

ESSENTIAL NORM AND A NEW CHARACTERIZATION  
OF WEIGHTED COMPOSITION OPERATORS  
FROM WEIGHTED BERGMAN SPACES AND HARDY SPACES  
INTO THE BLOCH SPACE

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*Abstract.* In this paper, we give some estimates for the essential norm and a new characterization for the boundedness and compactness of weighted composition operators from weighted Bergman spaces and Hardy spaces to the Bloch space.

*Keywords:* Bloch space; weighted Bergman space; Hardy space; essential norm; weighted composition operator

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## 1. INTRODUCTION

Let  $\mathbb{D}$  be the open unit disk in the complex plane  $\mathbb{C}$  and  $H(\mathbb{D})$  be the space of analytic functions on  $\mathbb{D}$ . For  $0 < p < \infty$  and  $\alpha > -1$ , the weighted Bergman space, denoted by  $A_\alpha^p$ , is the set of all functions  $f \in H(\mathbb{D})$  satisfying

$$\|f\|_{A_\alpha^p}^p = (\alpha + 1) \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) < \infty,$$

where  $A$  is the normalized Lebesgue area measure in  $\mathbb{D}$  such that  $A(\mathbb{D}) = 1$ . The Hardy space  $H^p$  is the space consisting of all  $f \in H(\mathbb{D})$  such that

$$\|f\|_{H^p}^p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

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The Bloch space, denoted by  $\mathcal{B} = \mathcal{B}(\mathbb{D})$ , is the space of all  $f \in H(\mathbb{D})$  such that

$$\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

Under the norm  $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_{\mathcal{B}}$ , the Bloch space is a Banach space. See [26] for more information on the Bloch space.

Let  $v: \mathbb{D} \rightarrow \mathbb{R}_+$  be a continuous, strictly positive and bounded function. An  $f \in H(\mathbb{D})$  is said to belong to the weighted space, denoted by  $H_v^\infty$ , if

$$\|f\|_v = \sup_{z \in \mathbb{D}} v(z) |f(z)| < \infty.$$

$H_v^\infty$  is a Banach space with the norm  $\|\cdot\|_v$ . The weight  $v$  is called radial, if  $v(z) = v(|z|)$  for all  $z \in \mathbb{D}$ . For a weight  $v$ , the associated weight  $\tilde{v}$  is defined as

$$\tilde{v} = (\sup\{|f(z)| : f \in H_v^\infty, \|f\|_v \leq 1\})^{-1}, \quad z \in \mathbb{D}.$$

When  $v = v_\alpha(z) = (1 - |z|^2)^\alpha$ ,  $0 < \alpha < \infty$ , it is easy to check that  $\tilde{v}_\alpha(z) = v_\alpha(z)$ . In this case, we denote  $H_v^\infty$  by  $H_{v_\alpha}^\infty$  and  $\|f\|_{v_\alpha} = \sup_{z \in \mathbb{D}} |f(z)|(1 - |z|^2)^\alpha$ .

Let  $S(\mathbb{D})$  denote the set of all analytic self-maps of  $\mathbb{D}$ . Let  $u \in H(\mathbb{D})$  and  $\varphi \in S(\mathbb{D})$ . For  $f \in H(\mathbb{D})$ , the composition operator  $C_\varphi$  and the multiplication operator  $M_u$  are defined by

$$(C_\varphi f)(z) = f(\varphi(z)) \quad \text{and} \quad (M_u f)(z) = u(z)f(z),$$

respectively. The weighted composition operator  $uC_\varphi$  is defined by

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

It is clear that the weighted composition operator  $uC_\varphi$  is the generalization of  $C_\varphi$  and  $M_u$ . A basic and interesting problem concerning concrete operators (such as composition operator, multiplication operator, Volterra operator, Toeplitz operator, Hankel operator and other integral-type operators) is to relate operator-theoretic properties to the function-theoretic properties of their symbols, which attracted a lot of attention recently, we refer the reader to [3] and [26].

It is well known that  $C_\varphi$  is bounded on  $\mathcal{B}$  by the Schwarz-Pick lemma for any  $\varphi \in S(\mathbb{D})$ . The compactness of  $C_\varphi$  on  $\mathcal{B}$  was studied for example in [13], [19], [21]. In [21], Wulan, Zheng and Zhu proved that for any  $\varphi \in S(\mathbb{D})$ ,  $C_\varphi: \mathcal{B} \rightarrow \mathcal{B}$  is compact if and only if  $\lim_{j \rightarrow \infty} \|\varphi^j\|_{\mathcal{B}} = 0$ . This result has been generalized to Bloch-type spaces by Zhao in [25] and shows that  $C_\varphi: \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$  is compact if and only if

$\lim_{j \rightarrow \infty} j^{\alpha-1} \|\varphi^j\|_{\mathcal{B}^\beta} = 0$ . For some results on composition operator and related operators mapping into the Bloch space see, for example, [1], [2], [7]–[14], [16]–[18], [22]–[25], [27] and the related references therein.

