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COPIES OF l_p^n'S UNIFORMLY IN THE SPACES $\Pi_2(C[0,1],X)$ AND $\Pi_1(C[0,1],X)$

Dumitru Popa, Constanţa

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Abstract. We study the presence of copies of l_p^n's uniformly in the spaces $\Pi_2(C[0,1],X)$ and $\Pi_1(C[0,1],X)$. By using Dvoretzky's theorem we deduce that if X is an infinite-dimensional Banach space, then $\Pi_2(C[0,1],X)$ contains $\lambda\sqrt{2}$-uniformly copies of l_∞^n's and $\Pi_1(C[0,1],X)$ contains λ-uniformly copies of l_2^n's for all $\lambda > 1$. As an application, we show that if X is an infinite-dimensional Banach space then the spaces $\Pi_2(C[0,1],X)$ and $\Pi_1(C[0,1],X)$ are distinct, extending the well-known result that the spaces $\Pi_2(C[0,1],X)$ and $\mathcal{N}(C[0,1],X)$ are distinct.

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1. Introduction and notation

The main purpose of this paper is to study the presence of copies of l_p^n's uniformly in the spaces $\Pi_2(C[0,1],X)$ and $\Pi_1(C[0,1],X)$. Let us fix some notation and concepts used below. The scalar field $\mathbb{R}$ (or $\mathbb{C}$) is denoted by $\mathbb{K}$ and if $n \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$, then $l_p^n = (\mathbb{K}^n, \|\cdot\|_p)$, where $\|(\alpha_1,\ldots,\alpha_n)\|_p = \Bigl(\sum\limits_{i=1}^n |\alpha_i|^p\Bigr)^{1/p}$ if $p < \infty$ and $\|(\alpha_1,\ldots,\alpha_n)\|_\infty = \max\limits_{1\leqslant i\leqslant n} |\alpha_i|$. By $(e_i)_{1\leqslant i\leqslant n}$ we denote the standard unit vectors in $\mathbb{K}^n$, i.e. $e_i = (0,\ldots,0,1,0,\ldots,0)$. For $1 \leqslant p \leqslant \infty$ we write, as usual, p^* for the conjugate of p, i.e. $1/p + 1/p^* = 1$. If $\alpha = (\alpha_i)_{1\leqslant i\leqslant n} \in \mathbb{K}^n$, $1 \leqslant p, q \leqslant \infty$, $M_\alpha\colon l_p^n \to l_q^n$ is the multiplication operator, i.e. $M_\alpha((\xi_i)_{1\leqslant i\leqslant n}) := (\alpha_i\xi_i)_{1\leqslant i\leqslant n}$. By $r_n\colon [0,1] \to \mathbb{R}$, $r_n(t) = (-1)^{[2^n t]}$ we denote the Rademacher functions ($[\cdot]$ denotes the integer part) and $C[0,1]$ is the space of all scalar-valued continuous functions on $[0,1]$ under the uniform norm.

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Let $1 \leqslant p \leqslant \infty$ and $1 < \lambda < \infty$. We say that a Banach space X *contains l_p^n's λ-uniformly* or that X *contains λ-uniformly copies of l_p^n* if for every $n \in \mathbb{N}$ there exists a linear operator $J_n\colon\ l_p^n \to X$ such that

$$\|\alpha\|_p \leqslant \|J_n(\alpha)\|_X \leqslant \lambda\|\alpha\|_p, \quad \alpha \in l_p^n$$

(see [3], page 260). Let X, Y be Banach spaces and $1 \leqslant p < \infty$. A linear operator $T\colon\ X \to Y$ is *p-summing* if there exists a constant $C \geqslant 0$ such that for every $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ the relation $\Bigl(\sum\limits_{i=1}^{n} \|T(x_i)\|^p\Bigr)^{1/p} \leqslant C \sup\limits_{\|x^*\| \leqslant 1} \Bigl(\sum\limits_{i=1}^{n} |x^*(x_i)|^p\Bigr)^{1/p}$ holds and the p-summing norm of T is defined by $\pi_p(T) := \min\{C\colon\ C \text{ as above}\}$. We denote by $\Pi_p(X,Y)$ the class of all p-summing operators from X into Y (see [2], [3], [4], [6]). Let X and Y be Banach spaces. If A is a set, the notation $(x_n)_{n\in\mathbb{N}} \subset A$ means that $x_n \in A$ for every $n \in \mathbb{N}$. A bounded linear operator $T\colon\ X \to Y$ is called *nuclear* if there exist $(x_n^*)_{n\in\mathbb{N}} \subset X^*$, $(y_n)_{n\in\mathbb{N}} \subset Y$such that $\sum\limits_{n=1}^{\infty} \|x_n^*\|\|y_n\| < \infty$ and $T(x) = \sum\limits_{n=1}^{\infty} x_n^*(x) y_n$ for $x \in X$; such a representation is called a *nuclear representation* of T and the nuclear norm of T is defined by $\|T\|_{\text{nuc}} := \inf\Bigl\{\sum\limits_{n=1}^{\infty} \|x_n^*\|\|y_n\|\Bigr\}$, where the infimum is taken over all the nuclear representations of T. We denote by $\mathcal{N}(X,Y)$ the space of all nuclear operators from X into Y (see [2], [3], [4], [6]). In [10], Theorem 4.2, it was shown that, if X is an infinite-dimensional Banach space, then $\mathcal{N}(C[0,1],X) \neq \Pi_2(C[0,1],X)$. As a natural consequence of our results, we recover the folklore result that if X is an infinite dimensional Banach space, then $\Pi_1(C[0,1],X) \neq \Pi_2(C[0,1],X)$, and hence $\mathcal{N}(C[0,1],X) \neq \Pi_2(C[0,1],X)$, see Corollary 1.

