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SOME PROPERTIES OF GENERALIZED REDUCED VERMA MODULES OVER $\mathbb{Z}$-GRADED MODULAR LIE SUPERALGEBRAS

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Abstract. We study some properties of generalized reduced Verma modules over $\mathbb{Z}$-graded modular Lie superalgebras. Some properties of the generalized reduced Verma modules and coinduced modules are obtained. Moreover, invariant forms on the generalized reduced Verma modules are considered. In particular, for $\mathbb{Z}$-graded modular Lie superalgebras of Cartan type we prove that generalized reduced Verma modules are isomorphic to mixed products of modules.

Keywords: modular Lie superalgebra; generalized reduced Verma module; coinduced module; invariant form; mixed product

MSC 2010: 17B50, 17B10, 17B05

1. Introduction

Verma modules proposed by Verma in [20] and Bernshtein, Gel'fand and Gel'fand in [1] are important objects in the representation theory of Lie algebras and superalgebras. The main results on the structure of Verma modules were obtained in [2], [6], [20]. As a natural generalization of Verma modules, generalized Verma modules are modules induced from a parabolic subalgebra and a complex semisimple Lie algebra (see [3], [5], [12], [13]). The theory of generalized Verma modules is rather similar to that of Verma modules. Some results of Verma modules were extended to certain class of generalized Verma modules in [9], [11], [14].

In 1990, Farnsteiner in [7] constructed generalized reduced Verma modules over modular Lie algebras. Hereafter, some properties of these generalized reduced Verma

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modules were obtained in [4], [8]. Since generalized reduced Verma modules are closely related to mixed products of modules, the structure of mixed products seems to be important and interesting. In [17], [18], [19], Shen classified $\mathbb{Z}$-graded irreducible representations of graded Lie algebras of Cartan type. His approach rests on the notion of the mixed product. In [4], graded modules of graded Cartan type Lie algebras which possess nondegenerate invariant form were determined by Chiu. In the case of modular Lie superalgebras of Cartan type, $\mathbb{Z}$-graded modules of Lie superalgebras $W(n)$ and $S(n)$, $H(n)$, mixed products of modules of infinite-dimensional Lie superalgebras and $\mathbb{Z}$-graded modules of finite-dimensional Hamiltonian Lie superalgebras were obtained in [22], [23], [25], [26], respectively.

In this paper, we generalize some beautiful results about generalized reduced Verma modules over modular Lie algebras in [4], [7], [8]. In Section 2, we review some necessary notions. In Section 3, some relations between generalized reduced Verma modules and coinduced modules are given. In Section 4, invariant forms on generalized reduced Verma modules are considered. In Section 5, we prove that generalized reduced Verma modules are isomorphic to mixed products for modules of $\mathbb{Z}$-graded modular Lie superalgebras of Cartan type.

All Lie superalgebras and modules treated in the present paper are assumed to be finite dimensional. All notations and notions of Lie superalgebras and modular representations are the same as in papers [10], [16], [24], readers can find the precise definitions in the corresponding references.

2. Preliminaries

Throughout this paper we assume that $\mathbb{F}$ is a field of prime characteristic and $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ is the residue class ring mod 2. Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a Lie superalgebra over $\mathbb{F}$. Then $\mathbb{F}$ has a trivial structure of a $\mathbb{Z}_2$-graded L-module: $\mathbb{F}_{\bar{0}} = \mathbb{F}$, $\mathbb{F}_{\bar{1}} = 0$. Furthermore, we always assume that the representation of L in $\mathbb{F}$ is equal to zero.

The standard notation $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ are used for the set of integers, the set of positive integers and the set of nonnegative integers, respectively. Denote by $\mathbb{N}_0^k$ the k-tuples with nonnegative integers as entries. For any Lie superalgebra L over $\mathbb{F}$, let $U(L)$ denote the universal enveloping algebra of L. If $L = \bigoplus_{i\in\mathbb{Z}} L_i$ is a $\mathbb{Z}$-graded Lie superalgebra over $\mathbb{F}$, we customarily put $L^+ = \bigoplus_{i>0} L_i$ and $L^- = \bigoplus_{i<0} L_i$. Then $L = L^+ \oplus L_0 \oplus L^-$ and $U(L) = U(L^+)U(L_0)U(L^-)$.

Without explicitly mentioning, if $d(x)$ $(zd(x))$ occurs in some expression in this paper, then x is assumed to be a $\mathbb{Z}_2$-homogeneous ($\mathbb{Z}$-homogeneous) element and $d(x)$ $(zd(x))$ is the $\mathbb{Z}_2$-degree ($\mathbb{Z}$-degree) of x.

Definition 1 ([21])**.** Let V and W be L-modules and suppose that f is a $\mathbb{Z}_2$-homogeneous element of $\mathrm{Hom}_{\mathbb{F}}(V, W)$. The mapping f is called a *homomorphism* of L-modules if $(x \cdot f)(v) = (-1)^{d(x)d(f)} f(x \cdot v)$ for all $x \in L$ and $v \in V$. The mapping f is said to be an *isomorphism* of L-modules if f is a homomorphism and if, furthermore, f is a bijection.

Let V be an L-module. The vector space $V^* := \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ obtains the structure of an L-module by means of $(x \cdot f)(v) = -(-1)^{d(x)d(f)} f(x \cdot v)$, where $x \in L$, $v \in V$, $f \in V^*$. Clearly, $d(x \cdot f) = d(x) + d(f)$.

