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THE SIZE OF THE LERCH ZETA-FUNCTION AT PLACES SYMMETRIC WITH RESPECT TO THE LINE $\Re(s) = 1/2$

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Abstract. Let $\zeta(s)$ be the Riemann zeta-function. If $t \geqslant 6.8$ and $\sigma > 1/2$, then it is known that the inequality $|\zeta(1-s)| > |\zeta(s)|$ is valid except at the zeros of $\zeta(s)$. Here we investigate the Lerch zeta-function $L(\lambda, \alpha, s)$ which usually has many zeros off the critical line and it is expected that these zeros are asymmetrically distributed with respect to the critical line. However, for equal parameters $\lambda = \alpha$ it is still possible to obtain a certain version of the inequality $|L(\lambda, \lambda, 1-\overline{s})| > |L(\lambda, \lambda, s)|$.

Keywords: Lerch zeta-function; functional equation; zero distribution

MSC 2010: 11M35

1. Introduction

Let $s = \sigma + it$ be a complex variable. In 1965 Spira in [20], Theorem 2, proved that the Riemann hypothesis is true if and only if

$$|\zeta(1-s)| > |\zeta(s)|, \quad t \geqslant 10, \ \frac{1}{2} < \sigma < 1. \tag{1}$$

Dixon and Schoenfeld in [4] showed that if $|t| > 6.8$ and $\sigma > 1/2$, then $|\zeta(1-s)| > |\zeta(s)|$ except at the zeros of $\zeta(s)$. Inequality (1) was studied by Saidak and Zvengrowski in [18], Nazardonyavi and Yakubovich in [17], and Trudgian in [25]. Also it was investigated for other zeta-functions. Berndt in [3] generalized Spira's inequality for some functions of the class of general Dirichlet series, Spira in [21] proved it in the case of the Ramanujan τ-Dirichlet series, Garunkštis and Grigutis in [6] considered the analog of inequality (1) for the Selberg zeta-functions. We discuss a monotonicity of the modulus of the Riemann zeta-function. Matiyasevich, Saidak,

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and Zvengrowski in [16] note that "... strict decrease of the modulus of any continuous complex function f along any curve in the complex plane clearly implies that f can have no zero along that curve." The monotonicity of the modulus of a complex function $|f|$ is related to the sign of the real part of the logarithmic derivative $\Re f'/f$; see [16], Lemma 2.3.

The well known Rieman ξ-function is defined as

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s). \tag{2}$$

It satisfies $\xi(s) = \xi(1-s)$, and it is an entire function, whose zeros coincide with nontrivial zeros of $\zeta(s)$. It is known that

$$\Re\frac{\xi'}{\xi}(s) > 0 \quad \text{when } \Re s > 1$$

and the Riemann hypothesis is equivalent to

$$\Re\frac{\xi'}{\xi}(s) > 0 \quad \text{when } \Re s > \frac{1}{2}.$$

The proofs of the last two inequalities can be found by Hinkkanen in [13] or Lagarias in [14]; see also Garunkštis [5]. Sondow and Dumitrescu in [19] also investigated the relation between the monotonicity of $|\xi(s)|$ and the Riemann hypothesis. Matiyasevich, Saidak, and Zvengrowski in [16] showed that

$$\Re\frac{\zeta'}{\zeta}(s) < \Re\frac{\xi'}{\xi}(s)$$

for $|t| > 8$ and $\sigma < 1$. Moreover, they proved that the modulus of the function $\zeta(s)$ is decreasing with respect to σ in the region $\sigma \leqslant 0$, $|t| \geqslant 8$; extending this region to $\sigma \leqslant 1/2$ is equivalent to the Riemann hypothesis. The similar modulus monotonicity properties and the sign of the real part of the logarithmic derivative of the Selberg zeta-functions are investigated by Grigutis and Šiaučiūnas in [12]. See also Alzer [1] for the monotonicity of the function $F_a(\sigma) = (1 - 1/\zeta(\sigma))^{1/(\sigma-a)}$, where $a \leqslant 1$ and $\sigma > 1$.