All notation and terminology, not otherwise explained, are as in [2], [3], [4], [6].

Preliminary results

The next Lemma is essentially well-known (see [8], Lemma 10).

Lemma 1. *Let $1 \leqslant p \leqslant \infty$, $n \in \mathbb{N}$, $\alpha = (\alpha_i)_{1\leqslant i\leqslant n} \in \mathbb{K}^n$ and let $U_\alpha^n\colon\ C[0,1] \to l_p^n$ be the operator defined by $U_\alpha^n(f) = \bigl(\alpha_i \int_0^1 f(t) r_i(t)\,\mathrm{d}t\bigr)_{1\leqslant i\leqslant n}$. Then:*

(i) $2^{-1/2}\|\alpha\|_r \leqslant \|U_\alpha^n\| \leqslant \pi_2(U_\alpha^n) \leqslant \|\alpha\|_r$ *if* $1 \leqslant p < 2$, *where* $1/p = 1/2 + 1/r$ *and* $2^{-1/2}\|\alpha\|_\infty \leqslant \|U_\alpha^n\| \leqslant \pi_2(U_\alpha^n) \leqslant \|\alpha\|_\infty$ *if* $2 \leqslant p \leqslant \infty$.

(ii) $\pi_1(U_\alpha^n) = \|\alpha\|_p$.

P r o o f. The representing measure of U_α^n is $G_\alpha^n\colon\ \Sigma \to l_p^n$ defined by $G_\alpha^n(E) := \bigl(\alpha_i \int_E r_i(t)\,\mathrm{d}t\bigr)_{1\leqslant i\leqslant n}$, where Σ is the σ-algebra of all borelian subsets of $[0,1]$, see [4], Theorem 1, page 152. Let $h_\alpha^n\colon [0,1] \to l_p^n$ be given by $h_\alpha^n(t) = (\alpha_i r_i(t))_{1\leqslant i\leqslant n}$ and observe that $G_\alpha^n(E) = \int_E h_\alpha^n(t)\,\mathrm{d}t$ for $E \in \Sigma$ (the Bochner integral).

(i) From [4], Theorem 1, page 152, and Proposition 11, page 4, we have

$$\|U_\alpha^n\| = \|G_\alpha^n\|([0,1]) = \sup_{\|y^*\|\leqslant 1} |y^* \circ G_\alpha^n|([0,1]) = \sup_{\|y^*\|\leqslant 1} \int_0^1 |\langle y^*, h_\alpha^n(t)\rangle|\,\mathrm{d}t$$

because $(y^* \circ G_\alpha^n)(E) = \int_E \langle y^*, h_\alpha^n(t)\rangle\,\mathrm{d}t$ and $|y^* \circ G_\alpha^n|([0,1]) = \int_0^1 |\langle y^*, h_\alpha^n(t)\rangle|\,\mathrm{d}t$. However, for any $y^* = (\xi_i)_{1\leqslant i\leqslant n} \in (l_p^n)^* = l_{p^*}^n$ we have $\langle y^*, h_\alpha^n(t)\rangle = \sum\limits_{i=1}^n \xi_i\alpha_i r_i(t)$ and by Khinchin's inequality $2^{-1/2}\Big(\sum\limits_{i=1}^n |\xi_i\alpha_i|^2\Big)^{1/2} \leqslant \int_0^1 |\langle y^*, h_\alpha^n(t)\rangle|\,\mathrm{d}t$, hence $2^{-1/2}\|M_\alpha\| \leqslant \|G_\alpha^n\|([0,1])$, where $M_\alpha\colon l_{p^*}^n \to l_2^n$ is the multiplication operator. Thus we have shown that $2^{-1/2}\|M_\alpha\colon l_{p^*}^n \to l_2^n\| \leqslant \|U_\alpha^n\|$. Let us note that always $\|U_\alpha^n\| \leqslant \pi_2(U_\alpha^n)$. Further, $U_\alpha^n\colon C[0,1] \overset{J}{\hookrightarrow} L_2[0,1] \overset{R}{\to} l_2^n \overset{M_\alpha}{\to} l_p^n$ is a factorization of U_α^n, where J is the canonical inclusion and $R(f) = \big(\int_0^1 f(t)r_i(t)\,\mathrm{d}t\big)_{1\leqslant i\leqslant n}$. Since J is 2-summing with $\pi_2(J) = 1$ and $\|R\| = 1$, we deduce that $\pi_2(U_\alpha^n) \leqslant \|M_\alpha\colon l_2^n \to l_p^n\|$. Now, as is well known, $\|M_\alpha\colon l_{p^*}^n \to l_2^n\| = \|M_\alpha\colon l_2^n \to l_p^n\| = \|\alpha\|_r$ if $1 \leqslant p < 2$, where $1/p = 1/2 + 1/r$ and $\|M_\alpha\colon l_{p^*}^n \to l_2^n\| = \|M_\alpha\colon l_2^n \to l_p^n\| = \max\limits_{1\leqslant i\leqslant n} |\alpha_i| = \|\alpha\|_\infty$ if $2 \leqslant p \leqslant \infty$, see [1], page 218, and the proof of (i) is finished.