We consider the subalgebra $K := L_0 \oplus L^+$ of a $\mathbb{Z}$-graded Lie superalgebra $L = \bigoplus_{i \in \mathbb{Z}} L_i$. Let $\{e_1, \ldots, e_k\}$ be a basis of $L^- \cap L_{\bar{0}}$ and $\{\xi_1, \ldots, \xi_l\}$ be a basis of $L^- \cap L_{\bar{1}}$. As $L^- \cap L_{\bar{0}}$ operates on L by nilpotent transformation, there exist $m_i \in \mathbb{N}_0$, $1 \leqslant i \leqslant k$ such that

$$z_i := e_i^{p^{m_i}} \in U(L^-) \cap Z(U(L)), \quad 1 \leqslant i \leqslant k,$$

where $Z(U(L))$ is the center of $U(L)$. In particular, $\{z_i \colon 1 \leqslant i \leqslant k\}$ are homogeneous elements relative to the $\mathbb{Z}$-gradation inherited by $U(L_{\bar{0}})$. An application of the Poincaré-Birkhoff-Witt theorem (PBW theorem), (see [15]), reveals that the subalgebra $\theta(L, K)$ of $U(L)$, which is generated by K and $\{z_1, \ldots, z_k\}$, is isomorphic to $\mathbb{F}[z_1, \ldots, z_k] \bigotimes_{\mathbb{F}} U(K)$, where $\mathbb{F}[z_1, \ldots, z_k]$ is a polynomial ring in k indeterminates. Then $\theta(L, K)$ is a $\mathbb{Z}$-graded subalgebra of $U(L)$.

Given $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}_0^k$, we put $|\alpha| := \sum_{i=1}^{m} \alpha_i$, $e^\alpha := e_1^{\alpha_1} e_2^{\alpha_2} \ldots e_k^{\alpha_k}$ and $\pi := (\pi_1, \ldots, \pi_k) = (p^{m_1} - 1, \ldots, p^{m_k} - 1)$. Set

$$\mathbb{B}_s := \{\langle i_1, i_2, \ldots, i_s \rangle \colon 1 \leqslant i_1 < i_2 < \ldots < i_s \leqslant l\}$$

and $\mathbb{B} := \bigcup_{s=0}^{l} \mathbb{B}_s$, where $\mathbb{B}_0 := \emptyset$ and $l \in \mathbb{N}$. For $u = \langle i_1, i_2, \ldots, i_s \rangle \in \mathbb{B}_s$, set $|u| := s$, $|\emptyset| := 0$, $\xi^\emptyset := 1$, $\xi^u := \xi_{i_1} \xi_{i_2} \ldots \xi_{i_s}$ and $\xi^E := \xi_1 \xi_2 \ldots \xi_l$, u is also used to stand for the index set $\{i_1, i_2, \ldots, i_s\}$. Then $U(L)$ is a $\mathbb{Z}$-graded $\theta(L, K)$-module with the basis

$$\{e^\alpha \xi^u \colon 0 \leqslant \alpha \leqslant \pi, \ u \in \mathbb{B}\}.$$

Any K-module V obtains the structure of a $\theta(L, K)$-module by letting $\mathbb{F}[z_1, \ldots, z_k]$ act via its canonical supplementation which sends z_i to 0. Henceforth, K-module will be regarded as $\theta(L, K)$-module in this fashion. Let ϱ be the natural representation of K in L/K. Then there exists a unique homomorphism $\sigma \colon U(K) \to \mathbb{F}$ of $\mathbb{F}$-superalgebra such that $\sigma(x) = \mathrm{str}(\varrho(x))$, where x is an arbitrary element of K and

$\mathrm{str}(\varrho(x))$ is the supertrace of $\varrho(x)$, see [10], [16]. We introduce a twisted action on K-module V by setting

$$x \circ v = x \cdot v + \sigma(x)v, \quad x \in K, \ v \in V.$$

Note that $\sigma(x) = 0$ for $x \in K_{\bar{1}}$, then

$$\begin{aligned}[x,y] \circ v &= [x,y] \cdot v + \sigma([x,y])v \\ &= x \cdot (y \cdot v) - (-1)^{d(x)d(y)} y \cdot (x \cdot v) + \sigma(x)\sigma(y)v - (-1)^{d(x)d(y)}\sigma(y)\sigma(x)v \\ &= x \cdot (y \cdot v) + \sigma(y)x \cdot v + \sigma(x)y \cdot v + \sigma(x)\sigma(y)v \\ &\quad - (-1)^{d(x)d(y)} y \cdot (x \cdot v) - (-1)^{d(x)d(y)}\sigma(y)x \cdot v \\ &\quad - (-1)^{d(x)d(y)}\sigma(x)y \cdot v - (-1)^{d(x)d(y)}\sigma(y)\sigma(x)v \\ &= x \cdot (y \circ v) + \sigma(y)(x \circ v) - (-1)^{d(x)d(y)} y \cdot (x \circ v) - (-1)^{d(x)d(y)}\sigma(y)(x \circ v) \\ &= x \circ (y \circ v) - (-1)^{d(x)d(y)} y \circ (x \circ v),\end{aligned}$$

i.e. V is a new K-module by the twisted action. The new K-module will be denoted by V_σ. If V is an L_0-module, then we can extend the operations on V to K by letting L^+ act trivially and regard it as a K-module.

3. Generalized reduced Verma modules and coinduced modules

Let L be a $\mathbb{Z}$-graded Lie superalgebra over $\mathbb{F}$ and V be a K-module. Following [7], we give a definition

Definition 2. The induced module $\mathrm{Ind}_K(V) := U(L) \bigotimes_{\theta(L,K)} V$ is called a *generalized reduced Verma module*. The coinduced module $\mathrm{Hom}_{\theta(L,K)}(U(L), V)$ will be denoted by $\mathrm{Coind}_K(V)$.

This definition shows that the modules $\mathrm{Ind}_K(V)$ and $\mathrm{Coind}_K(V)$ are annihilated by z_i.

Consider $\mathrm{Coind}_K(V)$ with $U(L)$-action given via

$$(y \cdot f)(x) := (-1)^{d(y)(d(f)+d(x))} f(xy), \quad x, y \in U(L).$$

For $v \in V$, $0 \leqslant \beta \leqslant \pi$ and $u, t \in \mathbb{B}$, let $\chi_v^{(\beta,t)}$ be the element of $\mathrm{Coind}_K(V)$ which sends $e^\alpha \xi^u$ onto $(-1)^{d(\chi_v^{(\beta,t)})d(\xi^u)}\delta(\alpha,\beta)\delta(u,t)v$, where $\delta(i,j)$ is Kronecker delta. It suffices to verify that

$$\chi_v^{(\beta,t)}(e^\beta \xi^t \vartheta) = (-1)^{d(\vartheta)(d(\chi_v^{(\beta,t)})+d(\xi^t))+d(\chi_v^{(\beta,t)})d(\xi^t)}\vartheta \circ v \tag{3.1}$$

and $d(\chi_v^{(\beta,t)}) = d(\xi^t) + d(v)$ for all $\vartheta \in \theta(L,K)$ and $v \in V_\sigma$.

