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SOME FINITE GENERALIZATIONS OF EULER'S PENTAGONAL NUMBER THEOREM

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Abstract. Euler's pentagonal number theorem was a spectacular achievement at the time of its discovery, and is still considered to be a beautiful result in number theory and combinatorics. In this paper, we obtain three new finite generalizations of Euler's pentagonal number theorem.

Keywords: q-binomial coefficient; q-binomial theorem; pentagonal number theorem

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1. Introduction

One of Euler's most profound discoveries, the pentagonal number theorem, see [1], Corollary 1.7, page 11, is stated as follows:

$$\prod_{k=1}^{\infty}(1-q^k)=\sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k+1)/2}. \tag{1.1}$$

For some connections between the pentagonal number theorem and the theory of partitions, one refers to [1], page 10, and [2].

Throughout this paper, we assume $|q|<1$ and use the following q-series notation:

$$(a;q)_0=1,\quad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\quad (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k),$$

and

$$\begin{bmatrix} n\\ m\end{bmatrix}=\begin{bmatrix} n\\ m\end{bmatrix}_q=\begin{cases}\dfrac{(q;q)_n}{(q;q)_m(q;q)_{n-m}} & \text{if } 0\leqslant m\leqslant n,\\ 0, & \text{otherwise.}\end{cases}$$

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Some finite forms of Euler's pentagonal number theorem have been already studied by several authors. Shanks in [7] proved that

$$\sum_{k=0}^{n}(-1)^k\frac{(q;q)_n}{(q;q)_k}q^{\binom{k+1}{2}+nk}=\sum_{k=-n}^{n}(-1)^k q^{k(3k+1)/2}, \tag{1.2}$$

which was a truncated version of (1.1). Note that (1.2) reduces to (1.1) when $n\to\infty$.

Berkovich and Garvan in [3] have found some finite generalizations of Euler's pentagonal number theorem. For example, they showed that

$$\sum_{j=-\infty}^{\infty}(-1)^j\begin{bmatrix}2L-j\\ L+j\end{bmatrix}q^{j(3j+1)/2}=1. \tag{1.3}$$

By using a well-known cubic summation formula, Warnaar in [8] obtained another finite generalization of Euler's pentagonal number theorem:

$$\sum_{j=-\infty}^{\infty}(-1)^j\begin{bmatrix}2L-j+1\\ L+j\end{bmatrix}q^{j(3j-1)/2}=1. \tag{1.4}$$

Note that

$$\lim_{L\to\infty}\begin{bmatrix}2L-j\\ L+j\end{bmatrix}=\lim_{L\to\infty}\begin{bmatrix}2L-j+1\\ L+j\end{bmatrix}=\frac{1}{(q;q)_\infty}.$$

Then both (1.3) and (1.4) reduce to (1.1) when $L\to\infty$.

The first aim of the paper is to show the following finite form of (1.1):

Theorem 1.1. *Let n be any non-negative integer. Then*

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\begin{bmatrix}n-k\\ k\end{bmatrix}q^{\binom{k+1}{2}}=\sum_{k=-\lfloor (n+1)/3\rfloor}^{\lfloor n/3\rfloor}(-1)^k q^{k(3k+1)/2}, \tag{1.5}$$

where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to a real number x.

Observe that

$$\lim_{n\to\infty}\begin{bmatrix}n-k\\ k\end{bmatrix}=\lim_{n\to\infty}\frac{(q;q)_{n-k}}{(q;q)_k(q;q)_{n-2k}}=\frac{1}{(q;q)_k}.$$

By Tannery's theorem, see [4], page 136, letting $n\to\infty$ in (1.5) reduces it to

$$\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k+1}{2}}}{(q;q)_k}=\sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k+1)/2}. \tag{1.6}$$

By [1], (2.2.6), we have

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}}}{(q;q)_k} = (q;q)_\infty, \tag{1.7}$$

which is a special case of the q-binomial theorem, see [1], Theorem 2.1, page 17. Combining (1.6) and (1.7), we are led to (1.1).

The second result consists of the following two finite generalizations of (1.1):

Theorem 1.2. *Suppose m is a positive integer. Then*

$$\frac{1-q^{3m}}{1+q^m} \sum_{k=-m}^{\lfloor m/2 \rfloor} (-1)^k \begin{bmatrix} 2m-k \\ m+k \end{bmatrix} \frac{q^{k(3k-1)/2}}{1-q^{2m-k}} = 1, \tag{1.8}$$

$$(1-q^{3m-1}) \sum_{k=-m}^{\lfloor (m-1)/2 \rfloor} (-1)^k \begin{bmatrix} 2m-k-1 \\ m+k \end{bmatrix} \frac{q^{k(3k+1)/2}}{1-q^{2m-k-1}} = 1. \tag{1.9}$$

Note that $|q| < 1$ and

$$\lim_{m\to\infty} \begin{bmatrix} 2m-k \\ m+k \end{bmatrix} = \lim_{m\to\infty} \begin{bmatrix} 2m-k-1 \\ m+k \end{bmatrix} = \frac{1}{(q;q)_\infty}.$$

By Tannery's theorem, see [4], page 136, we conclude that both (1.8) and (1.9) reduce to (1.1) when $m \to \infty$.

