeq \mathbb{D}_k^1$. Finally, consider the product map $i_A^m\colon (\mathbb{D}_k^1)^m \to A^m$ associated to $i_A\colon \mathbb{D}_k^1 \to A$. Clearly, this is a linear map again and since $k = \operatorname{width} A$, the space $J_0^1(\mathbb{R}^k, \mathbb{R}^m)_0 \simeq i_A^m(J_0^1(\mathbb{R}^k, \mathbb{R}^m)_0)$ can be considered as a linear subspace of $T_0^A\mathbb{R}^m$. Clearly, $\iota_k^r = i_{\mathbb{D}_k^r}$ and $\iota_k^{r,m} = i_{\mathbb{D}_k^r}^m$.

Let us consider $j^A\eta \in T_0^A\mathbb{R} \simeq N_A$ in the form of $j^A\eta = T_0^A f(j^A\varphi)$ for some $j^A\varphi \in \operatorname{reg} T_0^A\mathbb{R}^m$ and $T_0^A f \in T_0^{A*}\mathbb{R}^m$. Clearly, $j^A\eta$ can be decomposed (not necessarily uniquely) to

$$
\begin{aligned}
(2.3) \qquad j^A\eta &= p_A(j_0^r\eta) = p_A\left(\sum_{i=1}^{r} j_0^r\eta_{(i)}\right) = \sum_{i=1}^{r} j^A\eta_{(i)} \\
&= p_A(j_0^r\eta_{(r)}) + p_A(j_0^r\eta^{(r-1)}) = j^A\eta_{(r)} + j^A\eta^{(r-1)},
\end{aligned}
$$

where $j_0^r\eta_{(i)}$ coincides with a homogenous polynomial of order i and $j_0^r\eta^{(i)}$ with the sum $\sum_{j=1}^{i} j_0^r\eta_{(j)}$. It follows from the jet composition formula applied to any $(T_k^r)_0 f$ from (2.2) that $T_0^A f_{(r)}$ acts only on $j_0^1\varphi \simeq i_A^m(j_0^1\varphi)$ and that $T_0^A f_{(r)}$ affects only $j^A\eta_{(r)}$.

In the following investigations we will work with permutations, permutation matrices and their associated linear maps. Clearly, any permutation $\varrho \in S_m$ may be via its permutation matrix identified with the linear transformation s_ϱ on $\mathbb{R}^m$ defined by $(x^1, \dots, x^m) \mapsto (x^{\sigma(1)}, \dots, x^{\sigma(m)})$ and consequently with an element of $\operatorname{reg} J_0^1(\mathbb{R}^m, \mathbb{R}^m)_0$. Analogously we can do with the matrices $\sigma(c,l)$ determining the linear transformations on $\mathbb{R}^m$ denoted by $s_{\sigma(c,l)}$.

The following modifications of σ and s_σ are applied in Proposition 2.1, namely in its second part concerning the case of $m > k$. Nevertheless, the principal part of Proposition 2.1 and its proof will correspond to the case of $m = k$, for which we can do only with σ and s_σ.

As for $m \geqslant k$ and $\sigma \in S_m$, denote by $\widetilde{\sigma}\colon \{1, \dots, m\} \to \{0, 1, \dots, m\}$ the map assigning $\sigma(l)$ to l for $l \in \{1, \dots, k\}$ and 0 otherwise. Analogously to σ, the map $\widetilde{\sigma}$ can be identified with the linear transformation $s_{\widetilde{\sigma}}$ of $\mathbb{R}^m$ defined by the assignment $(x^1, \dots, x^m) \mapsto s_\sigma(x^1, \dots, x^k, 0, \dots, 0)$. Finally, let us define the map $\overline{\sigma} : \{1, \dots, k\} \to \{1, \dots, m\}$ by the assignment $\overline{\sigma}(l) = \sigma(l)$ for $l = 1, \dots, k$ and the map $j : \mathbb{R}^k \to \mathbb{R}^m$ by

$$
(2.4) \qquad (x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0).
$$

Clearly, $\overline{\sigma}$ can be identified with the linear map $s_{\overline{\sigma}}\colon \mathbb{R}^k \to \mathbb{R}^m$ defined by $s_{\overline{\sigma}} = s_{\widetilde{\sigma}} \circ j$ and consequently with the corresponding element of $\operatorname{reg} J^1_0(\mathbb{R}^k, \mathbb{R}^m)_0$. In other words, we remove the last $m-k$ columns from the matrix of the linear transformation $s_{\widetilde{\sigma}}$. The same can be done with the matrices $\sigma(c,l)$, obtaining the matrices $\overline{\sigma}(c,l)$ and linear maps $\mathbb{R}^k \to \mathbb{R}^m$ denoted by $s_{\overline{\sigma}(c,l)}$ together with the corresponding elements of $J^1_0(\mathbb{R}^k, \mathbb{R}^m)_0$.

In order to be brief we simplify the notation from $s_{\overline{\sigma}}$ to $\overline{\sigma}$, from $s_{\widetilde{\sigma}}$ to $\widetilde{\sigma}$ and from $s_{\sigma(c,l)}$ to $\sigma(c,l)$. As for matrices in general, we apply the same notation for them and the associated linear transformations of numerical spaces as well as for the corresponding 1-jets of zero-preserving maps.

Proposition 2.1. *Let A be in the normal form, $m \geqslant k = \operatorname{width} A$ and $\sigma_1, \dots, \sigma_m$ be the cyclic permutations. Then every $T^A_0 f_{(r)}$ contained in decomposition* (2.2) *is determined by its m images over $i^m_A(\overline{\sigma}_1), \dots, i^m_A(\overline{\sigma}_m)$.*

P r o o f. It follows from the deductions connected with (2.3) that for the purpose of our proof we can consider

$$T^A_0 f_{(r)}(i^m_A(j^1_0\varphi)) \simeq T^A_0 f_{(r)}(j^1_0\varphi) \tag{2.5}$$

instead of $T^A_0 f_{(r)}(j^A\varphi)$, keeping in mind the normal form of I. By Lemma 2.2, a regular matrix $j^1_0\varphi \in \operatorname{reg} J^1_0(\mathbb{R}^m, \mathbb{R}^m)_0$ is identified with some linear combination of the matrices $\sigma(c,l)$ from (2.1), i.e., $j^1_0\varphi = p^{\sigma,l}\sigma(c,l)$ for suitable coefficients $p^{\sigma,l} \in \mathbb{R}$, $\sigma \in C_m$ and $l = 1, \dots, m$.