Lemma 1. *There is a natural isomorphism of functors*

$$\Phi\colon \operatorname{Ind}_K(V_\sigma) \to \operatorname{Coind}_K(V)$$

such that $\Phi(y\otimes v) = (-1)^{d(y)d(\Phi)} y\cdot \chi_v^{(\pi,E)}$, *where* $y\in U(L)$ *and* $v\in V_\sigma$.

Proof. Assume that the bilinear mapping $\psi\colon U(L)\times V_\sigma \to \operatorname{Hom}_{\mathbb{F}}(U(L),V)$ is defined by $\psi(y,v) = (-1)^{d(y)d(\psi)} y\cdot \chi_v^{(\pi,E)}$. Let $\vartheta\in\theta(L,K)$ and $u'\in U(L)$. Then equation (3.1) and $d(\chi_v^{(\pi,E)}) = d(\psi)+d(v)$ imply that

$$\begin{aligned}
\psi(y\vartheta,v)(u') &= (-1)^{(d(y)+d(\vartheta))d(\psi)} y\vartheta\cdot\chi_v^{(\pi,E)}(u')\\
&= (-1)^{(d(y)+d(\vartheta))(d(v)+d(u'))}\chi_v^{(\pi,E)}(u'y\vartheta)\\
&= (-1)^{d(y)(d(v)+d(\vartheta)+d(u'))+d(\vartheta)d(\psi)+(d(\psi)+d(v))(d(u')+d(y))}\vartheta\circ v\\
&= (-1)^{d(y)(d(v)+d(\vartheta)+d(u'))+(d(\vartheta)+d(\psi)+d(v))(d(u')+d(y))}\vartheta\circ v\\
&= (-1)^{d(y)(d(v)+d(\vartheta)+d(u'))}\chi_{\vartheta\circ v}^{(\pi,E)}(u'y)\\
&= (-1)^{d(y)d(\psi)} y\cdot\chi_{\vartheta\circ v}^{(\pi,E)}(u')\\
&= \psi(y,\vartheta\circ v)(u').
\end{aligned}$$

Consequently, ψ is $\theta(L,K)$-balanced and induces a mapping

$$\Phi\colon U(L)\bigotimes_{\theta(L,K)} V_\sigma \to \operatorname{Hom}_{\mathbb{F}}(U(L),V).$$

The verification of the inclusion $\operatorname{im}\psi\subseteq \operatorname{Hom}_{\theta(L,K)}(U(L),V)$ is routine.

For any $x,y\in U(L)$ and $v\in V_\sigma$ we have

$$(x\cdot\Phi)(y\otimes v) = (-1)^{d(y)d(\Phi)}((xy)\cdot\chi_v^{(\pi,E)}) = (-1)^{d(x)d(\Phi)}\Phi(x\cdot(y\otimes v)).$$

Hence, Φ is a homomorphism of $U(L)$-modules.

For any $f\in\operatorname{Coind}_K(V)$ there exists $e^\alpha\xi^u\in U(L)$ such that

$$f = \sum_{\alpha,u}(-1)^{d(f)d(\xi^u)}\chi_{f(e^\alpha\xi^u)}^{(\alpha,u)},$$

where $0\leqslant\alpha\leqslant\pi$ and $u\in\mathbb{B}$. Then $\Phi\Big(\sum\limits_{\alpha,u}(-1)^{d(f)d(\xi^u)}y\otimes f(e^\alpha\xi^u)\Big) = f$, i.e. Φ is a surjection.

If $0 = y\cdot X_v^{(\pi,E)}\in\operatorname{Coind}_K(V)$ and $y = e^\alpha\xi^u\in U(L)$, then there exists $u' = e^\beta\xi^t\in U(L)$ such that $\alpha+\beta=\pi$ and $u+t=E$. It follows that

$$0 = y\cdot\chi_v^{(\pi,E)}(u') = (-1)^{d(y)(d(u')+d(\chi_v^{(\pi,E)}))+d(\chi_v^{(\pi,E)})(d(u')+d(y))}v.$$

Therefore, $y\otimes v = 0$, i.e., Φ is an injection.

Now we show that Φ is a natural homomorphism. If W is a K-module and $\varphi\colon V \to W$ is a homomorphism of K-module, then φ is also a homomorphism between V_σ and W_σ. We claim that the following diagram is commutative.

$$\begin{array}{ccc} \mathrm{Ind}_K(V_\sigma) & \xrightarrow{\Phi} & \mathrm{Coind}_K(V) \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes\varphi} & & \Big\downarrow{\scriptstyle \varphi^*} \\ \mathrm{Ind}_K(W_\sigma) & \xrightarrow{\Phi'} & \mathrm{Coind}_K(W) \end{array}$$

Note that φ^* and $\mathrm{id} \otimes \varphi$ are homomorphisms of $U(L)$-modules, the assertion follows from the ensuing calculation:

$$\varphi^* \circ \Phi(1 \otimes v)(u') = \chi^{(\pi,E)}_{\varphi(v)}(u') = (\Phi' \circ (\mathrm{id} \otimes \varphi))(1 \otimes v)(u'), \quad u' \in U(L).$$

In conclusion, the proof is completed. $\square$

Remark 1. (1) If the above result is applied to the module $V_{-\sigma}$, then we obtain natural isomorphism $\mathrm{Ind}_K(V) \cong \mathrm{Coind}_K(V_{-\sigma})$.

(2) Suppose that K acts nilpotently on L/K or $(\varrho(K))^{(1)} = \varrho(K)$. Then $\sigma = 0$ and every K-module V gives an isomorphism $\mathrm{Ind}_K(V) \cong \mathrm{Coind}_K(V)$.