2. Proof of Theorems 1.1 and 1.2

In order to prove the main results, we need some lemmas.

Lemma 2.1 ([1], page 35)**.** *Let $0 \leqslant m \leqslant n$ be integers. Then*

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^m \begin{bmatrix} n-1 \\ m \end{bmatrix},$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}.$$

The next two lemmas play important roles in our proof of Theorem 1.1 and 1.2. We shall prove these two lemmas together with Theorem 1.1.

Lemma 2.2. *Suppose n is a non-negative integer. Then*

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\begin{bmatrix}n-k\\k\end{bmatrix}q^{\binom{k}{2}}=\begin{cases}(-1)^m q^{m(3m-1)/2} & \text{if } n=3m,\\ (-1)^m q^{m(3m+1)/2} & \text{if } n=3m+1,\\ 0 & \text{if } n=3m-1.\end{cases}\tag{2.1}$$

Ekhad and Zeilberger in [5] proved (2.1) by Zeilberger's algorithm, see [6]. Warnaar in [8] gave another proof of (2.1) using a well-known cubic summation formula. We will present an essentially different proof by establishing relationships with other two results and using mathematical induction.

Lemma 2.3. *For any non-negative integer n, we have*

$$\begin{aligned}(1-q^n)\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\begin{bmatrix}n-k\\k\end{bmatrix}\frac{q^{\binom{k}{2}}}{1-q^{n-k}}\\ =\begin{cases}(-1)^m(1+q^m)q^{m(3m-1)/2} & \text{if } n=3m,\\ (-1)^m q^{m(3m+1)/2} & \text{if } n=3m+1,\\ (-1)^m q^{m(3m-1)/2} & \text{if } n=3m-1.\end{cases}\end{aligned}\tag{2.2}$$

P r o o f of Theorem 1.1, Lemma 2.2 and Lemma 2.3. Denote the left-hand sides of (2.1), (2.2) and (1.5) by U_n, V_n and W_n, respectively. We shall prove (2.1), (2.2) and (1.5) by establishing the following relationships:

$$W_n=W_{n-1}-q^{n-1}U_{n-2},\tag{2.3}$$

$$V_n=U_n-q^{n-1}U_{n-2},\tag{2.4}$$

$$V_n=W_n-W_{n-2}.\tag{2.5}$$

Substituting (2.3) into (2.5) gives

$$V_n=W_{n-1}-W_{n-2}-q^{n-1}U_{n-2}.\tag{2.6}$$

By (2.4) and (2.6), we have

$$U_n=W_{n-1}-W_{n-2}.\tag{2.7}$$

Replacing n by $n-1$ in (2.3), we get

$$W_{n-1}-W_{n-2}=-q^{n-2}U_{n-3}.\tag{2.8}$$

By (2.7) and (2.8), we get

$$U_n = -q^{n-2}U_{n-3} \quad \text{for } n \geqslant 3.$$

We can deduce (2.1) by induction from the initial values $U_0 = 1$, $U_1 = 1$ and $U_2 = 0$. Substituting (2.1) into (2.4), we get (2.2) directly.

We will prove (1.5) by using induction on n. It is easy to verify that (1.5) is true for $n = 0, 1, 2$. Assume (1.5) is true for $N \leqslant n$. By (2.3), we have

$$W_{n+1} = W_n - q^n U_{n-1}. \tag{2.9}$$

If $n = 3m$, by (2.1) and (2.9), we have $W_{n+1} = W_n$. It follows from the induction that

$$W_{n+1} = W_n = \sum_{k=-\lfloor (n+1)/3 \rfloor}^{\lfloor n/3 \rfloor} (-1)^k q^{k(3k+1)/2} = \sum_{k=-\lfloor (n+2)/3 \rfloor}^{\lfloor (n+1)/3 \rfloor} (-1)^k q^{k(3k+1)/2},$$

which implies that (1.5) is also true for $N = n + 1$.

If $n = 3m + 1$, it follows from (2.1) and (2.9) that

$$W_{n+1} = W_n + (-1)^{m+1} q^{(m+1)(3m+2)/2}.$$

So we have

$$\begin{aligned} W_{n+1} &= \sum_{k=-\lfloor (n+1)/3 \rfloor}^{\lfloor n/3 \rfloor} (-1)^k q^{k(3k+1)/2} + (-1)^{m+1} q^{(m+1)(3m+2)/2} \\ &= \sum_{k=-\lfloor (n+2)/3 \rfloor}^{\lfloor (n+1)/3 \rfloor} (-1)^k q^{k(3k+1)/2}, \end{aligned}$$

which proves (1.5) for the case $N = n + 1$.