In the first step, giving the principle idea of the proof, suppose $m = k$. In this case $\overline{\sigma}_i = \sigma_i$. Then for every $c \notin \{-1, 0, 1\}$ the value $T^A_0 f_{(r)}(j^1_0\varphi)$ can be expressed in the form of

$$T^A_0 f_{(r)}(i^k_A(p^{\sigma,l}\sigma(c,l))), \quad \sigma \in C_k \text{ and } l = 1, \dots, k. \tag{2.6}$$

Fixing σ, the sum $p^{\sigma,l}\sigma(c,l)$ contained in (2.6) can be expressed in the form of $\sigma(1,l) \cdot D_{\sigma,c} = \sigma \cdot D_{\sigma,c}$ for some matrix $D_{\sigma,c}$ having all elements zeros except the diagonal ones. Clearly,

$$D_{\sigma,c} = \sigma^{-1} \cdot d(p^{\sigma,l}c^{0\cdot l}, p^{\sigma,l}c^{1\cdot l}, p^{\sigma,l}c^{2\cdot l}, \dots, p^{\sigma,l}c^{(k-1)\cdot l}) \cdot \sigma, \tag{2.7}$$

where the components of the diagonal-like matrix in (2.7) are in the form of sums for $l = 1, \dots, k$. Clearly, for a dense subset of $\mathbb{R}^k$ of all k-tuples $(p^{\sigma,1}, \dots, p^{\sigma,k})$, the matrix $D_{\sigma,c} \simeq \iota^{r,k}_k(D_{\sigma,c}) \in G^A$. Applying (2.5) and recalling the notation (1.2) for automorphisms of $\mathbb{D}^r_k$ and the left action from (1.4), we deduce that for the dense subset of $\mathbb{R}^k$ under discussion, every summand of (2.6) is of the form

$$(2.8)\quad \begin{aligned} T_0^A f_{(r)}(i_A^k(\sigma\cdot D_{\sigma,c})) &= T_0^A f_{(r)}(l(\iota_k^{r,k}(D_{\sigma,c}^{-1}), i_A^k(\sigma)) \\ &= t_{\iota_k^{r,k}(D_{\sigma,c}^{-1})}\circ T_0^A f_{(r)}(i_A^k(\sigma)) = T_0^A f_{(r)}(i_A^k(\sigma))\circ \iota_k^r(D_{\sigma,c}). \end{aligned}$$

The standard density argument and the coincidence of the expressions in the second line of (2.8) prove our assertion for a fixed cycle $\sigma\in C_k$ in the sum (2.6). Since every partial summand from (2.6) corresponding to a fixed $\sigma\in C_k$ corresponds to a matrix of the same kind as σ in its nonzero positions, it follows from the well-known jet composition formula that $T_0^A f_{(r)}$ in (2.6) acts linearly and, consequently, (2.6) can be divided into k summands corresponding to the individual cycles σ. This completes the proof of the first step.

In the case of $m>k$, let us extend A to $\widetilde{A}=\mathbb{D}^r_m/(I\vee\langle\tau_{k+1},\dots,\tau_m\rangle\langle\tau_1,\dots,\tau_m\rangle)$, which is of width m. Further, A can be identified with

$$(2.9)\qquad \bar{A}=\mathbb{D}^r_m/(I\vee\langle\tau_{k+1},\dots,\tau_m\rangle),$$

adding formally $m-k$ indeterminates. Clearly, there is the projection homomorphism $\bar{p}\colon\widetilde{A}\to\bar{A}\simeq A$. Consider $j_0^1\varphi\in J_0^1(\mathbb{R}^k,\mathbb{R}^m)_0$, which can be also considered as an element of $J_0^1(\mathbb{R}^m,\mathbb{R}^m)_0$. First we observe that for any $c\notin\{-1,0,1\}$ the equality $i^m_{\widetilde{A}}(j_0^1\varphi)=i^m_{\widetilde{A}}(p^{\sigma,l}\sigma(c,l))$ is valid if and only if so is the equality $i^m_A(j_0^1\varphi)=i^m_A(p^{\sigma,l}\overline{\sigma}(c,l))$ in which all $\sigma\in C_m$ are considered. One can verify easily that $i^m_A(j_0^1\varphi)$ equals to the sum $\overline{\sigma}\cdot D_{\overline{\sigma},c}$ for $\sigma\in C_m$ and the matrices $D_{\overline{\sigma},c}$ obtained from $D_{\sigma,c}$ as their first diagonal blocks of order k. Further, $T_0^A f_{(r)}(i^m_A(j_0^1\varphi))=T_0^A f_{(r)}(i^m_A(p^{\sigma,l}\overline{\sigma}(c,l)))=\bar{p}\circ T_0^{\widetilde{A}} f_{(r)}(i^m_{\widetilde{A}}(p^{\sigma,l}\sigma(c,l)))$.

Applying the first step of the present proof to $\widetilde{A}$ we obtain the above expression in the form of the sum $\bar{p}\circ T_0^{\widetilde{A}} f_{(r)}(l(\iota_m^{r,m}(D_{\sigma,c}^{-1})),i^m_{\widetilde{A}}(\sigma))=\bar{p}\circ(t_{\iota_k^{r,m}(D_{\sigma,c}^{-1})}\circ T_0^{\widetilde{A}} f_{(r)}(i^m_{\widetilde{A}}(\sigma)))=\bar{p}\circ T_0^{\widetilde{A}} f_{(r)}(i^m_{\widetilde{A}}(\sigma))\circ\iota^r_m(D_{\sigma,c})$ (for $\sigma\in C_m$) on the dense subset of all m^2-tuples consisting of the coefficients $p^{\sigma,l}$. It is easy to see that this expression equals to $T_0^A f_{(r)}(i^m_A(\overline{\sigma}))\circ\iota^r_k(D_{\overline{\sigma},c})$. Thus we have proved that $T_0^A f_{(r)}(i^m_A(j_0^1\varphi))$ is really determined by the values $T_0^A f_{(r)}(i^m_A(\overline{\sigma}))$ and the coefficients $D_{\overline{\sigma},c}$ constructed from $D_{\sigma,c}$. This completes the proof. □

Proposition 2.2. *Let $m=\dim M$, $A=\mathbb{D}^r_k/I$, $k=\operatorname{width}A$ and $K=\max\{k,m\}$. Then every $T_x^A f\in T_x^{A*}M$ is determined by its values over arbitrary, in the first jet projection linearly independent regular elements $j^A\varphi_{1,x},\dots,j^A\varphi_{K,x}\in T_x^A M$.*

P r o o f. First assume $m\geqslant k$. We prove that for every $T_0^A f\in T_0^{A*}\mathbb{R}^m$ and $j^A\varphi\in T_0^A\mathbb{R}^m$, the value $T_0^A f(j^A\varphi)=j^A\eta$ is determined by m values $T_0^A f(i^m_A(\overline{\sigma}_i))$. For $s=$

$1, \ldots, r-1$ consider the chain of Weil algebra homomorphisms $\pi_{s,A_{s+1}}\colon A_{s+1} \to A_s$ of truncated Weil algebras associated to $A = A_r$, recalling the notation introduced immediately after the proof of Lemma 2.1. Further, consider the corresponding chain of Weil algebra homomorphisms $\pi_{s,B_{s+1}}\colon B_{s+1} \to B_s$ setting $B_s = T_0^{A_s *}\mathbb{R}^m$, see (1.12). By Proposition 2.1, for any $s = 1, \ldots, r$ the map $T_0^{A_s} f_{(s)}$ is determined by its m values over $i_A^m(\overline{\sigma}_i)$ and consequently over $\pi_{s,A} \circ i_A^m(\overline{\sigma}_i)$.