In [7], Li and Stević obtained a characterization of the boundedness and compactness of the weighted composition operator  $uC_\varphi: A_\alpha^p \rightarrow \mathcal{B}$ . Among others, we proved the following result.

**Theorem A.** *Let  $1 \leq p < \infty$ ,  $\alpha > -1$ ,  $u \in H(\mathbb{D})$  and  $\varphi \in S(\mathbb{D})$  such that  $uC_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  is bounded. Then  $uC_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  is compact if and only if*

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|u'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha)/p}} = 0 \quad \text{and} \quad \lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|u(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}} = 0.$$

In [2], Colonna obtained a new characterization by using two families of functions, among others, she obtained the following result.

**Theorem B.** *Let  $1 \leq p < \infty$ ,  $\alpha > -1$ ,  $u \in H(\mathbb{D})$  and  $\varphi \in S(\mathbb{D})$  such that  $uC_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  is bounded. Then  $uC_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  is compact if and only if*

$$\lim_{|a| \rightarrow 1} \|uC_\varphi f_a\|_{\mathcal{B}} = 0 \quad \text{and} \quad \lim_{|a| \rightarrow 1} \|uC_\varphi g_a\|_{\mathcal{B}} = 0,$$

where

$$f_a(z) = \frac{(1 - |a|^2)^{1+(2+\alpha)(1-1/p)}}{(1 - \bar{a}z)^{3+\alpha}}, \quad g_a(z) = \frac{(1 - |a|^2)^{1+(2+\alpha)(1-1/p)+1/p}}{(1 - \bar{a}z)^{3+\alpha+1/p}}.$$

In [2], Colonna also obtained two characterizations for the compactness of weighted composition operator  $uC_\varphi: H^p \rightarrow \mathcal{B}$ .

**Theorem C.** *Let  $1 \leq p < \infty$ ,  $u \in H(\mathbb{D})$  and  $\varphi \in S(\mathbb{D})$  such that  $uC_\varphi: H^p \rightarrow \mathcal{B}$  is bounded. Then the following statements are equivalent:*

- (a)  $uC_\varphi: H^p \rightarrow \mathcal{B}$  is compact.
- (b)

$$\lim_{|a| \rightarrow 1} \|uC_\varphi p_a\|_{\mathcal{B}} = 0 \quad \text{and} \quad \lim_{|a| \rightarrow 1} \|uC_\varphi q_a\|_{\mathcal{B}} = 0,$$

where

$$p_a(z) = \frac{(1 - |a|^2)^{2-1/p}}{(1 - \bar{a}z)^2}, \quad q_a(z) = \frac{(1 - |a|^2)^2}{(1 - \bar{a}z)^{2+1/p}}.$$

- (c)

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|u'(z)|}{(1 - |\varphi(z)|^2)^{1/p}} = 0 \quad \text{and} \quad \lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|u(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(1+p)/p}} = 0.$$

The purpose of this paper is to give some estimates for the essential norm of the operator  $uC_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  (as well as  $uC_\varphi: H^p \rightarrow \mathcal{B}$ ), in particular, by using  $\|uC_\varphi f_a\|_{\mathcal{B}}$  and  $\|uC_\varphi g_a\|_{\mathcal{B}}$  (as well as  $\|uC_\varphi p_a\|_{\mathcal{B}}$  and  $\|uC_\varphi q_a\|_{\mathcal{B}}$ ). Moreover, we give a new characterization for the boundedness, compactness and essential norm of the operator  $uC_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  (as well as  $uC_\varphi: H^p \rightarrow \mathcal{B}$ ) by using  $\varphi^j$ .

Recall that the essential norm of a bounded linear operator  $T: X \rightarrow Y$  is its distance to the set of compact operators  $K$  mapping  $X$  into  $Y$ , that is,

$$\|T\|_{\text{es}, X \rightarrow Y} = \inf\{\|T - K\|_{X \rightarrow Y} : K \text{ is compact}\},$$

where  $X, Y$  are Banach spaces and  $\|\cdot\|_{X \rightarrow Y}$  is the operator norm.