In this paper we consider the Lerch zeta-function. We always assume that $0 < \lambda$, $\alpha \leqslant 1$ are fixed parameters. The Lerch zeta-function is given by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi \mathrm{i}\lambda m}}{(m+\alpha)^s}, \quad \sigma > 1.$$

This function has an analytic continuation to the whole complex plane except for a possible simple pole at $s = 1$. Let $\zeta(s)$ and $L(s,\chi)$ denote the Riemann zeta-function and the Dirichlet L-function, respectively. We have that

$$L(1,1,s) = \zeta(s), \quad L(1,1/2,s) = (2^s - 1)\zeta(s),$$
$$L(1/2,1,s) = (1 - 2^{1-s})\zeta(s), \quad \text{and} \quad L(1/2,1/2,s) = 2^s L(s,\chi),$$

where χ is an odd Dirichlet character mod 4. For these four cases certain versions of the Riemann hypothesis can be formulated. For all the other cases, it is expected that the real parts of zeros of the Lerch zeta-function form a dense subset of the interval $(1/2, 1)$. This has been proved for any λ and transcendental α and some other cases; for more details see Laurinčikas and Garunkštis [15], Chapter 8. Only for these four above mentioned cases the Euler product and a symmetry (for most zeros) with respect to the critical line is expected.

Next we discus zero free regions and define trivial and nontrivial zeros of $L(\lambda,\alpha,s)$. For $\lambda \neq 1/2$ and $\lambda \neq 1$ let

$$l\colon\ \sigma = \frac{\pi t}{\log \frac{1-\lambda}{\lambda}} + 1 \tag{3}$$

be a line in the complex plane $\mathbb{C}$. If $\lambda = 1/2$ or $\lambda = 1$, then let l be a real line of the complex plane. Denote by $h(s,l)$ the distance from point s to l. Define for $\varepsilon > 0$,

$$L_\varepsilon(\lambda) = \bigl\{s \in \mathbb{C}\colon\ h(s,l) < \varepsilon\bigr\}.$$

Let $0 < \lambda < 1$ and $\lambda \neq 1/2$; then $L(\lambda,\alpha,s) \neq 0$ if $\sigma < -1$ and $s \notin L_{\log 4/\pi}(\lambda)$. Moreover, there exists a constant $\delta_1 \leqslant -1$ such that $L(\lambda,\alpha,s)$ has exactly one zero with real part between

$$\sigma_k := 1 - \frac{2\pi(\alpha + k)}{\pi + \frac{1}{\pi}\log^2 \frac{1-\lambda}{\lambda}} \tag{4}$$

and σ_{k+1} for $\sigma_k \leqslant \delta_1$ (see Garunkštis and Laurinčikas in [7], Garunkštis and Steuding in [9], Lemma 6, and [10]). For $\lambda = 1/2, 1$, from Spira in [22] and [7] we see that $L(\lambda,\alpha,s) \neq 0$ if $\sigma < -1$ and $|t| \geqslant 1$. Also, in [7] it is shown that $L(\lambda,\alpha,s) \neq 0$ for $\sigma \geqslant 1 + \alpha$. By this we say that a zero of $L(\lambda,\alpha,s)$ is *nontrivial* if it lies in the strip $-1 \leqslant \sigma < 1 + \alpha$. If a zero lies outside the strip $-1 \leqslant \sigma < 1 + \alpha$, then we call it *trivial*.

Denote by $N(\lambda,\alpha,T)$ the number of nontrivial zeros of the function $L(\lambda,\alpha,s)$ in the region $0 < t < T$. For $0 < \lambda$, $\alpha \leqslant 1$ we have [7]

$$N(\lambda,\alpha,T) = \frac{T}{2\pi}\log\frac{T}{2\pi e\alpha\lambda} + O(\log T), \quad T \to \infty. \tag{5}$$

Let $\varrho = \varrho(\lambda, \alpha) = \beta + \mathrm{i}\gamma$ always denote a zero of $L(\lambda, \alpha, s)$. For a positive constant C we define the following region related to the zeros of $L(\lambda, \alpha, s)$.

$$R(\lambda, \alpha, C) = \bigcup_{\substack{\varrho(\lambda,\alpha)\\ \beta \geqslant 1/2}} \{s\colon\ |s - \varrho(\lambda, \alpha)| < \mathrm{e}^{-C\gamma/\log\gamma},\ t \geqslant 2\} \cup \bigcup_{\substack{\varrho(\lambda,\alpha)\\ \beta \leqslant -1}} \Big\{s\colon\ |s - (1 - \overline{\varrho(\lambda, \alpha)})| < \frac{1}{\gamma},\ t \geqslant 2\Big\}. \tag{6}$$

We prove the following theorem.