In the first step observe that $T_0^A f$ is determined by its m values over $i_A^m(\overline{\sigma}_i)$ up to $T_0^{A_1} f_{(1)} = T_0^{A_1} f^{(1)}$, which is actually determined by its values over $\pi_{1,A} \circ i_A^m(\overline{\sigma}_i) = \overline{\sigma}_i$. We step by step particularize the determination of $T_0^A f$ by its values over $i_A^m(\overline{\sigma}_i)$ by means of $T_0^{A_s} f = T_0^{A_s} f^{(s)}$, increasing s up to r.

More exactly, let us deduce the $(s+1)$-st step from the assumption of $T_0^{A_s} f$ being determined by its values over $i_A^m(\overline{\sigma}_i)$. Consider $T_0^A f$ as an element of $T_0^{A_s} f = T_0^{A_s} f^{(s)} \in B_s$ and some of its decomposition (2.2) together with the corresponding decomposition (2.3) of $j^A \eta$. We observe from the well-known jet composition formula that $T_0^A f_{(s)}$ affects not only $j^A \eta_{(s)}$ but also $j^A \eta^{(t)}$ for any $s \leqslant t \leqslant r$. However, the remainder of $j^A \eta$ in the form of $j^A \varrho_{s+1} = T_0^A f(j^A \varphi) - T_0^A f^{(s)}(j^A \varphi)$ is affected only by $T_0^A f_{(t)}$ for $t \geqslant s+1$. Then $j^A \varrho_{s+1}$ is determined exactly by the values of $T_0^A f_{(s+1)}$ over $i_A^m(\overline{\sigma}_i)$ and consequently, $j_{s+1}^A \eta^{(s+1)}$ is determined by the values of $T^{A_{s+1}} f^{(s+1)} = T_0^{A_{s+1}} f$ over $i_A^m(\overline{\sigma}_i)$.

Replacing $\mathbb{R}^m$ by M, 0 by x and applying $T_x^A f(j^A \varphi_x) = T_0^A(f \circ u_x^{-1}) \circ j^A(u_x \circ \varphi_x)$ for any coordinate map $u_x\colon M_x \to \mathbb{R}_0^m$ completes the proof for $m \geqslant k$. For $m < k$ the assertion is obtained immediately if we identify every $f\colon M \to \mathbb{R}$ with the map $\widehat{f}\colon M \times \mathbb{R}^{k-m} \to \mathbb{R}$ defined by $\widehat{f}(x,y) = f(x)$. $\square$

Remark 2.1. An element $j^A \varphi_x \in T_x^A M$ is said to be regular if near $0 \in \mathbb{R}^k$ and for $k \leqslant m$ the map φ_x is an immersion while for $k \geqslant m$ it is a submersion.

Remark 2.2. In Proposition 2.2, the assumption of $\operatorname{width} A = k$ may be omitted. To verify this assertion, assume $m \geqslant h \geqslant k = \operatorname{width} A$ and consider $\widehat{A} = \mathbb{D}_h^r/(I \vee \langle \tau_{k+1}, \ldots, \tau_h \rangle)$. We remark that for $h = m$ the algebra $\widehat{A}$ coincides with $\bar{A}$ from (2.9). Let $j\colon \mathbb{R}^k \to \mathbb{R}^h$ be the map defined in (2.4), substituting h for m. Clearly, the map $t\colon A \to \widehat{A}$ defined by the assignment $j^A \eta \mapsto j^{\widehat{A}} \widehat{\eta}$ for any $j^{\widehat{A}} \widehat{\eta}$ satisfying $\eta = \widehat{\eta} \circ j$ is an isomorphism. Applying Lemma 2.1 we immediately deduce our claim formulated as follows.

Corollary 2.1. *Let $A = \mathbb{D}_k^r/I$ be a Weil algebra,* $\dim M = m$ *and* $m \geqslant k$. *Then every* $T_x^A f \in T_x^{A*} M$ *is determined by its values over* m *arbitrary linearly independent regular elements* $j_0^1 \varphi_{1,x}, \ldots, j_0^1 \varphi_{m,x} \in J_0^1(\mathbb{R}^k, M)_x$.

In the very end of the present section we give a lemma applied in the proof of the main rigidity result below.

Lemma 2.3. *Let $\widetilde{t}_M\colon T^{A_1}M \to T^{A_2}M$ be the natural equivalence induced by an isomorphism $t\colon A_1 \to A_2$ and $\widetilde{t}_{x,M}$ be its restriction to $T^{A_1}_x M$. Then for any $x \in M$ there is an isomorphism $t^*_{x,M}\colon T^{A_1*}_x M \to T^{A_2*}_x M$ defined by one of the following two mutually equivalent conditions:*

(1) $t^*_{x,M}(T^{A_1}_x f)(\widetilde{t}_M(j^{A_1}\varphi_x)) = t \circ T^{A_1}_x f(j^{A_1}\varphi_x)$ *for any* $j^{A_1}\varphi_x \in T^{A_1}_x M$,
(2) $t^*_{x,M}(T^{A_1}_x f) = T^{A_2}_x f = t \circ T^{A_1}_x f \circ \widetilde{t}^{-1}_{x,M}$.