Following [18], we refer to a $\mathbb{Z}$-graded L-module V as positively graded if $V = \bigoplus\limits_{i\geqslant 0} V_i$ and $L_j \cdot V_i \subseteq V_{i+j}$. A positively graded module V is said to be transitive if $V_0 = \{v \in V\colon\ x \cdot v = 0 \text{ for all } x \in L^-\}$.

Proposition 1. *Let* $P = \mathrm{Coind}_K(V)$ *be an* L*-module and*

$$P_i := \{f \in P\colon\ f(U(L)_j) = 0,\ j \neq -i\}.$$

Then

(1) P *is a positively graded* L*-module;*
(2) P_0 *is isomorphic to* V *as an* L_0*-module;*
(3) P *is transitively graded.*

P r o o f. (1) Let f be an element of P_i and suppose that $y \in U(L)_q$, where $i, q \in \mathbb{Z}$. If $x \in U(L)_j$ for $j \neq -i-q$, then $xy \in U(L)_{j+q}$, where $j \in \mathbb{Z}$. It follows that

$$(y \cdot f)(x) = (-1)^{d(y)(d(f)+d(x))} f(xy) = 0.$$

Consequently, $y \cdot f$ belongs to P_{i+q}.

Let $\{x_1,\dots,x_n\}$ be a basis of $U(L)$ over $\theta(L,K)$ and induced by $\{e_1,\dots,e_k\}$ and $\{\xi_1,\dots,\xi_l\}$. In accordance with the basis of $U(L)$, we may assume that $x_r = e^\alpha\xi^u \in U(L)_{i(r)}$, where $i(r) \leqslant 0$ and $1 \leqslant r \leqslant n$. Any element of $U(L)_q$ is a sum of elements $x = \sum_{r=1}^{n} h_r x_r$, $h_r \in \theta(L,K)_{q-i(r)}$. Given $r \in \{1,2,\dots,n\}$, we have $\chi_v^{(\alpha,u)}(x) = (-1)^{(d(x)+d(v))d(x)} h_r v$. If $q \neq i(r)$, then $\chi_v^{(\alpha,u)}(x) = 0$. It follows that $\chi_v^{(\alpha,u)}$ is an element of $P_{-i(r)}$. For every $f \in P$ we have $f = \sum_{\alpha,u}(-1)^{d(f)d(\xi^u)}\chi^{(\alpha,u)}_{f(e^\alpha\xi^u)}$. Consequently, $P = \bigoplus_{r=1}^{n} P_{-i(r)}$ and P is a positively graded module.

(2) We proceed by showing that $\mu\colon\ P_0 \to V$; $\mu(f) = f(1)$ is an isomorphism of L_0-modules. If $x \in L_0$, then

$$\mu(x\cdot f) = (x\cdot f)(1) = (-1)^{d(x)d(f)} f(x) = x\cdot f(1) = x\cdot\mu(f),$$

i.e. μ is a homomorphism of L_0-modules.

Since $1 := e^\alpha\xi^u \in U(L)_0$ is contained in $\{x_1,\dots,x_n\}$, $(-1)^{(d(\xi^u)+d(v))d(\xi^u)}\chi_v^{(\alpha,u)}$ is a pre-image of $v \in V$ under μ.

Suppose that $f \in \ker\mu$. Owing to the PBW theorem, for every element $x \in U(L)_0$ we may assume that $x = \sum_{i+j=0} a_i b_j$, where $a_i \in U(K)_i$ and $b_j \in U(L^-)_j$. Since $a_i = 0$ for $i < 0$ and $a_i \in U(L_0)U(L^+)$ for $i > 0$, we obtain

$$\begin{aligned} f(x) &= \sum_{i+j=0}(-1)^{d(a_i)d(f)} a_i f(b_j) = (-1)^{d(a_0)d(f)} a_0 f(b_0)\\ &= (-1)^{(d(a_0)+d(b_0))d(f)} a_0 b_0 f(1) = 0. \end{aligned}$$

As a result, $f = 0$ on $U(L_0)$ and thereby on all of $U(L)$. Therefore μ is an isomorphism of L_0-modules.

(3) Suppose that f is an element of P such that $x\cdot f = 0$ for every $x \in L^-$. Then each $\mathbb{Z}$-homogeneous constituent of f enjoys the same property. Since $q \in \mathbb{N}$ and y is an element of $U(L)_{-q}$, we assume that $f \in P_q$ and $y = \sum_{i+j=-q} a_i b_j$, where $a_i \in U(K)_i$ and $b_j \in U(L^-)_j$. As $a_i \cdot V = 0$ for $i > 0$, we have

$$f(y) = \sum_{i+j=-q}(-1)^{d(a_i)d(f)} a_i f(b_j) = (-1)^{d(a_0)d(f)} a_0 f(b_{-q}).$$

Then $f(y) = (-1)^{(d(a_0)+d(b_{-q}))d(f)} a_0 b_{-q} f(1)$. Since b_{-q} belongs to $U(L^-)$, we obtain $b_{-q}\cdot f = 0$. Thus $f(y) = 0$. Similarly, if $q < 0$, then $f(y)$ also equals zero. Therefore $f \in P_0$.

Conversely, if $f \in P_0$, then $f(U(L)_i) = 0$ for $i \neq 0$. For any $x \in L^-$ we have

$$(x\cdot f)(y) = (-1)^{d(x)(d(f)+d(y))} f(yx) = (-1)^{d(x)(d(y))} y\cdot f(x) = 0, \quad y \in U(L)^+$$

and

$$(x \cdot f)(y) = (-1)^{d(x)(d(f)+d(y))} f(yx) = 0, \quad y \in U(L)^- \oplus U(L)_0.$$

Therefore $x \cdot f = 0$ for all $x \in L^-$. □

For $x_1, \ldots, x_n \in L$ set

$$(x_1 \ldots x_n)^{\mathrm{T}} := (-1)^{n + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d(x_i)d(x_j)} x_n \ldots x_1.$$

A direct verification shows that $x_i^{\mathrm{T}} = -x_i$ and $d(x_i^{\mathrm{T}}) = d(x_i)$ for $i \in \{1, \ldots, n\}$. Then the principal anti-automorphism of $U(L)$ is defined by $x \mapsto x^{\mathrm{T}}$ for all $x \in U(L)$.