(ii) From [4], Theorem 3, page 162, $\pi_1(U_\alpha^n) = |G_\alpha^n|([0,1]) = \int_0^1 \|h_\alpha^n(t)\|_p\,\mathrm{d}t = \|\alpha\|_p$. $\square$

In the sequel the technique named Average of a finite number of elements, introduced in [7], [9] is used to construct a useful kind of operators. Let us now fix some notation and recall this concept.

Let n be a natural number. For $(\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n$ we define the finite system denoted by Average$(\lambda_i\colon 1 \leqslant i \leqslant n)$ as being the system with 2^n elements obtained by arranging in the lexicographical order of $D_n := \{-1,1\}^n$ the elements $\varepsilon_1\lambda_1 + \ldots + \varepsilon_n\lambda_n$ for $(\varepsilon_1, \ldots, \varepsilon_n) \in D_n$. (On $\{-1,1\}$ we consider the natural order). Thus, as sets we have

$$\text{Average}(\lambda_i\colon 1 \leqslant i \leqslant n) = \{\varepsilon_1\lambda_1 + \ldots + \varepsilon_n\lambda_n\colon (\varepsilon_1, \ldots, \varepsilon_n) \in D_n\}.$$

Let us note that if $(\lambda_i)_{1\leqslant i\leqslant n} \in \mathbb{K}^n$ and $(e_{(\varepsilon_1,\ldots,\varepsilon_n)})_{(\varepsilon_1,\ldots,\varepsilon_n)\in D_n}$ are the standard unit vectors in $\mathbb{K}^{2^n}$ ordered in the lexicographical order of D_n, then the following equality in $\mathbb{K}^{2^n}$ holds:

$$(1)\qquad \text{Average}(\lambda_i\colon 1 \leqslant i \leqslant n) = \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in D_n} (\varepsilon_1\lambda_1 + \ldots + \varepsilon_n\lambda_n)e_{(\varepsilon_1,\ldots,\varepsilon_n)}.$$

If $1 \leqslant p < \infty$, by Khinchin's inequality we have

$$(2) \qquad A_p\|(\lambda_1,\dots,\lambda_n)\|_2 \leqslant \Big\|\mathrm{Average}\Big(\frac{1}{2^{n/p}}\lambda_i\colon\ 1\leqslant i\leqslant n\Big)\Big\|_p = \Big(\frac{1}{2^n}\sum_{(\varepsilon_1,\dots,\varepsilon_n)\in D_n}|\varepsilon_1\lambda_1+\ldots+\varepsilon_n\lambda_n|^p\Big)^{1/p} \leqslant B_p\|(\lambda_1,\dots,\lambda_n)\|_2.$$

Above and in the sequel A_p, B_p are Khinchin's constants (see [3]).

Lemma 2. *Let $1 \leqslant p < \infty$, $n\in\mathbb{N}$, $\alpha = (\alpha_i)_{1\leqslant i\leqslant n}\in\mathbb{K}^n$ and let $Av_\alpha^n\colon C[0,1]\to l_p^{2^n}$ be the operator defined by*

$$Av_\alpha^n(f) = \mathrm{Average}\Big(\frac{\alpha_i}{2^{n/p}}\int_0^1 f(t)r_i(t)\,\mathrm{d}t\colon\ 1\leqslant i\leqslant n\Big).$$

Then:

(i) $A_p 2^{-1/2}\|\alpha\|_\infty \leqslant \pi_2(Av_\alpha^n)\leqslant B_p\|\alpha\|_\infty$.

(ii) $A_p\|\alpha\|_2\leqslant \pi_1(Av_\alpha^n)\leqslant B_p\|\alpha\|_2$.

Proof. Let $f\in C[0,1]$. From the relation (2) we have

$$A_p\|U_\alpha^n(f)\|_2\leqslant\|Av_\alpha^n(f)\|\leqslant B_p\|U_\alpha^n(f)\|_2$$

where $U_\alpha^n\colon C[0,1]\to l_2^n$ is defined by $U_\alpha^n(f)=\big(\alpha_i\int_0^1 f(t)r_i(t)\,\mathrm{d}t\big)_{1\leqslant i\leqslant n}$. Thus

$$A_p\pi_2(U_\alpha^n)\leqslant\pi_2(Av_\alpha^n)\leqslant B_p\pi_2(U_\alpha^n)\quad\text{and}\quad A_p\pi_1(U_\alpha^n)\leqslant\pi_1(Av_\alpha^n)\leqslant B_p\pi_1(U_\alpha^n).$$

The conclusion follows, because in this case, by Lemma 1, $2^{-1/2}\|\alpha\|_\infty\leqslant\pi_2(U_\alpha^n)\leqslant\|\alpha\|_\infty$ and $\pi_1(U_\alpha^n)=\|\alpha\|_2$. □

We need also the second average which we describe next. Let n be a natural number. Let us note that if $(\lambda_1,\dots,\lambda_n)\in\mathbb{K}^n$ then

$$(3) \qquad c_{\mathbb{K}}\sum_{i=1}^n|\lambda_i|\leqslant\|\mathrm{Average}(\lambda_i\colon\ 1\leqslant i\leqslant n)\|_\infty\leqslant\sum_{i=1}^n|\lambda_i|$$

where $c_{\mathbb{K}}=1$ if $\mathbb{K}:=\mathbb{R}$; $c_{\mathbb{K}}=1/2$ if $\mathbb{K}:=\mathbb{C}$ (in this case consider the real and the imaginary part).