Throughout this paper, we say that  $A \lesssim B$  if there exists a constant  $C$  such that  $A \leq CB$ . The symbol  $A \approx B$  means that  $A \lesssim B \lesssim A$ .

## 2. ESSENTIAL NORM OF $uC_\varphi$

In this section, we give two estimates for the essential norm of the operator  $uC_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  and the operator  $uC_\varphi: H^p \rightarrow \mathcal{B}$ , respectively.

**Theorem 2.1.** *Let  $1 \leq p < \infty$ ,  $\alpha > -1$ ,  $u \in H(\mathbb{D})$  and  $\varphi \in S(\mathbb{D})$  such that  $uC_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  is bounded. Then*

$$\|uC_\varphi\|_{\text{es}, A_\alpha^p \rightarrow \mathcal{B}} \approx \max\{A, B\} \approx \max\{P, Q\},$$

where

$$\begin{aligned} A &:= \limsup_{|a| \rightarrow 1} \|uC_\varphi(f_a)\|_{\mathcal{B}}, & B &:= \limsup_{|a| \rightarrow 1} \|uC_\varphi(g_a)\|_{\mathcal{B}}, \\ P &:= \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|u'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha)/p}}, & Q &:= \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|u(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}}. \end{aligned}$$

**Proof.** First we prove that

$$\max\{A, B\} \lesssim \|uC_\varphi\|_{\text{es}, A_\alpha^p \rightarrow \mathcal{B}}.$$

Let  $a \in \mathbb{D}$ . It is easy to check that  $f_a, g_a \in A_\alpha^p$  and  $\|f_a\|_{A_\alpha^p} \lesssim 1$ ,  $\|g_a\|_{A_\alpha^p} \lesssim 1$  for all  $a \in \mathbb{D}$  and  $f_a, g_a$  converge to zero uniformly on compact subsets of  $\mathbb{D}$  as  $|a| \rightarrow 1$ . Thus, for any compact operator  $K: A_\alpha^p \rightarrow \mathcal{B}$ , by Lemma 3.7 of [20] we have

$$\lim_{|a| \rightarrow 1} \|Kf_a\|_{\mathcal{B}} = 0, \quad \lim_{|a| \rightarrow 1} \|Kg_a\|_{\mathcal{B}} = 0.$$

Hence

$$\|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim \|(uC_\varphi - K)f_a\|_{\mathcal{B}} \geq \|uC_\varphi f_a\|_{\mathcal{B}} - \|Kf_a\|_{\mathcal{B}},$$

and

$$\|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim \|(uC_\varphi - K)g_a\|_{\mathcal{B}} \geq \|uC_\varphi g_a\|_{\mathcal{B}} - \|Kg_a\|_{\mathcal{B}}.$$

Taking  $\limsup_{|a| \rightarrow 1}$  to the last two inequalities on both sides, we obtain

$$\|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim A, \quad \|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim B.$$

Therefore, by the definition of the essential norm, we get

$$\|uC_\varphi\|_{\text{es}, A_\alpha^p \rightarrow \mathcal{B}} = \inf_K \|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim \max\{A, B\}.$$

Next, set

$$h_a(z) = f_a - g_a, \quad k_a(z) = f_a - \frac{3 + \alpha}{3 + \alpha + 1/p} g_a.$$

It is also easy to check that  $h_a, k_a \in A_\alpha^p$  and  $\|h_a\|_{A_\alpha^p} \lesssim 1, \|k_a\|_{A_\alpha^p} \lesssim 1$  for all  $a \in \mathbb{D}$  and  $h_a, k_a$  converge to zero uniformly on compact subsets of  $\mathbb{D}$  as  $|a| \rightarrow 1$ . Hence, for any  $b_j \in \mathbb{D}$  such that  $|\varphi(b_j)| \rightarrow 1$  and any compact operator  $K: A_\alpha^p \rightarrow \mathcal{B}$ , we have

$$\|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim \|(uC_\varphi - K)h_{\varphi(b_j)}\|_{\mathcal{B}} \geq \|uC_\varphi h_{\varphi(b_j)}\|_{\mathcal{B}} - \|Kh_{\varphi(b_j)}\|_{\mathcal{B}},$$

and

$$\|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim \|(uC_\varphi - K)k_{\varphi(b_j)}\|_{\mathcal{B}} \geq \|uC_\varphi k_{\varphi(b_j)}\|_{\mathcal{B}} - \|Kk_{\varphi(b_j)}\|_{\mathcal{B}}.$$