In the following proposition, the property of adjoint isomorphism will be investigated.

Proposition 2. *There is a natural isomorphism*

$$\Psi\colon (\mathrm{Ind}_K(V))^* \to \mathrm{Coind}_K(V^*),$$

namely, for $\varphi \in (\mathrm{Ind}_K(V))^*$, $x \in U(L)$ *and* $v \in V$,

$$\Psi\colon \varphi \mapsto \Psi(\varphi), \text{ where } \Psi(\varphi)(x)\colon v \mapsto \varphi(x^{\mathrm{T}} \otimes v).$$

Proof. Firstly, we prove that Ψ is a homomorphism of $U(L)$-modules. Let φ_1 and φ_2 be elements of $(\mathrm{Ind}_K(V))^*$. Then

$$\begin{aligned}\Psi(\varphi_1 + \varphi_2)(x)(v) &= (\varphi_1 + \varphi_2)(x^{\mathrm{T}} \otimes v)\\ &= (\varphi_1)(x^{\mathrm{T}} \otimes v) + (\varphi_2)(x^{\mathrm{T}} \otimes v)\\ &= \Psi(\varphi_1)(x)(v) + \Psi(\varphi_2)(x)(v)\\ &= (\Psi(\varphi_1) + \Psi(\varphi_2))(x)(v),\end{aligned}$$

where $x \in U(L)$ and $v \in V$. Therefore $\Psi(\varphi_1 + \varphi_2) = \Psi(\varphi_1) + \Psi(\varphi_2)$. For any $x, y \in U(L)$, $v \in V$ and $\varphi \in (\mathrm{Ind}_K(V))^*$ we have

$$\begin{aligned}y \cdot \Psi(\varphi)(x)(v) &= (-1)^{d(y)(d(\Psi)+d(\varphi)+d(x))} \Psi(\varphi)(xy)(v)\\ &= (-1)^{d(y)(d(\Psi)+d(\varphi)+d(x))} \varphi((xy)^{\mathrm{T}} \otimes v)\\ &= (-1)^{d(y)(d(\Psi)+d(\varphi))} \varphi(yx \otimes v)\\ &= (-1)^{d(y)d(\Psi)} y \cdot \varphi(x^{\mathrm{T}} \otimes v)\\ &= (-1)^{d(y)d(\Psi)} \Psi(y \cdot \varphi)(x)(v).\end{aligned}$$

Therefore $y \cdot \Psi(\varphi) = (-1)^{d(y)d(\Psi)} \Psi(y \cdot \varphi)$.

Next Ψ is injective. In fact, if $\Psi(\varphi)(x)(v) = 0$, then $0 = \Psi(\varphi)(x)(v) = \varphi(x^{\mathrm{T}} \otimes v)$ for all $x \in U(L)$ and $v \in V$. Thus $\varphi = 0$ because it vanishes on every generator of $\mathrm{Ind}_K(V)$.

Now we show that Ψ is surjective. Let $f \in \mathrm{Coind}_K(V^*)$. Define $\varphi(x \otimes v) := f(x^{\mathrm{T}})(v)$ for $x \in U(L)$ and $v \in V$. Then $\Psi(\varphi) = f$.

Since Ψ is a natural homomorphism, the proof is completed. □

Corollary 1. $\mathrm{Ind}_K(V_\sigma) \cong (\mathrm{Ind}_K(V_\sigma))^*$ *if and only if* $V \cong (V_\sigma)^*$.

Proof. If $\mathrm{Ind}_K(V_\sigma) \cong (\mathrm{Ind}_K(V_\sigma))^*$, by Lemma 1 and Proposition 2, then

$$\mathrm{Coind}_K(V) \cong \mathrm{Coind}_K((V_\sigma)^*).$$

Proposition 1 shows that $V \cong (V_\sigma)^*$. The sufficiency is obvious. □

4. Invariant forms on generalized reduced Verma modules

The results in this section generalize Chiu's results in [4] and determine generalized reduced Verma modules over modular Lie superalgebras which possess a nondegenerate super-symmetric or skew super-symmetric invariant bilinear form. Let L be a Lie superalgebra over $\mathbb{F}$ and V be an L-module. A bilinear form $\lambda\colon V \times V \to \mathbb{F}$ is called super-symmetric (skew super-symmetric) if $\lambda(v,w) = (-1)^{d(v)d(w)}\lambda(w,v)$ ($\lambda(v,w) = -(-1)^{d(v)d(w)}\lambda(w,v)$) for all $v,w \in V$. A super-symmetric (or skew super-symmetric) bilinear form $\lambda\colon V \times V \to \mathbb{F}$ is called invariant on L if $\lambda(x \cdot v, w) = -(-1)^{d(v)d(x)}\lambda(v, x \cdot w)$ for all $x \in L$ and $v, w \in V$. The subspace $\mathrm{rad}(\lambda) := \{v \in V\colon \lambda(v,w) = 0 \text{ for all } w \in V\}$ is called the radical of λ. The form λ is nondegenerate if $\mathrm{rad}(\lambda) = 0$.

Proposition 3. *There exists a nondegenerate super-symmetric (skew super-symmetric) invariant bilinear form λ on V if and only if there exists an isomorphism of L-modules $\phi\colon V \to V^*$ such that $\phi(v)(w) = (-1)^{d(v)d(w)}\phi(w)(v)$ ($\phi(v)(w) = -(-1)^{d(v)d(w)}\phi(w)(v)$) for all $v, w \in V$.*

Proof. Let λ be a nondegenerate super-symmetric (skew super-symmetric) invariant bilinear form on V. Define $\phi\colon V \to V^*$ such that $\phi(v)(w) := \lambda(v,w)$ for all $v, w \in V$. Then ϕ is a linear mapping such that $\ker\phi = \mathrm{rad}(\lambda) = 0$ and $\phi(v)(w) = (-1)^{d(v)d(w)}\phi(w)(v)$ ($\phi(v)(w) = -(-1)^{d(v)d(w)}\phi(w)(v)$). Hence ϕ is injective. Since $\dim V = \dim V^*$, ϕ is bijective. For $x \in L$ and $v, w \in V$ we have

$$\begin{aligned}\phi(x \cdot v)(w) &= \lambda(x \cdot v, w) = -(-1)^{d(x)d(v)}\lambda(v, x \cdot w) \\ &= -(-1)^{d(x)d(v)}\phi(v)(x \cdot w) = (-1)^{d(x)d(v)}(x \cdot \phi(v))(w).\end{aligned}$$

Thus, ϕ is the desired isomorphism of L-modules.