Theorem 1. *Let $0 < \lambda \leqslant 1$. Then there are constants $C > 0$ and $t_0 = t_0(\lambda) > 0$ such that for $\sigma > 1/2 + \mathrm{e}^{-t}$, $s \notin R(\lambda, \lambda, C)$, and $t \geqslant t_0$,*

$$|L(\lambda, \lambda, 1 - \overline{s})| > |L(\lambda, \lambda, s)|.$$

Theorem 1 implies the following corollary.

Corollary 2. *Suppose the conditions of Theorem 1 are satisfied. Then*

$$L(\lambda, \lambda, 1 - \overline{s}) \neq 0.$$

We expect that under the conditions similar to those given in Theorem 1 the function $f(\sigma) = |L(\lambda, \lambda, 1 - \sigma + \mathrm{i}t)|$ is increasing, similarly to the case of the Riemann zeta-function.

In the next section Theorem 1 is proved.

2. Proof of Theorem 1

First we formulate several useful lemmas.

Lemma 3. *We have*

$$|\Gamma(s)| = \sqrt{2\pi}\,|s|^{\sigma - 1/2}\mathrm{e}^{-\sigma - t\arg s}\Big(1 + O\Big(\frac{1}{|t|}\Big)\Big),$$

as $|s| \to \infty$, uniformly for $-\pi + \delta \leqslant \arg s \leqslant \pi - \delta$.

Proof. The lemma follows from Stirling's formula (see Titchmarsh [23], Section 4.42)

$$\log\Gamma(s) = \Big(s - \frac{1}{2}\Big)\log s - s + \frac{1}{2}\log 2\pi + O\Big(\frac{1}{|s|}\Big),$$

as $|s| \to \infty$, uniformly for $-\pi + \delta \leqslant \arg s \leqslant \pi - \delta$. $\square$

Lemma 4. *For $1/2 \leqslant \sigma \leqslant 1$ and $t \geqslant 2\pi + 1$ we have*

$$|\Gamma(s)| \geqslant \sqrt{2\pi}\,|s|^{\sigma-1/2}\mathrm{e}^{-\pi t/2}\Big(1 - \frac{4\sigma^3 - \sigma}{12t^2}\Big)(1 - \mathrm{e}^{-\pi t}).$$

P r o o f. To prove the lemma it is equivalent to show that

$$(2\pi)^{-\sigma}|\Gamma(s)|\mathrm{e}^{\pi t/2} \geqslant \Big(\frac{|s|}{2\pi}\Big)^{\sigma-1/2}\Big(1 - \frac{4\sigma^3 - \sigma}{12t^2}\Big)(1 - \mathrm{e}^{-\pi t}).$$

We observe that

$$(2\pi)^{-\sigma}|\Gamma(s)|\mathrm{e}^{\pi t/2} = \Big|(2\pi)^{-s}2\Gamma(s)\cos\frac{\pi s}{2}\Big|\Big|\frac{\mathrm{e}^{-\pi s/2}}{2\cos(\pi s/2)}\Big|.$$

Saidak and Zvengrowski in [18], Theorem 4, proved that

$$\Big|(2\pi)^{-s}2\Gamma(s)\cos\frac{\pi s}{2}\Big| \geqslant \Big(\frac{|s|}{2\pi}\Big)^{\sigma-1/2}\Big(1 - \frac{4\sigma^3 - \sigma}{12t^2}\Big)$$

if $1/2 \leqslant \sigma \leqslant 1$ and $t \geqslant 2\pi + 1$. By the inequality $1/(1+x) \geqslant 1 - x$, where $x > -1$, we get

$$\Big|\frac{\mathrm{e}^{-\pi s/2}}{2\cos(\pi s/2)}\Big| = \Big|\frac{1}{1 + \mathrm{e}^{\pi \mathrm{i} s}}\Big| \geqslant \frac{1}{1 + \mathrm{e}^{-\pi t}} \geqslant 1 - \mathrm{e}^{-\pi t}.$$