P r o o f. First we recall that any natural transformation of Weil functors covers the identity on base manifolds (see [8]). The equivalence of the conditions (1), (2) follows from the commutativity of natural transformations with morphisms and the surjectivity of $t_{x,M}\colon T^{A_1}_x M \to T^{A_2}_x M$. Further, every $t^*_{x,M}\colon T^{A_1*}_x M \to T^{A_2*}_x M$ is an isomorphism of Weil algebras. In order to verify this we check that $t^*_{x,M}$ is a homomorphism, i.e., $(T^{A_2}_x f_1 \cdot T^{A_2}_x f_2)(j^{A_2}\varphi_x) = t \circ (T^{A_1}_x f_1 \cdot T^{A_1}_x f_2)(j^{A_1}\varphi_x)$ for any $j^{A_1}\varphi_x \in T^{A_1}_x M$ and $j^{A_2}\varphi_x = t_{x,M}(j^{A_1}\varphi_x)$. Applying (1.8) and the commutativity of natural transformations with morphisms we immediately verify that $(T^{A_2}_x f_1 \cdot T^{A_2}_x f_2)(j^{A_2}\varphi_x) = T^{A_2}_{(0,0)}\mu \circ (T^{A_2}_x f_1(j^{A_2}\varphi_x), T^{A_2}_x f_2(j^{A_2}\varphi_x)) = T^{A_2}_{(0,0)}\mu \circ (T^{A_2}_x f_1 \circ \widetilde{t}_{x,M}(j^{A_1}\varphi_x), T^{A_2}_x f_2 \circ \widetilde{t}_{x,M}(j^{A_1}\varphi_x)) = T^{A_2}_{(0,0)}\mu \circ \widetilde{t}_{\mathbb{R}^2}\,(T^{A_1}_x f_1(j^{A_1}\varphi_x), T^{A_1}_x f_2(j^{A_1}\varphi_x)) = t \circ T^{A_1}_{(0,0)}\mu\,(T^{A_1}_x f_1(j^{A_1}\varphi_x), T^{A_1}_x f_2\ (j^{A_1}\varphi_x)) = t \circ (T^{A_1}_x f_1 \cdot T^{A_1}_x f_2)(j^{A_1}\varphi_x)$. Since $t^*_{x,M}$ is obviously invertible, it is really an isomorphism. $\square$

3. The general rigidity result

In the present section we prove the general rigidity result $T^{A*}M \simeq T^{r*}M \simeq P^r M[N^r_m, l]$. Besides Proposition 2.2 and its corollary we essentially apply the so-called rigidity of universally reparametrizable Weil algebras, formulated and proved in Proposition 4.1 below. The proof for the lower-dimensional cases of M is obtained by means of an essential application of Proposition 2.2 as well.

In the next Proposition 3.1 we are searching for a Weil algebra whose nilpotent ideal coincides with $T^{A*}_0\mathbb{R}^m$. In Subsection 1.2 before Lemma 1.1 and in Section 2 before (2.2) it was denoted by B and its defining ideal by J. Nevertheless, in the proof of Proposition 3.1 we use those symbols in a different meaning for convenience. Unlike the notation introduced after Lemma 2.1, the indices by symbols like B_i do not indicate subordinate Weil algebras but only their indexing.

In the whole section we assume height $A = r$. We will also work with the subalgebras $\mathbb{D}^r_{(i_1,\dots,i_l)} \subseteq \mathbb{D}^r_k$ consisting of polynomials in indeterminates $\tau_{i_1},\dots,\tau_{i_l}$ only. For the next proof, we recall the linear map $\iota^r_k\colon \mathbb{D}^1_k \to \mathbb{D}^r_k$, its associated product map $\iota^{r,m}_k\colon (\mathbb{D}^1_k)^m \to (\mathbb{D}^r_k)^m$, the linear map $i_A\colon \mathbb{D}^1_k \to A$ defined by $i_A = p_A \circ \iota^r_k$ and its associated product map $i^m_A\colon (\mathbb{D}^1_k)^m \to A^m$, all defined between (2.2) and (2.3).

Proposition 3.1. *For $m \geqslant \operatorname{width} A$ and $\operatorname{height} A = r$ the algebraic structure of $T_0^{A*}\mathbb{R}^m$ coincides with the nilpotent ideal N_m^r of the jet algebra $\mathbb{D}_m^r$.*

P r o o f. It follows from Lemma 2.3 applied to A and $\bar{A}$ defined by (2.9) that without loss of generality A can be assumed in the form of $\mathbb{D}_m^r/I$. Consider $T_0^A f \in T_0^{A*}\mathbb{R}^m$ in the form of the sum

$$\frac{1}{\alpha!} a_\alpha T_0^A \operatorname{pr}_{\mathbb{R}^m}^\alpha, \quad 1 \leqslant |\alpha| \leqslant r, \tag{3.1}$$

where $\operatorname{pr}_{\mathbb{R}^m}^\alpha \colon \mathbb{R}^m \to \mathbb{R}$ denotes a monomial constructed from the projections $\operatorname{pr}_{\mathbb{R}^m}^1, \dots, \operatorname{pr}_{\mathbb{R}^m}^m \colon \mathbb{R}^m \to \mathbb{R}$ corresponding to a multiindex α. We prove that all a_α are necessary for determining $T_0^A f$.

(i) In the first step consider $T_0^A f \big| \mathcal{O}rb(\operatorname{Aut} A, j^A \operatorname{id}_{\mathbb{R}^m})$, which can be identified with $j^A f = T_0^A f(j^A \operatorname{id}_{\mathbb{R}^m})$ (see (1.5)). Put $T_0^A f(j^A \operatorname{id}_{\mathbb{R}^m}) = j^A(a_\alpha \tau^\alpha/\alpha!) = [a_\alpha \tau^\alpha/\alpha!]_I$. This identification determines the coefficients a_α up to the decomposition classes $[(a_\alpha)_{|\alpha| \leqslant r}]_I$ corresponding to the I-classes $[(a_\alpha \tau^\alpha/\alpha!)]_I$. Notice that in the case of monomial A all a_α are uniquely determined except those corresponding to $\tau^\alpha \in I$ which may be for present arbitrary.

(ii) Given $j^A\varphi \in N_A$ we deduce that $\mathcal{O}rb(\operatorname{Aut} A, j^A\varphi) = N_A$ implies $A = \mathbb{D}_m^r$. It is easy to see that our claim is equivalent to $G^A = G_m^r$. This follows from the definition of G^A and G_A, the latter being identified with the stabilizing subgroup $\operatorname{St}(A) \subseteq G_m^r$ of all elements from A and from the identification $G^A/G_A \simeq \operatorname{Aut} A$ (see (1.3)). Thus every $j_0^r g \in G_m^r$ determines an A-admissible A-velocity. Applying Proposition 4.1 from Section 4 with m substituted for k yields immediately $A = \mathbb{D}_m^r$.

(iii) In the case of $A \neq \mathbb{D}_m^r$, select $j^A\psi \notin \mathcal{O}rb(\operatorname{Aut} A, j^A \operatorname{id}_{\mathbb{R}^m})$. Further, consider some $j_0^r\psi_0 \in j^A\psi$. Put $J = I \circ (j_0^r\psi_0)^{-1}$ and $B = \mathbb{D}_m^r/J$. Clearly, $(j_0^r\psi_0)^{-1}$ determines an A-admissible B-velocity and consequently a natural equivalence $\widetilde{t}_{j_0^r\psi_0} \colon T^A \to T^B$.