If $n = 3m + 2$, using (2.1) and (2.9), we get

$$W_{n+1} = W_n + (-1)^{m+1} q^{(m+1)(3m+4)/2},$$

and hence

$$\begin{aligned} W_{n+1} &= \sum_{k=-\lfloor (n+1)/3 \rfloor}^{\lfloor n/3 \rfloor} (-1)^k q^{k(3k+1)/2} + (-1)^{m+1} q^{(m+1)(3m+4)/2} \\ &= \sum_{k=-\lfloor (n+2)/3 \rfloor}^{\lfloor n+1/3 \rfloor} (-1)^k q^{k(3k+1)/2}, \end{aligned}$$

which implies that (1.5) is true for $N = n + 1$. This concludes the proof of (1.5).

It remains to prove (2.3)–(2.5). From Lemma 2.1, we have

$$\begin{bmatrix} n-k \\ k \end{bmatrix} = \begin{bmatrix} n-k-1 \\ k \end{bmatrix} + q^{n-2k} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}.$$

It follows that

$$\begin{aligned} W_n &= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{\binom{k+1}{2}} + q^{n-1} \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} q^{\binom{k-1}{2}} \\ &= W_{n-1} - q^{n-1} \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} (-1)^k \begin{bmatrix} n-k-2 \\ k \end{bmatrix} q^{\binom{k}{2}} \\ &= W_{n-1} - q^{n-1} U_{n-2}. \end{aligned}$$

This concludes the proof of (2.3).

Note that $1-q^n = 1-q^{n-k} + q^{n-k}(1-q^k)$. Then

$$\begin{aligned} V_n &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} n-k \\ k \end{bmatrix} q^{\binom{k}{2}} + q^{n-1} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} n-k \\ k \end{bmatrix} \frac{1-q^k}{1-q^{n-k}} q^{\binom{k-1}{2}} \\ &= U_n + q^{n-1} \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} q^{\binom{k-1}{2}} \\ &= U_n - q^{n-1} \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} (-1)^k \begin{bmatrix} n-k-2 \\ k \end{bmatrix} q^{\binom{k}{2}} \\ &= U_n - q^{n-1} U_{n-2}, \end{aligned}$$

which is (2.4).

Applying the fact:

$$\frac{1-q^n}{1-q^{n-k}} = \frac{1-q^k}{1-q^{n-k}} + q^k,$$

we get

$$\begin{bmatrix} n-k \\ k \end{bmatrix} \frac{1-q^n}{1-q^{n-k}} = \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-k \\ k \end{bmatrix} q^k. \tag{2.10}$$

Substituting (2.10) into the left-hand side of (2.2) gives

$$\begin{aligned} V_n &= \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} q^{\binom{k}{2}} + \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} n-k \\ k \end{bmatrix} q^{\binom{k+1}{2}} \\ &= -\sum_{k=0}^{\lfloor (n-2)/2 \rfloor} (-1)^k \begin{bmatrix} n-k-2 \\ k \end{bmatrix} q^{\binom{k+1}{2}} + W_n = -W_{n-2} + W_n. \end{aligned}$$

This proves (2.5). Now we complete the proof of (2.3)–(2.5). □

P r o o f of Theorem 1.2. Replacing n by $3m$ in (2.2) and then letting $k\to m+k$, we obtain

$$(1-q^{3m})\sum_{k=-m}^{m}(-1)^k\begin{bmatrix}2m-k\\ m+k\end{bmatrix}\frac{q^{\binom{m+k}{2}}}{1-q^{2m-k}}=(1+q^m)q^{m(3m-1)/2}. \tag{2.11}$$

Note that

$$\begin{aligned}\begin{bmatrix}n\\ m\end{bmatrix}_{q^{-1}}&=\frac{(1-q^{-1})(1-q^{-2})\dots(1-q^{-n})}{(1-q^{-1})\dots(1-q^{-m})(1-q^{-1})\dots(1-q^{-(n-m)})}\\&=\frac{(1-q)(1-q^2)\dots(1-q^n)}{(1-q)\dots(1-q^m)(1-q)\dots(1-q^{n-m})}q^{\binom{m+1}{2}+\binom{n-m+1}{2}-\binom{n+1}{2}}\\&=\begin{bmatrix}n\\ m\end{bmatrix}_q q^{m(m-n)}.\end{aligned}\tag{2.12}$$

Letting $q\to q^{-1}$ in (2.11) and then using (2.12), we obtain (1.8).

Similarly, replacing n by $3m-1$ in (2.2) and then letting $k\to m+k$ and $q\to q^{-1}$, we get (1.9). □

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