The lemma is proved. □

Lemma 5. *If $f(s)$ is regular, and*

$$\Big|\frac{f(s)}{f(s_0)}\Big| < \mathrm{e}^M$$

in $\{s\colon |s - s_0| \leqslant r\}$ with $M > 1$, then

$$\Big|\frac{f(s_0)}{f(s)}\prod_{\varrho}\frac{s - \varrho}{s_0 - \varrho}\Big| < \mathrm{e}^{CM}$$

for $|s - s_0| \leqslant 3r/8$, where C is some constant and ϱ runs through the zeros of $f(s)$ that $|\varrho - s_0| \leqslant r/2$.

P r o o f. The lemma follows immediately from the proof of Lemma α by Titchmarsh in [24], Paragraph 3.9. □

Lemma 6. *For any σ_0 there is a constant $A > 0$ such that*

$$L(\lambda, \alpha, \sigma + \mathrm{i}t) = O(t^A), \quad t \to \infty, \tag{7}$$

uniformly in $\sigma \geqslant \sigma_0$.

P r o o f. For $0 < \lambda < 1$, this is Theorem 1.4 in Chapter 3 of Laurinčikas and Garunkštis [15]. For $\lambda = 1$, the lemma follows from the Hurwitz formula (Apostol [2], Theorem 12.6) and the Phragmén-Lindelöf theorem (Titchmarsh [23], Paragraph 5.65). $\square$

Lemma 7. *Let $0 < \lambda$, $\alpha \leqslant 1$. Let $\sigma_0 \in \mathbb{R}$ and $\Re s \geqslant \sigma_0$. Let $L(\lambda, \alpha, s) \neq 0$ and d be the distance from s to the nearest zero of $L(\lambda, \alpha, s)$. Then*

$$\frac{1}{|L(\lambda, \alpha, s)|} < \exp(B(|\log d| + 1) \log t),$$

where B is a positive constant, which depends on σ_0 and parameters α and λ.

P r o o f. The lemma is proved by Garunkštis and Tamošiūnas in [11], Proposition 2. We reproduce the proof here for completeness.

In Lemma 5 we choose $f(s) = L(\lambda, \alpha, s)$, $s_0 = 3 + it$, and a sufficiently large but fixed radius r. In view of Lemma 6 we take $M = b \log T$, where $b = b(r)$. The function $1/L(\lambda, \alpha, s_0)$ is bounded. By the formula for the number of nontrivial zeros (5) and in view of the distribution of trivial zeros (see the discussion next to formula (4)), the number of zeros in the disc $|s - s_0| < r/2$ is less than $c \log \Im s_0$. This proves the lemma. $\square$

Lemma 8. *Let $0 < \lambda$, $\alpha \leqslant 1$ and $\varepsilon > 0$. Let s be such that*

$$\min_{\varrho(\lambda,\alpha)} |s - (1 - \overline{\varrho(\lambda, \alpha)})| > \frac{1}{\gamma}.$$

Then for a sufficiently large t and a sufficiently large σ,

$$|L(\lambda, \alpha, 1 - \overline{s})| > |s|^{(1-\varepsilon)\sigma}. \tag{8}$$

P r o o f. By the distribution of trivial zeros of the function $L(\lambda, \alpha, s)$ (see the discussion next to formula (3)), in the half-plane $\sigma \geqslant 2$ the zeros of the function $L(\lambda, \alpha, 1 - \overline{s})$ lie near to the line

$$k := \left\{ s\colon\ \sigma = \frac{\pi t}{\log \frac{\lambda}{1-\lambda}} \right\} \quad \text{if } \lambda \neq \frac{1}{2}, 1,$$

and

$$k := \{ s\colon\ \Im s = 0 \} \quad \text{if } \lambda = 1/2, 1.$$

First we consider the case $1/2 < \lambda < 1$. Then $\log(\lambda/(1 - \lambda))$ is positive and the line k intersects the area $\{ s\colon\ \sigma > 2,\ t > 0 \}$. We will investigate inequality (8) for

the following three subcases correspondingly: the value of s is (a) above, (b) below, and (c) near to the line k.