Taking  $\limsup_{|\varphi(b_j)| \rightarrow 1}$  to the last two inequalities on both sides we get

$$\|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim \limsup_{|\varphi(b_j)| \rightarrow 1} \|uC_\varphi h_{\varphi(b_j)}\|_{\mathcal{B}} \gtrsim \limsup_{|\varphi(b_j)| \rightarrow 1} \frac{(1 - |b_j|^2)|u'(b_j)|}{(1 - |\varphi(b_j)|^2)^{(2+\alpha)/p}} = P,$$

and

$$\|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim \limsup_{|\varphi(b_j)| \rightarrow 1} \|uC_\varphi k_{\varphi(b_j)}\|_{\mathcal{B}} \gtrsim \limsup_{|\varphi(b_j)| \rightarrow 1} \frac{(1 - |b_j|^2)|u(b_j)\varphi'(b_j)|}{(1 - |\varphi(b_j)|^2)^{(2+\alpha+p)/p}} = Q.$$

By the definition of the essential norm, we obtain

$$\|uC_\varphi\|_{\text{es}, A_\alpha^p \rightarrow \mathcal{B}} = \inf_K \|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim \max\{P, Q\}.$$

Finally, we prove that

$$\|uC_\varphi\|_{\text{es}, A_\alpha^p \rightarrow \mathcal{B}} \lesssim \max\{A, B\} \quad \text{and} \quad \|uC_\varphi\|_{\text{es}, A_\alpha^p \rightarrow \mathcal{B}} \lesssim \max\{P, Q\}.$$

For  $r \in [0, 1)$ , set  $K_r: H(\mathbb{D}) \rightarrow H(\mathbb{D})$  by

$$(K_r f)(z) = f_r(z) = f(rz), \quad f \in H(\mathbb{D}).$$

It is clear that  $K_r$  is compact on  $A_\alpha^p$  and  $\|K_r\|_{A_\alpha^p \rightarrow A_\alpha^p} \leq 1$ . Let  $\{r_j\} \subset (0, 1)$  be a sequence such that  $r_j \rightarrow 1$  as  $j \rightarrow \infty$ . Then for all positive integers  $j$ , the operator  $uC_\varphi K_{r_j}: A_\alpha^p \rightarrow \mathcal{B}$  is compact. By the definition of the essential norm we have

$$(2.1) \quad \|uC_\varphi\|_{\text{es}, A_\alpha^p \rightarrow \mathcal{B}} \leq \limsup_{j \rightarrow \infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{A_\alpha^p \rightarrow \mathcal{B}}.$$

Thus, we only need to show that

$$(2.2) \quad \limsup_{j \rightarrow \infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{A_\alpha^p \rightarrow \mathcal{B}} \lesssim \max\{A, B\},$$

and

$$(2.3) \quad \limsup_{j \rightarrow \infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{A_\alpha^p \rightarrow \mathcal{B}} \lesssim \max\{P, Q\}.$$

For any  $f \in A_\alpha^p$  such that  $\|f\|_{A_\alpha^p} \leq 1$ , we consider

$$\|(uC_\varphi - uC_\varphi K_{r_j})f\|_{\mathcal{B}} = |u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| + \|u(f - f_{r_j}) \circ \varphi\|_{\mathcal{B}}.$$

It is clear that  $\lim_{j \rightarrow \infty} |u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| = 0$ . Now we estimate

$$\begin{aligned} & \limsup_{j \rightarrow \infty} \|u(f - f_{r_j}) \circ \varphi\|_{\mathcal{B}} \\ & \leq \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| \leq r_N} (1 - |z|^2) |(f - f_{r_j})'(\varphi(z))| |\varphi'(z)| |u(z)| \\ & \quad + \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| > r_N} (1 - |z|^2) |(f - f_{r_j})'(\varphi(z))| |\varphi'(z)| |u(z)| \\ & \quad + \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| \leq r_N} (1 - |z|^2) |(f - f_{r_j})(\varphi(z))| |u'(z)| \\ & \quad + \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| > r_N} (1 - |z|^2) |(f - f_{r_j})(\varphi(z))| |u'(z)| \\ (2.4) \quad & = Q_1 + Q_2 + Q_3 + Q_4, \end{aligned}$$

where  $N \in \mathbb{N}$  is large enough such that  $r_j \geq 1/2$  for all  $j \geq N$ ,

$$Q_1 := \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| \leq r_N} (1 - |z|^2) |(f - f_{r_j})'(\varphi(z))| |\varphi'(z)| |u(z)|,$$

$$Q_2 := \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| > r_N} (1 - |z|^2) |(f - f_{r_j})'(\varphi(z))| |\varphi'(z)| |u(z)|,$$

$$Q_3 := \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| \leq r_N} (1 - |z|^2) |(f - f_{r_j})(\varphi(z))| |u'(z)|,$$

and

$$Q_4 := \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| > r_N} (1 - |z|^2) |(f - f_{r_j})(\varphi(z))| |u'(z)|.$$