Conversely, let ϕ be an isomorphism of L-modules such that

$$\phi(v)(w) = (-1)^{d(v)d(w)}\phi(w)(v)(\phi(v)(w) = -(-1)^{d(v)d(w)}\phi(w)(v))$$

for all $v, w \in V$. Put $\lambda(v, w) := \phi(v)(w)$. Thus, λ be a super-symmetric (skew super-symmetric) bilinear form on V. Since

$$\begin{aligned}\lambda(x \cdot v, w) &= \phi(x \cdot v)(w) = (-1)^{d(x)d(\phi)}(x \cdot \phi(v))(w) \\ &= -(-1)^{d(x)d(v)}\phi(v)(x \cdot w) = -(-1)^{d(x)d(v)}\lambda(v, x \cdot w)\end{aligned}$$

for all $x \in L$ and $v, w \in V$, λ is invariant. As $\operatorname{rad}(\lambda) = \ker \phi = 0$, λ is nondegenerate. □

Corollary 2. *Let V be an irreducible L-module. If V is isomorphic to V^* as L-module, then there exists a nondegenerate invariant bilinear form λ on V which is either super-symmetric or skew super-symmetric.*

Theorem 1. *Let L be a $\mathbb{Z}$-graded Lie superalgebra over $\mathbb{F}$ and V be an L_0-module. Then the following statements are equivalent.*

(1) *There exists a nondegenerate super-symmetric or skew super-symmetric invariant bilinear form on $\operatorname{Ind}_K(V_\sigma)$.*

(2) *There exists an isomorphism of L_0-modules $\zeta\colon V \to (V_\sigma)^*$ such that $\zeta(v)(w) = (-1)^{d(v)d(w)}\zeta(w)(v)$ or $\zeta(v)(w) = -(-1)^{d(v)d(w)}\zeta(w)(v)$, $v, w \in V$.*

Proof. Assume that there exists a nondegenerate super-symmetric or skew super-symmetric invariant bilinear form on $\operatorname{Ind}_K(V_\sigma)$. By Proposition 3, there exists an isomorphism of L-modules $\phi\colon \operatorname{Ind}_K(V_\sigma) \to (\operatorname{Ind}_K(V_\sigma))^*$ such that

$$\phi(x_1 \otimes v_1)(x_2 \otimes v_2) = (-1)^{(d(x_1)+d(v_1))(d(x_2)+d(v_2))}\phi(x_2 \otimes v_2)(x_1 \otimes v_1) \tag{4.1}$$

or

$$\phi(x_1 \otimes v_1)(x_2 \otimes v_2) = -(-1)^{(d(x_1)+d(v_1))(d(x_2)+d(v_2))}\phi(x_2 \otimes v_2)(x_1 \otimes v_1), \tag{4.2}$$

where $x_1, x_2 \in U(L)$ and $v_1, v_2 \in V$. Corollary 1 shows that there exists an isomorphism of L_0-modules $\zeta\colon V \to (V_\sigma)^*$.

Let $x_1 = e^\alpha \xi^u \in U(L^-)$ and $x_2 = e^\beta \xi^t \in U(L^-)$, where $0 \leqslant \alpha \leqslant \pi$, $0 \leqslant \beta \leqslant \pi$ and $u, t \in \mathbb{B}$. By the proof of Lemma 1 and Proposition 2, we have

$$
\begin{aligned}
(4.3)\quad \phi(x_1 \otimes v_1)(x_2 \otimes v_2) &= (-1)^{d(x_1)d(x_2)+d(x_1)d(v_1)} \chi^{(\pi,E)}_{\zeta(v_1)}(x_2^{\mathrm{T}} x_1)(v_2) \\
&= (-1)^{d(x_1)d(x_2)+d(x_1)d(v_1)+(d(\zeta)+d(v_1)+d(\xi^E))(d(x_1)+d(x_2))} \\
&\quad \times \delta(\pi, \alpha+\beta)\delta(E, u+t)\zeta(v_1)(v_2) \\
&= (-1)^{d(x_1)d(x_2)+d(x_2)d(v_1)+(d(\zeta)+d(\xi^E))(d(x_1)+d(x_2))}\zeta(v_1)(v_2).
\end{aligned}
$$

Combining (4.1), (4.2) and (4.3), we have

$$\zeta(v_1)(v_2) = (-1)^{d(v_1)d(v_2)}\zeta(v_2)(v_1) \text{ or } \zeta(v_1)(v_2) = -(-1)^{d(v_1)d(v_2)}\zeta(v_2)(v_1)$$

for all $v_1, v_2 \in V$.

The converse also follows from Lemma 1, Corollary 1, Propositions 2 and 3. □

Remark 2. Following the notations in the proof of Theorem 1, we have the following results:

(1) If $d(x_1)$ and $d(x_2)$ need not all $\bar{1}$, then there exists a nondegenerate super-symmetric (skew super-symmetric) invariant bilinear form on $\mathrm{Ind}_K(V_\sigma)$ if and only if there exists an isomorphism of L_0-modules $\zeta\colon V \to (V_\sigma)^*$ such that

$$\zeta(v_1)(v_2) = (-1)^{d(v_1)d(v_2)}\zeta(v_2)(v_1), \quad (\zeta(v_1)(v_2) = -(-1)^{d(v_1)d(v_2)}\zeta(v_2)(v_1))$$

for all $v_1, v_2 \in V$.