For $(\lambda_1,\dots,\lambda_n)\in\mathbb{K}^n$ let us denote the 2^n elements of the set $\mathrm{Average}(\lambda_i\colon\ 1\leqslant i\leqslant n)$ by $\{\beta_1,\beta_2,\dots,\beta_{2^n}\}$ and apply the same procedure; we define

$$\mathrm{Saverage}(\lambda_i\colon\ 1\leqslant i\leqslant n):=\mathrm{Average}(\beta_i\colon\ 1\leqslant i\leqslant 2^n) = \{\varepsilon_1\beta_1+\ldots+\varepsilon_{2^n}\beta_{2^n}\colon\ (\varepsilon_1,\dots,\varepsilon_{2^n})\in D_{2^n}\}\subset\mathbb{K}^{2^{2^n}}.$$

From the relation (3) we have

$$\frac{c_{\mathbb{K}}}{2^n}\|(\beta_1,\dots,\beta_{2^n})\|_1 \leqslant \frac{1}{2^n}\|\operatorname{Saverage}(\lambda_i\colon\ 1\leqslant i\leqslant n)\|_\infty \leqslant \frac{1}{2^n}\|(\beta_1,\dots,\beta_{2^n})\|_1$$

and since by Khinchin's inequality

$$\begin{aligned}\frac{1}{\sqrt{2}}\|(\lambda_1,\dots,\lambda_n)\|_2 \leqslant \frac{1}{2^n}\sum_{i=1}^{2^n}|\beta_i| = \frac{1}{2^n}\sum_{(\varepsilon_1,\dots,\varepsilon_n)\in D_n}|\varepsilon_1\lambda_1+\dots+\varepsilon_n\lambda_n|\\ \leqslant \|(\lambda_1,\dots,\lambda_n)\|_2\end{aligned}$$

we get

$$\frac{c_{\mathbb{K}}}{\sqrt{2}}\|(\lambda_1,\dots,\lambda_n)\|_2 \leqslant \frac{1}{2^n}\|\operatorname{Saverage}(\lambda_i\colon\ 1\leqslant i\leqslant n)\|_\infty \leqslant \|(\lambda_1,\dots,\lambda_n)\|_2. \tag{4}$$

Lemma 3. (a) *Let $n\in\mathbb{N}$, $\alpha=(\alpha_i)_{1\leqslant i\leqslant n}\in\mathbb{K}^n$ and let $Av_\alpha^n\colon\ C[0,1]\to l_\infty^{2^n}$ be the operator defined by*

$$Av_\alpha^n(f) = \operatorname{Average}\Big(\alpha_i\int_0^1 f(t)r_i(t)\,\mathrm{d}t\colon\ 1\leqslant i\leqslant n\Big).$$

Then:
(i) $c_{\mathbb{K}}2^{-1/2}\|\alpha\|_2 \leqslant \pi_2(Av_\alpha^n)\leqslant\|\alpha\|_2$.
(ii) $c_{\mathbb{K}}\|\alpha\|_1 \leqslant \pi_1(Av_\alpha^n)\leqslant\|\alpha\|_1$.

(b) *Let $n\in\mathbb{N}$, $\alpha=(\alpha_i)_{1\leqslant i\leqslant n}\in\mathbb{K}^n$ and let* $\operatorname{Sav}_\alpha^n\colon\ C[0,1]\to l_\infty^{2^{2^n}}$ *be the operator defined by*

$$\operatorname{Sav}_\alpha^n(f) := \operatorname{Saverage}\Big(\frac{1}{2^n}\alpha_i\int_0^1 f(t)r_i(t)\,\mathrm{d}t\colon\ 1\leqslant i\leqslant n\Big).$$

Then:
(i) $c_{\mathbb{K}}2^{-1}\|\alpha\|_\infty \leqslant \pi_2(\operatorname{Sav}_\alpha^n)\leqslant\|\alpha\|_\infty$.
(ii) $c_{\mathbb{K}}2^{-1/2}\|\alpha\|_2 \leqslant \pi_1(\operatorname{Sav}_\alpha^n)\leqslant\|\alpha\|_2$.

Proof. (a) Let $f\in C[0,1]$. From the relation (3) we have

$$c_{\mathbb{K}}\|U_\alpha^n(f)\|_1 \leqslant \|Av_\alpha^n(f)\|_\infty \leqslant \|U_\alpha^n(f)\|_1$$

where $U_\alpha^n\colon\ C[0,1]\to l_1^n$ is defined by $U_\alpha^n(f)=(\alpha_i\int_0^1 f(t)r_i(t)\,\mathrm{d}t)_{1\leqslant i\leqslant n}$. Thus, easily,

$$c_{\mathbb{K}}\pi_2(U_\alpha^n)\leqslant\pi_2(Av_\alpha^n)\leqslant\pi_2(U_\alpha^n) \text{ and } c_{\mathbb{K}}\pi_1(U_\alpha^n)\leqslant\pi_2(Av_\alpha^n)\leqslant\pi_1(U_\alpha^n).$$

The conclusion follows, because in this case, by Lemma 1, $2^{-1/2}\|\alpha\|_2 \leqslant \pi_2(U_\alpha^n) \leqslant \|\alpha\|_2$ and $\pi_1(U_\alpha^n) = \|\alpha\|_1$.