We start with the functional equation of the Lerch zeta-function (see for example Laurinčikas and Garunkštis [15], Chapter 2, or Garunkštis, Laurinčikas and Steuding [8], formula (1)), which we write in the form

$$\overline{L(\lambda,\alpha,1-\overline{s})} = (2\pi)^{-s}\Gamma(s)(\mathrm{e}^{-\pi\mathrm{i}s/2+2\pi\mathrm{i}\alpha\lambda}L(\alpha,\lambda,s) + \mathrm{e}^{\pi\mathrm{i}s/2-2\pi\mathrm{i}\alpha(1-\{\lambda\})}L(1-\alpha,1-\{\lambda\},s)). \tag{9}$$

where $\{\lambda\}$ denotes the fractional part of number λ. Let $\varepsilon>0$. By Lemma 3 for any sufficiently large σ and sufficiently large t,

$$|\Gamma(s)|(2\pi)^{-\sigma}\mathrm{e}^{\pi t/2} > \Big(\frac{|s|}{2\pi}\Big)^{\sigma-1/2}\mathrm{e}^{-\sigma}\Big(1+O\Big(\frac{1}{t}\Big)\Big) > |s|^{(1-\varepsilon)\sigma}. \tag{10}$$

Subcase (a). In view of formula (9) we see that for $\sigma>0$ and any t,

$$|L(\lambda,\alpha,1-\overline{s})| = |\Gamma(s)|(2\pi)^{-\sigma}\mathrm{e}^{\pi t/2}\lambda^{-\sigma}\Big|1+\mathrm{e}^{\pi\mathrm{i}s-2\pi\mathrm{i}\alpha}\Big(\frac{\lambda}{1-\{\lambda\}}\Big)^s + \sum_{m=1}^{\infty}\Big(\Big(\frac{\lambda}{\lambda+m}\Big)^s+\mathrm{e}^{\pi\mathrm{i}s-2\pi\mathrm{i}\alpha}\Big(\frac{\lambda}{1-\{\lambda\}+m}\Big)^s\Big)\Big|. \tag{11}$$

We consider the function

$$g(s) := 1+\mathrm{e}^{\pi\mathrm{i}s-2\pi\mathrm{i}\alpha}\Big(\frac{\lambda}{1-\{\lambda\}}\Big)^s + \sum_{m=1}^{\infty}\Big(\Big(\frac{\lambda}{\lambda+m}\Big)^s+\mathrm{e}^{\pi\mathrm{i}s-2\pi\mathrm{i}\alpha}\Big(\frac{\lambda}{1-\{\lambda\}+m}\Big)^s\Big). \tag{12}$$

This is an analytic function for $\sigma>1$. If s lies on the line k, then

$$\Big|\mathrm{e}^{\pi\mathrm{i}s-2\pi\mathrm{i}\alpha}\Big(\frac{\lambda}{1-\{\lambda\}}\Big)^s\Big| = 1.$$

If $s\in k$ and $t>0$, then there is $\delta>0$ (independent on s) such that for $\Im z \geqslant \delta+\Im s$ (i.e. z is above the line k),

$$\Big|\mathrm{e}^{\pi\mathrm{i}z-2\pi\mathrm{i}\alpha}\Big(\frac{\lambda}{1-\{\lambda\}}\Big)^z\Big| \leqslant \exp\Big(-\frac{\pi\delta}{\log\frac{\lambda}{1-\{\lambda\}}}\Big) \leqslant \frac{1}{3}.$$

Moreover, for sufficiently large σ and any $t>0$,

$$\Big|\sum_{m=1}^{\infty}\Big(\Big(\frac{\lambda}{\lambda+m}\Big)^s+\mathrm{e}^{\pi\mathrm{i}s-2\pi\mathrm{i}\alpha}\Big(\frac{\lambda}{1-\{\lambda\}+m}\Big)^s\Big)\Big| \leqslant \frac{1}{3}. \tag{13}$$