Since  $uC_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  is bounded, applying the operator  $uC_\varphi$  to 1 and  $z$ , we easily get that  $u \in \mathcal{B}$  and

$$\tilde{K} := \sup_{z \in \mathbb{D}} (1 - |z|^2) |\varphi'(z)| |u(z)| < \infty.$$

Since  $r_j f'_{r_j} \rightarrow f'$  uniformly on compact subsets of  $\mathbb{D}$  as  $j \rightarrow \infty$ , we have

$$(2.5) \quad Q_1 \leq \tilde{K} \limsup_{j \rightarrow \infty} \sup_{|w| \leq r_N} |f'(w) - r_j f'(r_j w)| = 0.$$

Also, from the fact that  $u \in \mathcal{B}$  and  $f_{r_j} \rightarrow f$  uniformly on compact subsets of  $\mathbb{D}$  as  $j \rightarrow \infty$ , we have

$$(2.6) \quad Q_3 \leq \|u\|_{\mathcal{B}} \limsup_{j \rightarrow \infty} \sup_{|w| \leq r_N} |f(w) - f(r_j w)| = 0.$$

Next we consider  $Q_2$ . We have  $Q_2 \leq \limsup_{j \rightarrow \infty} (S_1^j + S_2^j)$ , where

$$S_1^j := \sup_{|\varphi(z)| > r_N} (1 - |z|^2) |f'(\varphi(z))| |\varphi'(z)| |u(z)|$$

and

$$S_2^j := \sup_{|\varphi(z)| > r_N} (1 - |z|^2) r_j |f'(r_j \varphi(z))| |\varphi'(z)| |u(z)|.$$

First we estimate  $S_1^j$ . Using the fact that  $\|f\|_{A_\alpha^p} \leq 1$ , we have

$$\begin{aligned} S_1^j &= \sup_{|\varphi(z)| > r_N} (1 - |z|^2) |f'(\varphi(z))| |\varphi'(z)| |u(z)| \\ &\lesssim \frac{1}{r_N} \|f\|_{A_\alpha^p} \sup_{|\varphi(z)| > r_N} (1 - |z|^2) |\varphi'(z)| |u(z)| \frac{|\varphi(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}} \\ &\lesssim \frac{1}{p} \sup_{|\varphi(z)| > r_N} \sup_{|a| > r_N} (1 - |z|^2) |\varphi'(z)| |u(z)| \frac{|\varphi(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}} \\ &\lesssim \sup_{|a| > r_N} \|uC_\varphi(f_a - g_a)\|_{\mathcal{B}} \\ &\lesssim \sup_{|a| > r_N} \|uC_\varphi f_a\|_{\mathcal{B}} + \sup_{|a| > r_N} \|uC_\varphi g_a\|_{\mathcal{B}}. \end{aligned}$$

Taking limit as  $N \rightarrow \infty$  we obtain

$$\begin{aligned} \limsup_{j \rightarrow \infty} S_1^j &\lesssim \limsup_{|a| \rightarrow 1} \frac{(1 - |z|^2)|\varphi'(z)||u(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}} = Q \\ &\lesssim \limsup_{|a| \rightarrow 1} \|uC_\varphi f_a\|_{\mathcal{B}} + \limsup_{|a| \rightarrow 1} \|uC_\varphi g_a\|_{\mathcal{B}}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \limsup_{j \rightarrow \infty} S_2^j &\lesssim \limsup_{|a| \rightarrow 1} \frac{(1 - |z|^2)|\varphi'(z)||u(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}} = Q \\ &\lesssim \limsup_{|a| \rightarrow 1} \|uC_\varphi f_a\|_{\mathcal{B}} + \limsup_{|a| \rightarrow 1} \|uC_\varphi g_a\|_{\mathcal{B}}, \end{aligned}$$

i.e., we get that

$$(2.7) \quad Q_2 \lesssim Q \lesssim A + B \lesssim \max\{A, B\}.$$

Next we consider  $Q_4$ . We have  $Q_4 \leq \limsup_{j \rightarrow \infty} (S_3^j + S_4^j)$ , where

$$S_3^j := \sup_{|\varphi(z)| > r_N} (1 - |z|^2)|f(\varphi(z))||u'(z)|, \quad S_4^j := \sup_{|\varphi(z)| > r_N} (1 - |z|^2)|f(r_j \varphi(z))||u'(z)|.$$