Then the commutativity of natural transformations with morphisms yields $T_0^A f(j^A\psi) = T_0^B f(j^B \operatorname{id}_{\mathbb{R}^m}) \circ j_0^r\psi_0$ and consequently $j^B f = T_0^B f(j^B \operatorname{id}_{\mathbb{R}^m}) = T_0^A f(j^A\psi) \circ (j_0^r\psi_0)^{-1}$. For $j_0^r\psi_0$ we select $\iota_m^{r,m}(\sigma)$ for some cycle $\sigma \in C_m$. Then the coefficients a_α of $T_0^A f$ are determined up to the decomposition J-classes $[(a_\alpha)_{|\alpha| \leqslant r}]_J$ corresponding to $[(a_\alpha \tau^\alpha/\alpha!)]_J = j^B(a_\alpha \tau^\alpha/\alpha!)$ and consequently to the decomposition $I \cap J$-classes $[(a_\alpha)_{|\alpha| \leqslant r}]_{I \cap J}$ corresponding to $j^C(a_\alpha \tau^\alpha/\alpha!)$, if we denote by C the Weil algebra $\mathbb{D}_m^r/I \cap J$.

(iv) Conversely, consider $T_0^B f$. We identify $T_0^B f \big| \mathcal{O}rb(\operatorname{Aut} B, j^B \operatorname{id}_{\mathbb{R}^m})$ with $j^B f = T_0^B f(j^B \operatorname{id}_{\mathbb{R}^m})$ analogously as in (i). This enables us to determine the coefficients a_α of $T_0^B f$ up to the decomposition classes $[(a_\alpha)_{|\alpha| \leqslant r}]_J$ corresponding to the J-classes $[(a_\alpha \tau^\alpha/\alpha!)]_J \simeq j^B(a_\alpha \tau^\alpha/\alpha!)$. Further, consider $j_0^r\psi_0$ from (iii) and its inverse $j_0^r\psi_0^{-1}$. This is an $\iota_m^{r,m}$-image of some cycle again, which follows from the definition of $\iota_k^{r,m}$ presented after (2.2) and the coordinate expression of the multiplication in G_m^r. Clearly, $j_0^r\psi_0$ determines a B-admissible A-velocity and hence a natural equivalence $\widetilde{t}_{j_0^r\psi_0^{-1}}\colon T^B \to T^A$. Then the commutativity of natural transformations with morphisms yields $T_0^B f(j^B\psi^{-1}) = T_0^A f(j^A \operatorname{id}_{\mathbb{R}^m}) \circ j_0^r\psi_0^{-1}$ and consequently $j^A f = T_0^A f(j^A \operatorname{id}_{\mathbb{R}^m}) = T_0^B f(j^B\psi) \circ (j_0^r\psi_0)$.

Analogously as in (iii) we obtain that the coefficients of $T_0^B f$ are determined by its evaluation over $j^B \operatorname{id}_{\mathbb{R}^m}$ and $j^B\psi_0^{-1}$ up to the decomposition $I \cap J$-classes $[(a_\alpha)_{|\alpha|\leqslant r}]_{I\cap J}$ corresponding to $j^C(a_\alpha\tau^\alpha/\alpha!)$ for $C = \mathbb{D}_m^r/I\cap J$ again.

(v) Consider the elements $j_0^r\psi_i = \iota_m^{r,m}(\sigma_i)$ for all cycles $\sigma_i \in C_m$. Then $j_0^r \operatorname{id}_{\mathbb{R}^m}$ coincides with $j_0^r\psi_1$ in this notation, while $j_0^r\psi_0$ from (iii) corresponds to the minimal $i \in \{1,\dots,m\}$ and $\iota_m^{r,m}(\sigma_i)$ for which $\widetilde{p}_{A,\mathbb{R}^m} \circ \iota_m^{r,m}(\sigma_i) = i_A^m(\sigma_i) \notin \mathcal{O}rb(\operatorname{Aut} A, j^A \operatorname{id}_{\mathbb{R}^m})$. For each $j_0^r\psi_i$ we put $J_i = I \circ (j_0^r\psi_i)^{-1}$ and $B_i = \mathbb{D}_m^r/J_i$. We step by step particularize the coefficients a_α of $T_0^A f$ from (3.1). We do so with the coefficients of $T_0^{B_i} f$ and thus extend the step (iv), applying the fact of $\{\iota_m^{r,m}(\sigma)\colon \sigma \in C_m\}$ being closed with respect to inverses in G_m^r. We conclude that the coefficients of the individual elements $T_0^{B_i} f$, $i = 1,\dots,m$, are determined up to the $J_1 \cap \dots \cap J_m$-classes $[(a_\alpha)_{|\alpha|\leqslant r}]_{J_1\cap\dots\cap J_m}$ corresponding to $j^C(a_\alpha\tau^\alpha/\alpha!)$ for $C = \mathbb{D}_m^r/J_1 \cap \dots \cap J_m$.

(vi) Applying Proposition 2.2 and Corollary 2.1 we conclude after m steps that every element of the form (3.1) with the set of the coefficients $(a_\alpha)_{|\alpha|\leqslant r}$ contained in a $J_1 \cap \dots \cap J_m$-class $[(a_\alpha)_{|\alpha|\leqslant r}]_{J_1\cap\dots\cap J_m} \simeq j^C(a_\alpha\tau^\alpha/\alpha!)$ determines each of the maps $T_0^{B_i} f$. Moreover, each of $T_0^{B_i} f$ coincides with $T_0^C f$ for $C = \mathbb{D}_m^r/J_1 \cap \dots \cap J_m$. This follows from the fact that the maps $T_0^{B_i} f$ mutually coincide and that they respect J_i-classes. Further, suppose $J = J_1 \cap \dots \cap J_m \neq \{j_0^r 0_{\mathbb{R}^m}\}$. By (ii) there is $j_0^r\chi \in G_m^r$ such that $J \circ (j_0^r\chi)^{-1} \nsubseteq J$. This enforces another limiting condition on the coefficients a_α from (3.1). Nevertheless, this is a contradiction with Proposition 2.2 and Corollary 2.1.

(vii) In the final step we are to verify the algebraic isomorphism $T_0^{A*}\mathbb{R}^m \cong N_m^r$. Its existence is obtained directly from $T_0^{A*}\mathbb{R}^m = T_0^{\mathbb{D}_m^r}\mathbb{R}^m$ and Lemma 1.1. This completes the proof. □

Remark 3.1. The assumption of height $A = r$ has been applied in (ii) by means of Proposition 4.1 from Section 4. This assumption is necessary for its validity and it is explicitly formulated there.