By the last two inequalities we conclude that there are $\delta > 0$, $\sigma_0 > 0$ and $t_0 > 0$ such that

$$|g(s)| \geqslant \frac{1}{3} \tag{14}$$

if $t \geqslant t_0$ and

$$\sigma_0 \leqslant \sigma \leqslant \frac{\pi(t-\delta)}{\log \frac{\lambda}{1-\{\lambda\}}}. \tag{15}$$

Thus, if s satisfies condition (15), then by formulas (9) and (10) we get for a sufficiently large t and a sufficiently large σ,

$$|L(\lambda, \alpha, 1-\overline{s})| \geqslant \frac{1}{3}|\Gamma(s)|(2\pi)^{-\sigma}\mathrm{e}^{\pi t/2}\lambda^{-\sigma} > |s|^{(1-\varepsilon)\sigma}. \tag{16}$$

Subcase (b). Further we consider the half-plane beneath the line k. By the functional equation (9) we see that

$$\begin{aligned} |L(\lambda, \alpha, 1-\overline{s})| > |\Gamma(s)|(2\pi)^{-\sigma}\mathrm{e}^{-\pi t/2}(1-\{\lambda\})^{-\sigma} \\ \times \Bigg(1 - \mathrm{e}^{\pi t}\Big(\frac{\lambda}{1-\{\lambda\}}\Big)^{-\sigma} \\ - \sum_{m=1}^{\infty}\Big(\mathrm{e}^{\pi t}\Big(\frac{\lambda+m}{1-\{\lambda\}}\Big)^{-\sigma} + \Big(\frac{1-\{\lambda\}+m}{1-\{\lambda\}}\Big)^{-\sigma}\Big)\Bigg). \end{aligned} \tag{17}$$

Similarly as above, there are $\delta_1 > 0$, and $t_1 > 0$ such that

$$\left|1 - \mathrm{e}^{\pi t}\Big(\frac{\lambda}{1-\{\lambda\}}\Big)^{-\sigma} - \sum_{m=1}^{\infty}\Big(\mathrm{e}^{\pi t}\Big(\frac{\lambda+m}{1-\{\lambda\}}\Big)^{-\sigma} + \Big(\frac{1-\{\lambda\}+m}{1-\{\lambda\}}\Big)^{-\sigma}\Big)\right| \geqslant \frac{1}{3}$$

if $t \geqslant t_1$ and

$$\sigma \geqslant \frac{\pi(t+\delta_1)}{\log \frac{\lambda}{1-\{\lambda\}}}.$$

From the inequalities above, inequality (8) follows for Subcase (b).

Subcase (c). Let δ and δ_1 be as in Subcases (a) and (b), respectively. Let $s_1 = \sigma_1 + \mathrm{i}t_1$ be such that

$$\frac{\pi(t_1-\delta)}{\log \frac{\lambda}{1-\{\lambda\}}} \leqslant \sigma_1 \leqslant \frac{\pi(t_1+\delta_1)}{\log \frac{\lambda}{1-\{\lambda\}}} \tag{18}$$

and

$$\min_{\varrho(\lambda,\alpha)} |s_1 - (1 - \overline{\varrho(\lambda,\alpha)})| = d > 0.$$

We will apply Lemma 5 to prove that there is a constant $C_1 = C_1(\lambda, \alpha, \delta, \delta_1) > 0$ such that

$$g(s_1) > \exp(C_1(-|\log d| - 1)). \tag{19}$$

First we show that $g(s)$ is bounded in any fixed width neighborhood of the line k. More precisely, if $\delta_2 > 0$ and

$$\sigma \leqslant \frac{\pi(t+\delta_2)}{\log \frac{\lambda}{1-\{\lambda\}}}, \tag{20}$$

then

$$\Big|\mathrm{e}^{\pi \mathrm{i} s - 2\pi \mathrm{i}\alpha}\Big(\frac{\lambda}{1-\{\lambda\}}\Big)^s\Big| \leqslant \exp\Big(\frac{\pi\delta_2}{\log \frac{\lambda}{1-\{\lambda\}}}\Big).$$