Similarly, we have

$$\begin{aligned} S_3^j &\lesssim \sup_{|\varphi(z)| > r_N} \sup_{|a| > r_N} (1 - |z|^2)|u'(z)| \frac{1}{(1 - |\varphi(z)|^2)^{(2+\alpha)/p}} \\ &\lesssim \sup_{|a| > r_N} \left\| uC_\varphi f_a - \frac{3 + \alpha}{3 + \alpha + 1/p} uC_\varphi g_a \right\|_{\mathcal{B}} \\ &\leq \sup_{|a| > r_N} \|uC_\varphi f_a\|_{\mathcal{B}} + \frac{3 + \alpha}{3 + \alpha + 1/p} \sup_{|a| > r_N} \|uC_\varphi g_a\|_{\mathcal{B}} \\ &\leq \sup_{|a| > r_N} \|uC_\varphi f_a\|_{\mathcal{B}} + \sup_{|a| > r_N} \|uC_\varphi g_a\|_{\mathcal{B}}. \end{aligned}$$

Taking limit as  $N \rightarrow \infty$  we obtain

$$\begin{aligned} \limsup_{j \rightarrow \infty} S_3^j &\lesssim \limsup_{|a| \rightarrow 1} \frac{(1 - |z|^2)|u'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha)/p}} = P \\ &\lesssim \limsup_{|a| \rightarrow 1} \|uC_\varphi f_a\|_{\mathcal{B}} + \limsup_{|a| \rightarrow 1} \|uC_\varphi g_a\|_{\mathcal{B}} = A + B. \end{aligned}$$

Similarly, we have  $\limsup_{j \rightarrow \infty} S_4^j \lesssim P \lesssim A + B$ , i.e., we get that

$$(2.8) \quad Q_4 \lesssim P \lesssim A + B.$$



Hence, by (2.4), (2.5), (2.6), (2.7) and (2.8) we get

$$\begin{aligned}
 (2.9) \quad \limsup_{j \rightarrow \infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{A_\alpha^p \rightarrow \mathcal{B}} &= \limsup_{j \rightarrow \infty} \sup_{\|f\|_{A_\alpha^p} \leq 1} \|(uC_\varphi - uC_\varphi K_{r_j})f\|_{\mathcal{B}} \\
 &= \limsup_{j \rightarrow \infty} \sup_{\|f\|_{A_\alpha^p} \leq 1} \|u(f - f_{r_j}) \circ \varphi\|_{\mathcal{B}} \\
 &\lesssim P + Q \lesssim A + B.
 \end{aligned}$$

Therefore, by (2.1) and (2.9), we obtain

$$\|uC_\varphi\|_{\text{es}, A_\alpha^p \rightarrow \mathcal{B}} \lesssim P + Q \lesssim \max\{P, Q\}$$

and

$$\|uC_\varphi\|_{\text{es}, A_\alpha^p \rightarrow \mathcal{B}} \lesssim A + B \lesssim \max\{A, B\}.$$

This completes the proof of the theorem.  $\square$

The Hardy space  $H^p$  can be viewed as the limiting space of  $A_\alpha^p$  as  $\alpha$  decreases to  $-1$ . In fact, carefully check the proof of Theorem 2.1 and replacing  $A_\alpha^p$  and  $\alpha$  by  $H^p$  and  $-1$ , respectively, we get the following result.

**Theorem 2.2.** *Let  $1 \leq p < \infty$ ,  $u \in H(\mathbb{D})$  and  $\varphi \in S(\mathbb{D})$  such that  $uC_\varphi: H^p \rightarrow \mathcal{B}$  is bounded. Then*

$$\begin{aligned}
 \|uC_\varphi\|_{\text{es}, H^p \rightarrow \mathcal{B}} &\approx \max\left\{ \limsup_{|a| \rightarrow 1} \|uC_\varphi(p_a)\|_{\mathcal{B}}, \limsup_{|a| \rightarrow 1} \|uC_\varphi(q_a)\|_{\mathcal{B}} \right\} \\
 &\approx \max\left\{ \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|u'(z)|}{(1 - |\varphi(z)|^2)^{1/p}}, \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|u(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(1+p)/p}} \right\}.
 \end{aligned}$$

From Theorems 2.1 and 2.2, we immediately get the following two corollaries.