(b) Let $f \in C[0,1]$. From the relation (4) we have

$$\frac{c_{\mathbb{K}}}{\sqrt{2}}\|U_\alpha^n(f)\|_2 \leqslant \|\operatorname{Sav}_\alpha^n(f)\|_\infty \leqslant \|U_\alpha^n(f)\|_2$$

where $U_\alpha^n\colon C[0,1] \to l_2^n$ is defined by $U_\alpha^n(f) = \bigl(\alpha_i \int_0^1 f(t)r_i(t)\,\mathrm{d}t\bigr)_{1\leqslant i\leqslant n}$. Thus

$$\frac{c_{\mathbb{K}}}{\sqrt{2}}\pi_2(U_\alpha^n) \leqslant \pi_2(\operatorname{Sav}_\alpha^n) \leqslant \pi_2(U_\alpha^n); \qquad \frac{c_{\mathbb{K}}}{\sqrt{2}}\pi_1(U_\alpha^n) \leqslant \pi_1(\operatorname{Sav}_\alpha^n) \leqslant \pi_1(U_\alpha^n).$$

The conclusion follows, because in this case, by Lemma 1, $2^{-1/2}\|\alpha\|_\infty \leqslant \pi_2(U_\alpha^n) \leqslant \|\alpha\|_\infty$ and $\pi_1(U_\alpha^n) = \|\alpha\|_2$. $\square$

The results

In the next theorem, which is the main result of this paper, we show how the local structure of the spaces $\Pi_2(C[0,1],X)$ and $\Pi_1(C[0,1],X)$ depends on the local structure of X.

Theorem 4. *Let $1 \leqslant p \leqslant \infty$, $1 < \lambda < \infty$ and let X be a Banach space which contains l_p^n's λ-uniformly. Then:*

(i) *For $1 \leqslant p < 2$, $\Pi_2(C[0,1],X)$ contains $\lambda\sqrt{2}$-uniformly copies of l_r^n's where $1/p = 1/2 + 1/r$.*

(ii) *For $2 \leqslant p \leqslant \infty$, $\Pi_2(C[0,1],X)$ contains $\lambda\sqrt{2}$-uniformly copies of l_∞^n's.*

(iii) *For $1 \leqslant p < \infty$, $\Pi_2(C[0,1],X)$ contains $\lambda B_p\sqrt{2}/A_p$-uniformly copies of l_∞^n's.*

(iv) *$\Pi_1(C[0,1],X)$ contains λ-uniformly copies of l_p^n's.*

(v) *For $1 \leqslant p < \infty$, $\Pi_1(C[0,1],X)$ contains $\lambda B_p/A_p$-uniformly copies of l_2^n's.*

(vi) *For $1 \leqslant p < \infty$, the spaces $\Pi_2(C[0,1],X)$ and $\Pi_1(C[0,1],X)$ are distinct; in particular, $\Pi_2(C[0,1],X) \neq \mathcal{N}(C[0,1],X)$.*

P r o o f. (i), (ii) and (iv). Let $n \in \mathbb{N}$ be arbitrary. By hypothesis there exists a bounded linear operator $J_n\colon l_p^n \to X$ such that

$$\|\alpha\|_p \leqslant \|J_n(\alpha)\|_X \leqslant \lambda\|\alpha\|_p, \quad \alpha \in l_p^n. \tag{5}$$

Let us define $A_n\colon \mathbb{K}^n \to L(C[0,1],X)$ by $A_n(\alpha) = J_n \circ U_\alpha^n$, where $U_\alpha^n\colon C[0,1] \to l_p^n$ is the operator from Lemma 1. Though not needed in the sequel, let us note that if $\alpha = (\alpha_i)_{1\leqslant i\leqslant n} \in \mathbb{K}^n$ and $f \in C[0,1]$ then

$$A_n(\alpha)(f) = \sum_{i=1}^n \alpha_i\Bigl(\int_0^1 f(t)r_i(t)\,\mathrm{d}t\Bigr)J_n(e_i).$$

Let $\alpha \in \mathbb{K}^n$. For every $f \in C[0,1]$ by (5) we have

$$\|U^n_\alpha(f)\|_p \leqslant \|[A_n(\alpha)](f)\|_X = \|J_n(U^n_\alpha(f))\|_X \leqslant \lambda \|U^n_\alpha(f)\|_p$$

and by the definition of p-summing operators we deduce that

$$\pi_2(U^n_\alpha) \leqslant \pi_2(A_n(\alpha)) \leqslant \lambda \pi_2(U^n_\alpha) \quad \text{and} \quad \pi_1(U^n_\alpha) \leqslant \pi_1(A_n(\alpha)) \leqslant \lambda \pi_1(U^n_\alpha). \tag{6}$$

From (6) and Lemma 1 we obtain

$$\begin{aligned}
&\|\alpha\|_r \leqslant \pi_2(\sqrt{2}A_n(\alpha)) \leqslant \lambda\sqrt{2}\|\alpha\|_r \quad \text{if } 1 \leqslant p < 2, \text{ where } \frac{1}{p} = \frac{1}{2} + \frac{1}{r},\\
&\|\alpha\|_\infty \leqslant \pi_2(\sqrt{2}A_n(\alpha)) \leqslant \lambda\sqrt{2}\|\alpha\|_\infty \quad \text{if } 2 \leqslant p < \infty,\\
&\|\alpha\|_p \leqslant \pi_1(A_n(\alpha)) \leqslant \lambda\|\alpha\|_p,
\end{aligned}$$

which ends the proof of (i), (ii) and (iv).