The last inequality, (12) and (13) give that for a sufficiently large t and a sufficiently large σ satisfying inequality (20),

$$|g(s)| \leqslant \frac{4}{3} + \exp\Big(\frac{\pi\delta_2}{\log \frac{\lambda}{1-\{\lambda\}}}\Big). \tag{21}$$

Let

$$s_0 = \sigma_1 + \mathrm{i}\frac{\sigma_1}{\pi}\log\frac{\lambda}{1-\{\lambda\}} + \mathrm{i}\delta, \tag{22}$$

i.e. s_0 has the same real part as s_1, and s_0 lies on the line

$$k + \mathrm{i}\delta := \Big\{s\colon\ \sigma = \frac{\pi(t-\delta)}{\log \frac{\lambda}{1-\{\lambda\}}}\Big\}.$$

Then by (14) we see that

$$g(s_0) > \frac{1}{3}. \tag{23}$$

The functional equation (9) and the definition (12) of the function $g(s)$ imply that for $\sigma > 2$, the set of zeros of $g(s)$ is equal to the set

$$\{1 - \overline{\varrho(\lambda,\alpha)}\colon\ \varrho(\lambda,\alpha) \text{ is a trivial zero of } L(\lambda,\alpha,s)\}. \tag{24}$$

We apply Lemma 5 with $f(s) = g(s)$, s_0 defined by (22), and $r = 8(\delta+\delta_1)/3$. By inequalities (21) and (23) we choose

$$M = 4 + 3\exp\Big(\frac{\pi 8(\delta+\delta_1)/3}{\log \frac{\lambda}{1-\lambda}}\Big).$$

In view of the distribution of trivial zeros of $L(\lambda,\alpha,s)$ (see the discussion next to formula (4)), and formula (24) we have that the number of zeros of $g(s)$ in the disc $|s-s_0|<r/2$ is less than or equel to $C_2=C_2(\lambda,\delta,\delta_1)$. Then Lemma 5 gives the desired inequality (19). Let $1-\overline{\varrho(\lambda,\alpha)}$ be the zero of $g(s)$ nearest to s_1. In view of the distribution of trivial zeros of $L(\lambda,\alpha,s)$, there is a constant $C_3=C_3(\lambda,\delta,\delta_1)$ such that $\gamma\leqslant t_1+C_3$. Thus, choosing $d\geqslant 1/\gamma$ we obtain from inequality (19) that for a sufficiently large t_1,

$$g(s_1)>\exp(C_1(-\log(t_1+C_3)).$$

Then bound (10) together with (11) gives for large σ_1 and large t_1,

$$|L(\lambda,\alpha,1-\overline{s_1})|>|s_1|^{(1-\varepsilon)\sigma}.$$

By this, (16), and Subcase (b) we prove Lemma 8 for $1/2<\lambda<1$.

If $0<\lambda\leqslant 1/2$ or $\lambda=1$, then s with a positive real and a positive imaginary part lies above the line k. In this case Lemma 8 follows by reasoning similar to that in Subcase (a). The lemma is proved. $\square$

Now we will prove Theorem 1. It is a consequence of the following more general proposition.

Proposition 9. *Let $\varepsilon>0$ and $0<\lambda$, $\alpha\leqslant 1$. Then there are constants $C>0$ and $t_0=t_0(\lambda,\alpha,\varepsilon)>0$ such that for $\sigma>1/2+\mathrm{e}^{-t}$, $s\notin R(\alpha,\lambda,C)$, and $t\geqslant t_0$,*

$$|L(\lambda,\alpha,1-\overline{s})|>|L(\alpha,\lambda,s)|.$$

P r o o f. The Dirichlet series of $L(\alpha,\lambda,s)$ converges absolutely for $\sigma>1$, thus $L(\alpha,\lambda,s)$ is bounded for $\sigma\geqslant 2$. Then in view of Lemma 8 we conclude that there is σ_0 such that for $\sigma\geqslant\sigma_0$ and a sufficiently large t, Proposition 9 is true.