**Corollary 2.1.** *Let  $1 \leq p < \infty$ ,  $\alpha > -1$  and  $\varphi \in S(\mathbb{D})$  such that  $C_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  is bounded. Then*

$$\begin{aligned}
 \|C_\varphi\|_{\text{es}, A_\alpha^p \rightarrow \mathcal{B}} &\approx \limsup_{|a| \rightarrow 1} \|C_\varphi(f_a)\|_{\mathcal{B}} \approx \limsup_{|a| \rightarrow 1} \|C_\varphi(g_a)\|_{\mathcal{B}} \\
 &\approx \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}}.
 \end{aligned}$$

**Corollary 2.2.** *Let  $1 \leq p < \infty$  and  $\varphi \in S(\mathbb{D})$  such that  $C_\varphi: H^p \rightarrow \mathcal{B}$  is bounded. Then*

$$\begin{aligned} \|C_\varphi\|_{\text{es}, H^p \rightarrow \mathcal{B}} &\approx \limsup_{|a| \rightarrow 1} \|C_\varphi(p_a)\|_{\mathcal{B}} \approx \limsup_{|a| \rightarrow 1} \|C_\varphi(q_a)\|_{\mathcal{B}} \\ &\approx \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(1+p)/p}}. \end{aligned}$$

### 3. NEW CHARACTERIZATION OF $uC_\varphi$

In this section, motivated by [4], we give a new characterization for the boundedness, compactness and essential norm for the weighted composition operators  $uC_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  and  $uC_\varphi: H^p \rightarrow \mathcal{B}$ . For this purpose, we state some lemmas which will be used.

**Lemma 3.1** ([15]). *Let  $v$  and  $w$  be radial, non-increasing weights tending to zero at the boundary of  $\mathbb{D}$ . Then the following statements hold.*

(a) *The weighted composition operator  $uC_\varphi: H_v^\infty \rightarrow H_w^\infty$  is bounded if and only if*

$$\sup_{z \in \mathbb{D}} \frac{w(z)}{\tilde{v}(\varphi(z))} |\varphi(z)| < \infty.$$

Moreover,

$$\|uC_\varphi\|_{H_v^\infty \rightarrow H_w^\infty} = \sup_{z \in \mathbb{D}} \frac{w(z)}{\tilde{v}(\varphi(z))} |\varphi(z)|.$$

(b) *Suppose  $uC_\varphi: H_v^\infty \rightarrow H_w^\infty$  is bounded. Then*

$$\|uC_\varphi\|_{\text{es}, H_v^\infty \rightarrow H_w^\infty} = \lim_{s \rightarrow 1^-} \sup_{|\varphi(z)| > s} \frac{w(z)}{\tilde{v}(\varphi(z))} |\varphi(z)|.$$

**Lemma 3.2** ([5]). *Let  $v$  and  $w$  be radial, non-increasing weights tending to zero at the boundary of  $\mathbb{D}$ . Then the following statements hold.*

(a)  *$uC_\varphi: H_v^\infty \rightarrow H_w^\infty$  is bounded if and only if*

$$\sup_{k \geq 0} \frac{\|u\varphi^k\|_w}{\|z^k\|_v} < \infty,$$

*with the norm comparable to the above supremum.*

(b) *Suppose  $uC_\varphi: H_v^\infty \rightarrow H_w^\infty$  is bounded. Then*

$$\|uC_\varphi\|_{\text{es}, H_v^\infty \rightarrow H_w^\infty} = \limsup_{k \rightarrow \infty} \frac{\|u\varphi^k\|_w}{\|z^k\|_v}.$$

**Lemma 3.3** ([6]). For  $\alpha > 0$ , we have  $\lim_{k \rightarrow \infty} k^\alpha \|z^{k-1}\|_{v_\alpha} = (2\alpha/e)^\alpha$ .