(iii) and (v). Let $n \in \mathbb{N}$ be arbitrary. By hypothesis there exists a bounded linear operator $J_{2^n}\colon l_p^{2^n} \to X$ such that

$$\|\xi\|_p \leqslant \|J_{2^n}(\xi)\|_X \leqslant \lambda\|\xi\|_p, \quad \xi \in l_p^{2^n}. \tag{7}$$

We define $Av_n\colon \mathbb{K}^n \to L(C[0,1], X)$ by $Av_n(\alpha) = J_{2^n} \circ Av^n_\alpha$, where $Av^n_\alpha\colon C[0,1] \to l_p^{2^n}$ is the operator from Lemma 2. Again, though not needed in the sequel, let us note that if $\alpha = (\alpha_i)_{1\leqslant i\leqslant n} \in \mathbb{K}^n$ and $f \in C[0,1]$ we have

$$\begin{aligned}
[Av_n(\alpha)](f) = \frac{1}{2^{n/p}} \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in D_n} \biggl(\varepsilon_1\alpha_1 \int_0^1 f(t)r_1(t)\,\mathrm{d}t + \ldots\\
+ \varepsilon_n\alpha_n \int_0^1 f(t)r_n(t)\,\mathrm{d}t\biggr) J_{2^n}(e_{(\varepsilon_1,\ldots,\varepsilon_n)}).
\end{aligned}$$

Let $\alpha \in \mathbb{K}^n$. For every $f \in C[0,1]$ by (7) we have

$$\|Av^n_\alpha(f)\|_p \leqslant \|[Av_n(\alpha)](f)\|_X = \|J_{2^n}(Av^n_\alpha(f))\|_X \leqslant \lambda\|Av^n_\alpha(f)\|_p$$

and by the definition of p-summing operators we deduce that

$$\pi_2(Av^n_\alpha) \leqslant \pi_2(Av_n(\alpha)) \leqslant \lambda\pi_2(Av^n_\alpha) \quad \text{and} \quad \pi_1(Av^n_\alpha) \leqslant \pi_1(Av_n(\alpha)) \leqslant \lambda\pi_1(Av^n_\alpha). \tag{8}$$

Since by Lemma 2

$$\frac{A_p}{\sqrt{2}}\|\alpha\|_\infty \leqslant \pi_2(Av_n(\alpha)) \leqslant B_p\|\alpha\|_\infty \quad \text{and} \quad A_p\|\alpha\|_2 \leqslant \pi_1(Av_n(\alpha)) \leqslant B_p\|\alpha\|_2,$$

from (8) we obtain

$$\|\alpha\|_\infty \leqslant \pi_2\Big(\frac{\sqrt{2}}{A_p}Av_n(\alpha)\Big) \leqslant \frac{\lambda B_p\sqrt{2}}{A_p}\|\alpha\|_\infty; \quad \|\alpha\|_2 \leqslant \pi_1\Big(\frac{Av_n(\alpha)}{A_p}\Big) \leqslant \frac{\lambda B_p}{A_p}\|\alpha\|_2,$$

which ends the proof of (iii) and (v).

(vi) If $\Pi_2(C[0,1],X) = \Pi_1(C[0,1],X)$, then by the open mapping theorem it follows that there exists $C>0$ such that $\pi_1(T) \leqslant C\pi_2(T)$ for all $T\in\Pi_1(C[0,1],X)$. In particular, $\pi_1(A_n(\alpha)) \leqslant C\pi_2(A_n(\alpha))$ for all natural numbers n and all $\alpha\in\mathbb{K}^n$. By (i), (ii) and (iv) for all natural numbers n and all $\alpha\in\mathbb{K}^n$ we have $\|\alpha\|_p \leqslant C\|\alpha\|_r$ if $1\leqslant p<2$, where $1/p = 1/2+1/r$, or $\|\alpha\|_p \leqslant C\|\alpha\|_\infty$ if $2\leqslant p<\infty$. Taking

$$\alpha = (\underbrace{1,\dots,1}_{n\text{-times}})$$

we get that for all natural numbers n we have $n\leqslant C^2$ if $1\leqslant p<2$, or $n\leqslant C^p$ if $2\leqslant p<\infty$, which is impossible. Let us note that a contradiction can be obtained if we use (iii) or (v). If $\Pi_2(C[0,1],X) = \mathcal{N}(C[0,1],X)$ then, since $\mathcal{N}(C[0,1],X) \subseteq \Pi_1(C[0,1],X) \subseteq \Pi_2(C[0,1],X)$, it follows that $\Pi_1(C[0,1],X) = \Pi_2(C[0,1],X)$, which as we have shown above is impossible. □

As a natural consequence of Theorem 4, we recover the folklore result that if X is an infinite-dimensional Banach space then the spaces $\Pi_2(C[0,1],X)$ and $\Pi_1(C[0,1],X)$ are distinct. This extends the well-known result that the spaces $\Pi_2(C[0,1],X)$ and $\mathcal{N}(C[0,1],X)$ are distinct, see [10], Theorem 4.2.