Let $1/2<\sigma\leqslant\sigma_0$. In view of the functional equation (9) and Lemma 4 we obtain

$$\begin{aligned}(25)\quad \Big|\frac{L(\lambda,\alpha,1-\overline{s})}{L(\alpha,\lambda,s)}\Big| &= (2\pi)^{-\sigma}|\Gamma(s)|\mathrm{e}^{\pi t/2}\Big|1+\mathrm{e}^{\pi\mathrm{i}(s-2\alpha(1+[\lambda]))}\frac{L(1-\alpha,1-\{\lambda\},s)}{L(\alpha,\lambda,s)}\Big|\\ &\geqslant\Big(\frac{t}{2\pi}\Big)^{\sigma-1/2}\Big(1-\frac{4\sigma^3-\sigma}{12t^2}\Big)(1-\mathrm{e}^{-\pi t})\\ &\quad\times\Big(1-\mathrm{e}^{-\pi t}\Big|\frac{L(1-\alpha,1-\{\lambda\},s)}{L(\alpha,\lambda,s)}\Big|\Big).\end{aligned}$$

Let d be the distance from s to the nearest zero of $L(\alpha,\lambda,s)$. Note that this nearest zero is a nontrivial zero of $L(\alpha,\lambda,s)$ if t is large. By Lemma 7, there is a positive

constant B such that

$$\frac{1}{|L(\alpha,\lambda,s)|} < \exp(B(|\log d|+1)\log t). \tag{26}$$

Moreover, by Lemma 6, we see that $L(1-\alpha, 1-\{\lambda\}, s) = O(t^A)$. This and formulas (25), (26) show that there is a positive constant D such that

$$\begin{aligned}\Big|\frac{L(\lambda,\alpha,1-\overline{s})}{L(\alpha,\lambda,s)}\Big| &> \Big(\frac{t}{2\pi}\Big)^{\sigma-1/2}\Big(1-\frac{4\sigma^3-\sigma}{12t^2}\Big)\\ &\quad\times(1-\mathrm{e}^{-\pi t})(1-D\mathrm{e}^{-\pi t+B(|\log d|+1)\log t+A\log t}).\end{aligned} \tag{27}$$

Next we will prove that, under conditions of the proposition,

$$\log\Big|\frac{L(\lambda,\alpha,1-\overline{s})}{L(\alpha,\lambda,s)}\Big| > 0.$$

It is easy to see that for $1/2 < \sigma \leqslant \sigma_0$,

$$\log\Big(1-\frac{4\sigma^3-\sigma}{12t^2}\Big) = O\Big(\frac{\sigma-1/2}{t^2}\Big), \quad t\to\infty \tag{28}$$

and

$$\log(1-\mathrm{e}^{-\pi t}) = O(\mathrm{e}^{-\pi t}), \quad t\to\infty. \tag{29}$$

The condition $s \notin R(\alpha,\lambda)$ implies that $|\log d| \leqslant Ct/\log t$, where C will be chosen later. Therefore,

$$\log(1-D\mathrm{e}^{-\pi t+B(|\log d|+1)\log t+A\log t}) = O(\mathrm{e}^{(-\pi+(BC+1))t}), \quad t\to\infty. \tag{30}$$

We choose $0 < C < (\pi-3)/B$. Then by the condition $\sigma - 1/2 > \mathrm{e}^{-t}$ and by formulas (27)–(30) we have

$$\begin{aligned}\log\Big|\frac{L(\lambda,\alpha,1-\overline{s})}{L(\alpha,\lambda,s)}\Big| &\geqslant \Big(\sigma-\frac{1}{2}\Big)\Big(\log\frac{t}{2\pi}+O\Big(\frac{1}{t^2}\Big)\Big)+O(\mathrm{e}^{(-\pi+(BC+1))t})\\ &> \mathrm{e}^{-t}+O(\mathrm{e}^{-2t}) > 0, \quad t\to\infty.\end{aligned}$$

This proves Proposition 9 for $1/2 < \sigma \leqslant \sigma_0$ and finishes the proof. □

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