(2) If $d(x_1) = d(x_2) = \bar{1}$, then there exists a nondegenerate super-symmetric (skew super-symmetric) invariant bilinear form on $\mathrm{Ind}_K(V_\sigma)$ if and only if there exists an isomorphism of L_0-modules $\zeta\colon V \to (V_\sigma)^*$ such that $\zeta(v_1)(v_2) = -(-1)^{d(v_1)d(v_2)}\zeta(v_2)(v_1)$ $(\zeta(v_1)(v_2) = (-1)^{d(v_1)d(v_2)}\zeta(v_2)(v_1))$ for all $v_1, v_2 \in V$.

5. Generalized reduced Verma modules and mixed products of modules

In this section, the relation between generalized reduced Verma modules and mixed products of modules over $\mathbb{Z}$-graded modular Lie superalgebras of Cartan type will be discussed.

Proposition 4. *Let L be a $\mathbb{Z}$-graded Lie superalgebra over $\mathbb{F}$ and $V = \bigoplus\limits_{i\geqslant 0} V_i$ be a positively and transitively graded L-module such that $z_i \cdot V = 0$, $1 \leqslant i \leqslant k$. Then the linear mapping $\psi\colon V \to \operatorname{Coind}_K(V_0)$ defined by $\psi(v)(x) = (-1)^{d(x)d(v)}\operatorname{pr}_0(x\cdot v)$ for all $x \in U(L)$ and $v \in V$ is an injective homomorphism of L-modules, where $\operatorname{pr}_0\colon V \to V_0$ denotes the canonical projection. In particular, $\psi(V_0) = \operatorname{Coind}_K(V_0)_0$ and $zd(\psi) = 0$.*

Proof. Note that pr_0 is a homomorphism of $\theta(L,K)$-modules. In fact, for any $h_j \in \theta(L,K)_j$ and $v_i \in V_i$ we have $\operatorname{pr}_0(h_j\cdot v_i) = (-1)^{d(h_j)d(\operatorname{pr}_0)}h_j\cdot\operatorname{pr}_0(v_i)$, where $i,j\in\mathbb{N}_0$. Since the mapping $U(L)\to V$ defined by $x\mapsto(-1)^{d(x)d(v)}x\cdot v$ also satisfies this property, ψ is well-defined. Moreover, for an arbitrary element $l\in L$ we obtain

$$\begin{aligned}\psi(l\cdot v)(x) &= (-1)^{d(x)(d(l)+d(v))}\operatorname{pr}_0(x\cdot(l\cdot v))\\ &= (-1)^{d(l)(d(x)+d(v))}\psi(v)(x\cdot l) = (-1)^{d(l)d(\psi)}(l\cdot\psi(v))(x).\end{aligned}$$

Therefore ψ is a homomorphism of L-modules. To prove that ψ is injective, we assume that $\ker\psi\neq 0$. Evidently, $zd(\psi)=0$ and thereby $\ker\psi$ is a $\mathbb{Z}$-homogeneous subspace of V. Then $\ker\psi\neq 0$ leads to the existence of a minimal $i\geqslant 0$ such that $\ker\psi\cap V_i\neq 0$. Let $v_i\in\ker\psi\cap V_i$ and $l\in L_{-j}$, $j>0$. This implies that $x\cdot v_i = \operatorname{pr}_0(x\cdot v_i) = (-1)^{d(x)d(v_i)}\psi(v_i)(x) = 0$ for every $x\in U(L)_{-i}$. If $q\neq j-i$, then

$$\psi(l\cdot v_i)(x) = (-1)^{d(x)(d(l)+d(v_i))}\operatorname{pr}_0(x\cdot(l\cdot v_i)) = 0,$$

where $x\in U(L)_q$. If $q=j-i$, then $xl\in U(L)_{-i}$ and $(xl)\cdot v_i = 0$. Consequently, $l\cdot v_i$ belongs to the trivial subspace $\ker\psi\cap V_{i-j}$. Since V is transitive, $v_i\in V_0$ and $i=0$. As a result, $x\cdot v_0 = 0$ for all $x\in U(L)_0$. It follows from $1\in U(L)_0$ that $v_0=0$. This conclusion refutes the assumption $\ker\psi\neq 0$ and thereby ψ is an injective homomorphism of L-modules.

Let $\mu\colon\operatorname{Coind}_K(V_0)_0\to V_0$ such that $\mu(f)=f(1)$. Let x be an element of $U(L)_j$. If $j\neq 0$, then $\operatorname{pr}_0(x\cdot f(1))=0$ and $f(x)=0$. In the case of $j=0$, the PBW theorem provides a presentation $x=\sum\limits_{j=1}^{n}\sum\limits_{i\geqslant 0}a_{ij}b_{ij}$, where $a_{ij}\in U(K)_i$ and $b_{ij}\in U(L^-)_{-i}$. Then

$$\begin{aligned}&f(x)-(-1)^{d(x)d(f)}\operatorname{pr}_0(x\cdot f(1))\\ &\quad=\sum_{j=1}^{n}\sum_{i\geqslant 0}((-1)^{d(a_{ij})d(f)}a_{ij}f(b_{ij})-(-1)^{d(x)d(f)}a_{ij}\operatorname{pr}_0(b_{ij}f(1)))\end{aligned}$$

$$= \sum_{j=1}^{n}((-1)^{d(a_{0j})d(f)}a_{0j}f(b_{0j}) - (-1)^{d(x)d(f)}a_{0j}\mathrm{pr}_0(b_{0j}f(1)))$$
$$= \sum_{j=1}^{n}(-1)^{d(x)d(f)}(a_{0j}b_{0j}f(1) - a_{0j}b_{0j}f(1)) = 0.$$

For an arbitrary element $x \in U(L)$, $f(x) = (-1)^{d(x)d(f)}\mathrm{pr}_0(x \cdot f(1))$. Consequently, $\psi \circ \mu = \mathrm{id}_{\mathrm{Coind}_K(V_0)_0}$ and $\psi(V_0) = \mathrm{Coind}_K(V_0)_0$. $\square$

For $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}_0^k$ we put $|\alpha| := \sum_{i=1}^{k} \alpha_i$. Let $\mathcal{O}(k, \underline{m})$ denote the divided power algebra over $\mathbb{F}$ with an $\mathbb{F}$-basis $\{x^{(\alpha)}\colon\ \alpha \in \mathbb{A}(k, \underline{m})\}$, where

$$\mathbb{A}(k, \underline{m}) := \{\alpha := (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}_0^k\colon\ 0 \leqslant \alpha_i \leqslant p^{m_i} - 1,\ i = 1, 2, \ldots, k\}.$$

Let $\Lambda(l)$ be the exterior superalgebra over $\mathbb{F}$ in l variables $\xi_1, \xi_2, \ldots, \xi_l$. Denote by $\mathcal{O}(k, l, \underline{m})$ the tensor product $\mathcal{O}(k, \underline{m}) \bigotimes_{\mathbb{F}} \Lambda(l)$.