Corollary 5. *Let X be an infinite dimensional Banach space. Then:*

(i) $\Pi_2(C[0,1],X)$ *contains $\lambda\sqrt{2}$-uniformly copies of l_∞^n's for all $\lambda>1$.*

(ii) $\Pi_1(C[0,1],X)$ *contains λ-uniformly copies of l_2^n's for all $\lambda>1$.*

(iii) *The spaces $\Pi_2(C[0,1],X)$ and $\Pi_1(C[0,1],X)$ are distinct; in particular, $\Pi_2(C[0,1],X) \neq \mathcal{N}(C[0,1],X)$.*

Proof. Since X is infinite-dimensional, by the famous Dvoretzky theorem, see [3], Chapter 19, X contains l_2^n's λ-uniformly for all $1<\lambda<\infty$. The statement follows by taking $p=2$ in Theorem 4. □

Let us note that for $p=\infty$ in Theorem 4 ((ii) and (iv)) it follows that if $1<\lambda<\infty$ and X is a Banach space which contains l_∞^n's λ-uniformly, then $\Pi_2(C[0,1],X)$ contains $\lambda\sqrt{2}$-uniformly copies of l_∞^n's and $\Pi_1(C[0,1],X)$ contains λ-uniformly copies of l_∞^n's, so in this case, there is no distinction between these classes.

We prove now a natural completion of Theorem 4. It shows that for $p=\infty$ in Theorem 4 we have also a distinction if we use the first and the second average.

Theorem 6. *Let $1 < \lambda < \infty$ and let X be a Banach space which contains l_∞^n's λ-uniformly. Then:*

(i) *$\Pi_2(C[0,1], X)$ contains $\lambda\sqrt{2}$-uniformly copies of l_2^n's in the real case ($2\lambda\sqrt{2}$-uniformly copies of l_2^n's in the complex case).*

(ii) *$\Pi_1(C[0,1], X)$ contains λ-uniformly copies of l_1^n's in the real case (2λ-uniformly copies of l_1^n's in the complex case).*

(iii) *$\Pi_2(C[0,1], X)$ contains 2λ-uniformly copies of l_∞^n's in the real case (4λ-uniformly copies of l_∞^n's in the complex case).*

(iv) *$\Pi_1(C[0,1], X)$ contains $\lambda\sqrt{2}$-uniformly copies of l_2^n's in the real case ($2\lambda\sqrt{2}$-uniformly copies of l_2^n's in the complex case).*

Proof. (i) and (ii). Let $n \in \mathbb{N}$ be arbitrary. By hypothesis there exists a bounded linear operator $J_{2^n} : l_\infty^{2^n} \to X$ such that

$$\|\xi\|_\infty \leqslant \|J_{2^n}(\xi)\|_X \leqslant \lambda \|\xi\|_\infty, \quad \xi \in l_\infty^{2^n}. \tag{9}$$

We define $Av_n : \mathbb{K}^n \to L(C[0,1], X)$ by $Av_n(\alpha) = J_{2^n} \circ Av_\alpha^n$, where $Av_\alpha^n : C[0,1] \to l_\infty^{2^n}$ is the operator from Lemma 3. Let us note (not used in the sequel) the explicit expression,

$$[Av_n(\alpha)](f) = \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in D_n} \left(\varepsilon_1\alpha_1 \int_0^1 f(t) r_1(t)\,\mathrm{d}t + \ldots \right.$$
$$\left. + \varepsilon_n \alpha_n \int_0^1 f(t) r_n(t)\,\mathrm{d}t\right) J_{2^n}(e_{(\varepsilon_1,\ldots,\varepsilon_n)})$$

where $\alpha = (\alpha_i)_{1\leqslant i\leqslant n} \in \mathbb{K}^n$ (see also the equality (1)). Let $\alpha \in \mathbb{K}^n$. For every $f \in C[0,1]$ by (9) we have

$$\|Av_\alpha^n(f)\|_\infty \leqslant \|[Av_n(\alpha)](f)\|_X = \|J_{2^n}(Av_\alpha^n(f))\|_X \leqslant \lambda \|Av_\alpha^n(f)\|_\infty,$$

and by the definition of p-summing operators we deduce that

$$\pi_2(Av_\alpha^n) \leqslant \pi_2(Av_n(\alpha)) \leqslant \lambda \pi_2(Av_\alpha^n) \tag{10}$$

and

$$\pi_1(Av_\alpha^n) \leqslant \pi_1(Av_n(\alpha)) \leqslant \lambda \pi_1(Av_\alpha^n).$$

Since by Lemma 3

$$\frac{c_{\mathbb{K}}}{\sqrt{2}}\|\alpha\|_2 \leqslant \pi_2(Av_n(\alpha)) \leqslant \|\alpha\|_2 \quad \text{and} \quad c_{\mathbb{K}}\|\alpha\|_1 \leqslant \pi_1(Av_n(\alpha)) \leqslant \|\alpha\|_1,$$

from (10) we obtain

$$\|\alpha\|_2 \leqslant \pi_2\Big(\frac{\sqrt{2}}{c_{\mathbb{K}}} Av_n(\alpha)\Big) \leqslant \frac{\lambda\sqrt{2}}{c_{\mathbb{K}}}\|\alpha\|_2 \quad \text{and} \quad \|\alpha\|_1 \leqslant \pi_1\Big(\frac{Av_n(\alpha)}{c_{\mathbb{K}}}\Big) \leqslant \frac{\lambda}{c_{\mathbb{K}}}\|\alpha\|_1$$

which ends the proof of (i) and (ii).