Let us return to the notation $B = \mathbb{R} \oplus T_0^{A*}\mathbb{R}^m = \mathbb{D}_m^r/J$. Clearly, $B = \mathbb{D}_m^r$ for any $m \geqslant k$. In what follows, consider the subalgebras $A_{(i_1,\dots,i_m)} = p_A(\mathbb{D}_{(i_1,\dots,i_m)}^r)$. We prove the coincidence of B with $\mathbb{D}_m^r$ for $m <$ width A.

Proposition 3.2. *For $m < k = \operatorname{width} A$ and $\operatorname{height} A = r$, the algebraic structure of $T_0^{A*}\mathbb{R}^m$ coincides with the nilpotent ideal N_m^r of $\mathbb{D}_m^r$.*

Proof. We prove that every $T_0^A f \in T_0^{A*}\mathbb{R}^m$, obviously identified with $T_0^A \widehat{f}\colon T_0^A\mathbb{R}^m \times T_0^A\mathbb{R}^{k-m} \to T_0^A\mathbb{R}$ free of the second set of arguments, coincides with $(a_\alpha/\alpha!)T_0^A \operatorname{pr}^\alpha_{\mathbb{R}^m}$ from (3.1), satisfying $a_\alpha = 0$ whenever α contains an index from the set $\{m+1,\dots,k\}$. We make the most of the fact that $T_0^A\widehat{f}$ is determined by its values over $i_A^k(\sigma_1),\dots,i_A^k(\sigma_k)$ for the cycles $\sigma_i \in C_k$ and analogously we do with the restrictions of $T_0^A f$ to $A^m_{(\sigma(1),\dots,\sigma(m))}$, cycling $\sigma(1),\dots,\sigma(m)$. Clearly, the restrictions of $T_0^A f$ to $A^m_{(\sigma(1),\dots,\sigma(m))}$ under discussion are identified with the restrictions of $T_0^A\widehat{f}$ to $A^m_{(\sigma(1),\dots,\sigma(m))} \times A^{k-m}_{(\sigma(m+1),\dots,\sigma(k))}$.

For every $\sigma \in C_k$, define the ideal $J_\sigma \subseteq \mathbb{D}_m^r$ by

$$J_\sigma(t_1,\dots,t_m) = (I\circ\sigma \vee \langle \tau_{\sigma(m+1)},\dots,\tau_{\sigma(k)}\rangle) \cap \mathbb{D}_m^r(\tau_{\sigma(1)},\dots,\tau_{\sigma(m)}), \tag{3.2}$$

where $t_1,\dots,t_m$ denote the indeterminates $\tau_{\sigma(1)},\dots,\tau_{\sigma(m)}$ and $\langle \tau_{\sigma(m+1)},\dots,\tau_{\sigma(k)}\rangle$ denotes the ideal in $\mathbb{D}_k^r$ generated by $\tau_{\sigma(m+1)},\dots,\tau_{\sigma(k)}$. Briefly speaking, we have substituted $t_1 = \tau_{\sigma(1)},\dots,t_k = \tau_{\sigma(k)}$ for $\tau_1,\dots,\tau_k$ in I and joining $I\circ\sigma$ obtained in this way with $\langle \tau_{\sigma(m+1)},\dots,\tau_{\sigma(k)}\rangle$ we receive an ideal of width m which can be considered only in $t_1,\dots,t_m$ after the obvious renaming of the indeterminates presented above. Finally, we put $B_\sigma = \mathbb{D}_m^r/J_\sigma$.

The elements $T_0^A f$ restricted to $A^m_{(\sigma(1),\dots,\sigma(m))}$ can be identified with the elements $T_0^{B_\sigma} f$ and so can be the corresponding restrictions of T^{A*}-maps with $T^{B_\sigma *}$-maps. By the partial rigidity result obtained in Proposition 3.1, the algebra of such elements coincides with the nilpotent ideal $N_m^{q_\sigma} \subseteq \mathbb{D}_m^{q_\sigma}$, putting $q_\sigma = \operatorname{height} A_{(\sigma(1),\dots,\sigma(m))}$. Nevertheless, for the restrictions of $T_0^A\widehat{f}$ to $A^m_{(\sigma(1),\dots,\sigma(m))} \times A^{k-m}_{(\sigma(m+1),\dots,\sigma(k))}$ the coefficients a_α corresponding to α satisfying $\alpha \cap \{m+1,\dots k\} \neq \emptyset$ may be put to zero.

This step can be done with every $\sigma \in C_k$. Clearly, the coefficients a_α corresponding to $\alpha \cap \{m+1,\dots,k\} \neq \emptyset$ may be put to zero in all of those cases. Since every $T_0^A f$ under discussion, identified with $T_0^A\widehat{f}$ free of the second argument, is by Proposition 2.2 determined by its values over $i_A^k(\sigma_1),\dots,i_A^k(\sigma_k)$, the coefficients a_α containing any index from $\{m+1,\dots,k\}$ are really zeros and our claim is proved. $\square$

We state the general rigidity result in the following theorem.

Theorem 3.1. *Let $A = \mathbb{D}_k^r/I$, $\operatorname{height} A = r$ and N_m^r be the nilpotent ideal of $\mathbb{D}_m^r$. Further, let $l\colon G_m^r \times N_m^r \to N_m^r$ be the left action defined by*

$$l(j_0^r g, j_0^r \alpha) = j_0^r(\alpha \circ g^{-1}). \tag{3.3}$$

Then the system of spaces $T^{A*}M$ and T^{A*}-maps from (1.6) *over m-dimensional manifolds and local diffeomorphisms forms the natural bundle $P^rM[N^r_m, l]$ identified with the r-th order covelocities bundle $T^{r*}M$.*

P r o o f. In the context of the general theory of natural bundles (see [8]) consider the formula

$$l(j^r_0 g, T^A_0 f) = T^A_0(f \circ g^{-1}). \tag{3.4}$$

Further, any $T^A_x f$ is identified with $\{j^r_0\alpha_x, T^A_x f \circ (T^A_0\alpha_x)\} = \{j^r_0\alpha_x, T^A_0(f \circ \alpha_x)\}$ and conversely $\{j^r_0\alpha_x, T^A_0 f\}$ is identified with $T^A_x(f \circ \alpha^{-1}_x)$. The fact that $\{j^r_0\alpha_x \circ j^r_0 g, T^A_0(f \circ g)\}$ and $\{j^r_0\alpha_x, T^A_0 f\}$ determine the same element of $T^{A*}_x M$ implies the identification $T^{A*}_x M \simeq P^r_x M[T^{A*}_0\mathbb{R}^m, l]$ with l defined by (3.4).