Put $\mathrm{Y}_0 := \{1, 2, \ldots, k\}$ and $\mathrm{Y}_1 := \{1, 2, \ldots, l\}$. Suppose that $u - \langle j \rangle \in \mathbb{B}_{s-1}$ and $\{u - \langle j \rangle\} = \{u\} \setminus \{j\}$, when $u \in \mathbb{B}_s$, $j \in \{u\}$. Let $u(j) = |\{l \in \{u\}\colon\ l < j\}|$. If $j \in \mathrm{Y}_1 \setminus \{u\}$, then we put $u(j) = 0$ and $\xi^{u - \langle j \rangle} = 0$. Thus, $\{x^{(\alpha)}\xi^u\colon\ \alpha \in \mathbb{A}(k, \underline{m}),\ u \in \mathbb{B}\}$ constitutes an $\mathbb{F}$-basis of $\mathcal{O}(k, l, \underline{m})$ and $zd(x^{(\alpha)}\xi^u) = |\alpha| + |u| \geqslant 0$.

Let $D_1, \ldots, D_k, d_1, \ldots, d_l$ be the linear transformations of $\mathcal{O}(k, l, \underline{m})$ and $\varepsilon_i := (\delta(i, 1), \ldots, \delta(i, k))$ such that

$$D_i(x^{(\alpha)}\xi^u) = x^{(\alpha - \varepsilon_i)}\xi^u, \quad i \in \mathrm{Y}_0,$$
$$d_j(x^{(\alpha)}\xi^u) = (-1)^{u(j)}x^{(\alpha)}\xi^{u - \langle j \rangle}, \quad j \in \mathrm{Y}_1.$$

Modular Lie superalgebras of Cartan type $L(k, l, \underline{m})$, $L = W, S, H, K$, are subalgebras of the derivation superalgebras of $\mathcal{O}(k, l, \underline{m})$. For the precise definitions please refer to [24]. If $L = W, S, H$, then $\{D_1, \ldots, D_k\}$ is the canonical basis of $L(k, l, \underline{m})^- \cap L(k, l, \underline{m})_{\bar{0}}$ and $\{d_1, \ldots, d_l\}$ is the canonical basis of $L(k, l, \underline{m})^- \cap L(k, l, \underline{m})_{\bar{1}}$. The definition of the product in $L(k, l, \underline{m})$ (see [24]) entails the vanishing $\mathrm{ad}\, D_i^{p^{m_i}}$ on $L(k, l, \underline{m})$, so we define $z_i := D_i^{p^{m_i}}$, $1 \leqslant i \leqslant k$.

Theorem 2. *Let $L(k, l, \underline{m})$, $L = W, S, H$, denote a $\mathbb{Z}$-graded Lie superalgebra of Cartan type. If V is an $L(k, l, \underline{m})_0$-module, then* $\mathrm{Ind}_K(V_\sigma)$ *is isomorphic to the mixed product $\mathcal{O}(k, l, \underline{m}) \otimes V$.*

P r o o f. Since $(\mathcal{O}(k, l, \underline{m}) \otimes V)_k := \langle a \otimes v\colon\ a \in \mathcal{O}(k, l, \underline{m})_k,\ v \in V \rangle$, the mixed product is a positively graded module. According to the definition of the mixed product, see [22], we have

$$D_i(x^{(\alpha)}\xi^u \otimes v) = x^{(\alpha-\varepsilon_i)}\xi^u \otimes v, \quad i \in \mathrm{Y}_0,$$
$$d_j(x^{(\alpha)}\xi^u \otimes v) = (-1)^{u(j)}x^{(\alpha)}\xi^{u-\langle j\rangle} \otimes v, \quad j \in \mathrm{Y}_1,$$

where $\alpha \in \mathbb{A}(k,\underline{m})$, $u \in \mathbb{B}$ and $v \in V$. The first equality shows $z_i(\mathcal{O}(k,l,\underline{m})\otimes V) = 0$, $1 \leqslant i \leqslant k$. The above equalities also ensure the transitivity of $\mathcal{O}(k,l,\underline{m}) \otimes V$. Proposition 4 furnishes an embedding from $\mathcal{O}(k,l,\underline{m}) \otimes V$ into $\mathrm{Coind}_K(V)$. Since

$$\dim(\mathrm{Coind}_K(V)) = \dim(\mathcal{O}(k,l,\underline{m}) \otimes V) = 2^l p^{m_1+\ldots+m_k} \dim V,$$

the mapping is bijective. Then Lemma 1 gives an isomorphism between $\mathrm{Ind}_K(V_\sigma)$ and $\mathcal{O}(k,l,\underline{m}) \otimes V$. □

Remark 3. Let the notation be as in Theorems 1 and 2. Then the following statements are equivalent.

(1) There exists a nondegenerate super-symmetric or skew super-symmetric invariant bilinear form on the mixed product $\mathcal{O}(k,l,\underline{m}) \otimes V$.

(2) There exists an isomorphism of $L(k,l,\underline{m})_0$-modules $\zeta\colon V \to (V_\sigma)^*$ such that $\zeta(v)(w) = (-1)^{d(v)d(w)}\zeta(w)(v)$ or $\zeta(v)(w) = -(-1)^{d(v)d(w)}\zeta(w)(v)$ for all $v, w \in V$.

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