(iii) and (iv). Let $n \in \mathbb{N}$ be arbitrary. By hypothesis there exists a bounded linear operator $J_{2^{2^n}}\colon l_\infty^{2^{2^n}} \to X$ such that

$$\|\xi\|_\infty \leqslant \|J_{2^{2^n}}(\xi)\|_X \leqslant \lambda\|\xi\|_\infty, \quad \xi \in l_\infty^{2^{2^n}}. \tag{11}$$

We define $\operatorname{Sav}_n\colon \mathbb{K}^n \to L(C[0,1], X)$ by $\operatorname{Sav}_n(\alpha) = J_{2^{2^n}} \circ \operatorname{Sav}_\alpha^n$ where $\operatorname{Sav}_\alpha^n\colon C[0,1] \to l_\infty^{2^{2^n}}$ is the operator from Lemma 3. We leave for the interested reader to write the explicit expresion for $[\operatorname{Sav}_n(\alpha)](f)$, which again is not used in the sequel. Let $\alpha \in \mathbb{K}^n$. For every $f \in C[0,1]$ by (11) we have

$$\|\operatorname{Sav}_\alpha^n(f)\|_\infty \leqslant \|[\operatorname{Sav}_n(\alpha)](f)\|_X = \|J_{2^{2^n}}(\operatorname{Sav}_\alpha^n(f))\|_X \leqslant \lambda\|\operatorname{Sav}_\alpha^n(f)\|_\infty$$

and by the definition of p-summing operators we deduce that

$$\pi_2(\operatorname{Sav}_\alpha^n) \leqslant \pi_2(\operatorname{Sav}_n(\alpha)) \leqslant \lambda\pi_2(\operatorname{Sav}_\alpha^n) \tag{12}$$

and

$$\pi_1(\operatorname{Sav}_\alpha^n) \leqslant \pi_1(\operatorname{Sav}_n(\alpha)) \leqslant \lambda\pi_1(\operatorname{Sav}_\alpha^n).$$

Since by Lemma 3

$$\frac{c_{\mathbb{K}}}{2}\|\alpha\|_\infty \leqslant \pi_2(\operatorname{Sav}_n(\alpha)) \leqslant \|\alpha\|_\infty \quad \text{and} \quad \frac{c_{\mathbb{K}}}{\sqrt{2}}\|\alpha\|_2 \leqslant \pi_1(\operatorname{Sav}_n(\alpha)) \leqslant \|\alpha\|_2,$$

from (12) we obtain

$$\|\alpha\|_\infty \leqslant \pi_2\Big(\frac{2}{c_{\mathbb{K}}} \operatorname{Sav}_n(\alpha)\Big) \leqslant \frac{2\lambda}{c_{\mathbb{K}}}\|\alpha\|_\infty \text{ and } \|\alpha\|_2 \leqslant \pi_1\Big(\frac{\sqrt{2}\operatorname{Sav}_n(\alpha)}{c_{\mathbb{K}}}\Big) \leqslant \frac{\lambda\sqrt{2}}{c_{\mathbb{K}}}\|\alpha\|_2,$$

which ends the proof of (iii) and (iv). □

In [5] was shown that the space $\Pi_1(C[0,1], X)$ can be identified with the so called space $l_1^{\text{tree}}(X)$; we refer the reader to the paper [5] for the definition of this space and more details. From Theorems 4, 6 and Corollary 5 we get

Corollary 7. (a) *Let $1 \leqslant p \leqslant \infty$, $1 < \lambda < \infty$ and let X be a Banach space which contains l_p^n's λ-uniformly. Then:*

(i) *$l_1^{\text{tree}}(X)$ contains λ-uniformly copies of l_p^n's.*

(ii) *For $1 \leqslant p < \infty$, $l_1^{\text{tree}}(X)$ contains $\lambda B_p/A_p$-uniformly copies of l_2^n's.*

(b) *Let $1 < \lambda < \infty$ and let X be a Banach space which contains l_∞^n's λ-uniformly. Then:*

(i) *$l_1^{\text{tree}}(X)$ contains $\lambda\sqrt{2}$-uniformly copies of l_1^n's in the real case ($2\lambda\sqrt{2}$-uniformly copies of l_1^n's in the complex case).*

(ii) *$l_1^{\text{tree}}(X)$ contains λ-uniformly copies of l_2^n's in the real case (2λ-uniformly copies of l_2^n's in the complex case).*

(c) *Let X be an infinite dimensional Banach space. Then $l_1^{\text{tree}}(X)$ contains λ-uniformly copies of l_2^n's for all $\lambda > 1$.*

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Author's address: Dumitru Popa, Department of Mathematics, Ovidius University of Constanţa, Bd. Mamaia 124, 900527 Constanţa, Romania, e-mail: `dpopa@univ-ovidius.ro`.