As for morphisms, it is easy to verify that the maps $T^{A*}g$ correspond to the morphisms of associated bundles which are of the form $\{P^r g, \mathrm{id}_{T^{A*}_0\mathbb{R}^m}\}$.

Let us denote by $\omega\colon T^{A*}_0\mathbb{R}^m \to N^r_m$ the algebraic identification introduced by (3.1) and verified by the proofs of Proposition 3.1 and Proposition 3.2. Clearly, ω is of the form $T^A_0 f \cong (1/\alpha!)a_\alpha T^A_0 \operatorname{pr}^\alpha_{\mathbb{R}^m} \cong (1/\alpha!)a_\alpha\tau^\alpha$, $1 \leqslant |\alpha| \leqslant r$.

Checking the equivariancy of ω with respect to (3.3) and (3.4), which reads $\omega \circ l(j^r_0 g, T^A_0 f) = l(j^r_0 g, \omega(T^A_0 f))$, completes the proof. $\square$

Corollary 3.1. *For $A = \mathbb{D}^r_k/I$ the system of spaces $J^A(M,N)$ and their $J^A(g,h)$-maps* (see (1.7)) *is identified with the jet functor J^r.*

P r o o f. For any $M \in \mathrm{Obj}(\mathcal{M}f_m)$ consider the bundle functor $G_M\colon \mathcal{M}f \to \mathcal{FM}$ defined by $G_M N = J^r(M,N)$ on objects and by $G_M h = J^r(\mathrm{id}_M, h)$ on morphisms. From the point of view of the objects, it follows from Theorem 3.1 that $J^r(M,N) = G_M N = P^r N[(T^{r*}M)^n, l_M] = P^r N[(T^{A*}M)^n, l_M]$, where $l_M\colon G^r_n \times (T^{r*}M)^n \to (T^{r*}M)^n$ is defined by $l_M(j^r_0 g, T^r_x f) = T^r_0 g \circ T^r_x f = T^A_0 g \circ T^A_x f$. Therefore $J^r(M,N) = J^A(M,N)$. For morphisms, we have $J^A(g,h)(T^A_x f) = T^A_{g(x)}(h \circ f \circ g^{-1}) = J^r_{g(x)}(h \circ f \circ g^{-1}) = J^r(g,h)(j^r_x f)$. By g^{-1} we denote the inverse to g considered near x. $\square$

4. The rigidity of universally reparametrizable Weil algebras

Let us consider a Weil algebra invariant to all reparametrizations. A Weil algebra of this property will be said to be universally reparametrizable. The following proposition states the rigidity of such Weil algebras.

Proposition 4.1. *Let $A = \mathbb{D}^r_k/I$ be a Weil algebra considered in the polynomial form $\mathbb{R}[\tau_1, \dots, \tau_k]/I$. Further, let any local diffeomorphism $f\colon \mathbb{R}^k_0 \to \mathbb{R}^k_0$ determine an A-admissible A-velocity. Then the existence of a nonzero $j^r_0\eta \in I$ implies $\langle \tau_1, \dots \tau_k\rangle^r \subseteq I$. In particular, $A = \mathbb{D}^r_k$ whenever* height $A = r$.

Proof. First we observe that the second assertion is a direct consequence of the first one. Indeed, $0 \neq j^r_0\eta \in I$ implies $\langle \tau_1, \dots \tau_k\rangle^r \subseteq I$. Nevertheless, this is a contradiction with height $A = r$.

As for the first assertion, we prove by induction with respect to l that the existence of a nonzero $j^r_0\eta \in I \cap \mathbb{D}_{(i_1,\dots,i_l)}$ implies $\langle \tau_{i_1}, \dots, \tau_{i_l}\rangle^r \subseteq I$ for any l-elementary subset $\{i_1, \dots, i_l\}$ of $\{1, \dots, k\}$. Clearly, the polynomial $j^r_0\eta$ may be assumed to be from $\langle \tau_{i_1}, \dots, \tau_{i_l}\rangle^r$. First we observe that for any permutation σ of $\{1, \dots, k\}$ and any polynomial p under discussion it holds that $p(\tau_{\sigma(1)}, \dots, \tau_{\sigma(k)}) \in I$ whenever $p(\tau_1, \dots, \tau_k) \in I$ independently of p being a polynomial of exactly k or less indeterminates. A symmetry of this kind follows immediately from the universal reparametrization property of A.

The first step corresponding to $l = 1$ is trivial. As for the induction step, suppose the assertion being satisfied for $l \leqslant k$. Further, assume $i_1 = 1, \dots, i_l = l$ and $j^r_0\eta$ in the form of a polynomial $p \in I$ in indeterminates $\tau_1, \dots, \tau_{l+1}$. The proof will be divided into two steps.

(a) Let p be a monomial. The cases of $r = 1$ and $k = 1$ are trivial. One observes easily that it suffices to discuss l only up to $K = \min\{k, r\}$. In the induction step we can suppose p without loss of generality in the form of $\tau^h_{l+1}\tau^{s-h}_l q_{r-s}(\tau_1, \dots, \tau_{l-1})$ for some $h \leqslant 1$ and $h < s$, where $s \in \{2, \dots, r - l + 1\}$ and $q_{r-s}(\tau_1, \dots, \tau_{l-1})$ is a monomial of order $r - s$ in the indicated indeterminates exactly. Consider $s + 1$ reparametrizations of the form

$$(4.1) \qquad \tau_{l+1} \mapsto c^j\tau_{l+1} + \tau_l,\ \tau_l \mapsto \tau_l,\ \dots,\ \tau_1 \mapsto \tau_1, \quad j = 0, 1, \dots, s.$$

By the universal reparametrization property we have the following system of $s + 1$ elements from I whose h-th element, $h = 0, \dots, s$, is of the form

$$(4.2) \qquad (c^h\tau_{l+1} + \tau_l)^s q_{r-s}(\tau_1, \dots, \tau_{l-1}) = q_{r-s}(\tau_1, \dots, \tau_{l-1}) \sum_{j=0}^{s} \binom{s}{j} c^{hj}\tau^j_{l+1}\tau^{s-j}_l.$$

We show that for $j = 0, 1, \dots, s$ each monomial $\tau^j_{l+1}\tau^{s-j}_l q_{r-s}(\tau_1, \dots, \tau_{l-1})$ is an element of I provided p is. In other words, we show that each monomial $\tau^j_{l+1}\tau^{s-j}_l$ can be expressed in the form a linear combination of $(\tau_{l+1} + \tau_l)^s, \dots, (c^s\tau_{l+1} + \tau_l)^s$. Consider the matrix of the coefficients of the monomials $q_{r-s}(\tau_1, \dots, \tau_{l-1}) \cdot \tau^j_{l+1}\tau^{s-j}_l$

from the right-hand sides of the system (4.2). Its determinant equals to

$$\binom{s}{0}\binom{s}{1}\cdots\binom{s}{s}\det M(s,c), \tag{4.3}$$

where the matrix $M(s,c) = (m(s,c))_{i,j}$ is defined by $(m(s,c))_{i,j} = c^{(i-1)(j-1)}$.

We particularly obtain that $\tau_l^s \cdot q_{r-s}(\tau_1,\dots,\tau_{l-1}) \in I$ and conclude that $p \in I$ implies that every monomial in the indeterminates $\tau_1,\dots,\tau_l$ only is an element of I. The same holds for all monomials in arbitrary l indeterminates.

Now we are to deduce that if any monomial of order r in $\tau_1,\dots,\tau_l$ is an element of I, then so is any monomial in $\tau_1,\dots,\tau_{l+1}$. However, one can deduce that applying the set of $s+1$ reparametrizations

$$\tau_{l+1} \mapsto \tau_{l+1},\ \tau_l \mapsto c^j\tau_{l+1} + \tau_l,\ \dots,\ \tau_1 \mapsto \tau_1, \quad j = 0,1,\dots,s \tag{4.4}$$

and the deductions connected with (4.2) applied in the converse direction, i.e., from $j=0$ to other cases of j.

(b) Without loss of generality, p may be assumed to be a sum of t monomials of order r, otherwise p could be multiplied by some monomial truncating the monomials of the maximal degree in the sum and consequently reducing its length. Iterating this step we would obtain either a sum of monomials of the same degree or one monomial, which would lead to the step (a). Consider p in the form of $q \cdot p_1 + p_2$ for a monomial q in $\tau_1,\dots,\tau_{l+1}$ exactly, a sum p_1 of t_1 monomials in the same indeterminates in which at least one summand does not contain τ_{l+1} and a residual sum of monomials in the indeterminates $\tau_1,\dots,\tau_l$ only.

Let us denote by s the maximal degree of τ_{l+1} in p. For $c \neq 0$, take the reparametrization $\tau_{l+1} \mapsto c\tau_{l+1}, \tau_l \mapsto \tau_l, \dots, \tau_1 \mapsto \tau_1$ and the polynomial $\bar{p}$ obtained from p by such reparametrization. Clearly, the polynomial $\bar{p} - c^s p \in I$ and there is a nonzero $c \in \mathbb{R}$ such that $\bar{p} - c^s p \neq j_0^r 0$. Clearly, its length is shorter than that of p. Iterating this step we achieve either a polynomial in $\tau_1,\dots,\tau_l$ only or a monomial in the indeterminates $\tau_1,\dots,\tau_{l+1}$, both of them being elements of I. The first case coincides with the already proved step in the induction proof, while the second one leads to the case (a). □

References

[1] *R. J. Alonso*: Jet manifolds associated to a Weil bundle. Arch. Math., Brno *36* (2000), 195–209. zbl MR

[2] *R. J. Alonso-Blanco, D. Blázquez-Sanz*: The only global contact transformations of order two or more are point transformations. J. Lie Theory *15* (2005), 135–143. zbl MR

[3] *W. Bertram*: Differential geometry, Lie groups and symmetric spaces over general base fields and rings. Mem. Am. Math. Soc. *192* (2008), 202 pages. zbl MR doi

[4] *G. N. Bushueva, V. V. Shurygin*: On the higher order geometry of Weil bundles over smooth manifolds and over parameter-dependent manifolds. Lobachevskii J. Math. (electronic only) *18* (2005), 53–105. zbl MR

[5] *D. J. Eck*: Product-preserving functors on smooth manifolds. J. Pure Appl. Algebra *42* (1986), 133–140. zbl MR doi

[6] *G. Kainz, P. W. Michor*: Natural transformations in differential geometry. Czech. Math. J. *37* (1987), 584–607. zbl MR

[7] *I. Kolář*: Covariant approach to natural transformations of Weil functors. Commentat. Math. Univ. Carol. *27* (1986), 723–729. zbl MR doi

[8] *I. Kolář, P. W. Michor, J. Slovák*: Natural Operations in Differential Geometry. Springer, Berlin, 1993. zbl MR doi

[9] *I. Kolář, W. M. Mikulski*: On the fiber product preserving bundle functors. Differ. Geom. Appl. *11* (1999), 105–115. zbl MR doi

[10] *M. Kureš*: Weil algebras associated to functors of third order semiholonomic velocities. Math. J. Okayama Univ. *56* (2014), 117–127. zbl MR

[11] *O. O. Luciano*: Categories of multiplicative functors and Weil's infinitely near points. Nagoya Math. J. *109* (1988), 63–89. zbl MR doi

[12] *W. M. Mikulski*: Product preserving bundle functors on fibered manifolds. Arch. Math., Brno *32* (1996), 307–316. zbl MR

[13] *J. Muñoz, J. Rodriguez, F. J. Muriel*: Weil bundles and jet spaces. Czech. Math. J. *50* (2000), 721–748. zbl MR doi

[14] *H. Nishimura*: Axiomatic differential geometry I–1—towards model categories of differential geometry. Math. Appl., Brno *1* (2012), 171–182. zbl MR doi

[15] *V. V. Shurygin*: The structure of smooth mappings over Weil algebras and the category of manifolds over algebras. Lobachevskii J. Math. *5* (1999), 29–55. zbl MR

[16] *V. V. Shurygin*: Some aspects of the theory of manifolds over algebras and Weil bundles. J. Math. Sci., New York *169* (2010), 315–341; translation from Itogi Nauki Tekh., Ser. Sovrem. Mat. Prilozh., Temat. Obz. *123* (2009), 211–255. zbl doi

[17] *J. M. Tomáš*: On bundles of covelocities. Lobachevskii J. Math. *30* (2009), 280–288. zbl MR doi

[18] *J. Tomáš*: Some results on bundles of covelocities. J. Appl. Math., Aplimat V *4* (2011), 297–306.

[19] *J. Tomáš*: Bundles of (p, A)-covelocities and (p, A)-jets. Miskolc Math. Notes *14* (2013), 547–555. zbl MR

[20] *A. Weil*: Théorie des points proches sur les variétés des différentiables. Colloques internat. Centre nat. Rech. Sci. *52* (1953), 111–117. (In French.) zbl MR

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