**Theorem 3.1.** Let  $1 \leq p < \infty$ ,  $\alpha > -1$ ,  $u \in H(\mathbb{D})$  and  $\varphi \in S(\mathbb{D})$ . Then the operator  $uC_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  is bounded if and only if

$$(3.1) \quad \sup_{j \geq 1} j^{(2+\alpha)/p} \|I_u(\varphi^j)\|_{\mathcal{B}} < \infty \quad \text{and} \quad \sup_{j \geq 1} j^{(2+\alpha)/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} < \infty,$$

where

$$I_u g(z) = \int_0^z g'(\xi) u(\xi) d\xi, \quad J_u g(z) = \int_0^z g(\xi) u'(\xi) d\xi, \quad z \in \mathbb{D}, \quad g \in H(\mathbb{D}).$$

*Proof.* By Theorem A,  $uC_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  is bounded if and only if

$$(3.2) \quad \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2) |u'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha)/p}} < \infty \quad \text{and} \quad \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2) |u(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}} < \infty,$$

which are equivalent to the conditions that the weighted composition operator  $u'C_\varphi: H_{v_{(2+\alpha)/p}}^\infty \rightarrow H_{v_1}^\infty$  is bounded and  $u\varphi'C_\varphi: H_{v_{(2+\alpha+p)/p}}^\infty \rightarrow H_{v_1}^\infty$  is bounded, respectively. By Lemma 3.2, we see that the two inequalities in (3.2) are equivalent to

$$\sup_{j \geq 1} \frac{\|u'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{(2+\alpha)/p}}} < \infty \quad \text{and} \quad \sup_{j \geq 1} \frac{\|u\varphi'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{(2+\alpha+p)/p}}} < \infty,$$

respectively. Since  $I_u f(0) = 0$ ,  $J_u f(0) = 0$ ,

$$(I_u(\varphi^j)(z))' = ju(z)\varphi'(z)\varphi^{j-1}(z), \quad (J_u(\varphi^{j-1})(z))' = u'(z)\varphi^{j-1}(z),$$

by Lemma 3.3, we see that  $uC_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  is bounded if and only if

$$(3.3) \quad \begin{aligned} \sup_{j \geq 1} j^{(2+\alpha)/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} &= \sup_{j \geq 1} j^{(2+\alpha)/p} \|u'\varphi^{j-1}\|_{v_1} \\ &\approx \sup_{j \geq 1} \frac{j^{(2+\alpha)/p} \|u'\varphi^{j-1}\|_{v_1}}{j^{(2+\alpha)/p} \|z^{j-1}\|_{v_{(2+\alpha)/p}}} < \infty \end{aligned}$$

and

$$(3.4) \quad \begin{aligned} \sup_{j \geq 1} j^{(2+\alpha)/p} \|I_u(\varphi^j)\|_{\mathcal{B}} &= \sup_{j \geq 1} j^{(2+\alpha+p)/p} \|u\varphi'\varphi^{j-1}\|_{v_1} \\ &\approx \sup_{j \geq 1} \frac{j^{(2+\alpha+p)/p} \|u\varphi'\varphi^{j-1}\|_{v_1}}{j^{(2+\alpha+p)/p} \|z^{j-1}\|_{v_{(2+\alpha+p)/p}}} < \infty. \end{aligned}$$

The proof is complete. □

**Theorem 3.2.** *Let  $1 \leq p < \infty$ ,  $\alpha > -1$ ,  $u \in H(\mathbb{D})$  and  $\varphi \in S(\mathbb{D})$  such that the operator  $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$  is bounded. Then*

$$\|uC_\varphi\|_{\text{es}, A_\alpha^p \rightarrow \mathcal{B}} \approx \max \left\{ \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|I_u(\varphi^j)\|_{\mathcal{B}}, \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} \right\}.$$

*Proof.* By Theorem A and Lemma 3.1,  $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$  is bounded if and only if the weighted composition operator  $u'C_\varphi : H_{v(2+\alpha)/p}^\infty \rightarrow H_{v_1}^\infty$  is bounded and  $u\varphi'C_\varphi : H_{v(2+\alpha+p)/p}^\infty \rightarrow H_{v_1}^\infty$  is bounded. By Lemmas 3.2 and 3.3, we get

$$\begin{aligned} (3.5) \quad \|u'C_\varphi\|_{\text{es}, H_{v(2+\alpha)/p}^\infty \rightarrow H_{v_1}^\infty} &= \limsup_{j \rightarrow \infty} \frac{\|u'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v(2+\alpha)/p}} \\ &= \limsup_{j \rightarrow \infty} \frac{j^{(2+\alpha)/p} \|u'\varphi^{j-1}\|_{v_1}}{j^{(2+\alpha)/p} \|z^{j-1}\|_{v(2+\alpha)/p}} \\ &\approx \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|u'\varphi^{j-1}\|_{v_1} \\ &= \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} \end{aligned}$$

and

$$\begin{aligned} (3.6) \quad \|u\varphi'C_\varphi\|_{\text{es}, H_{v(2+\alpha+p)/p}^\infty \rightarrow H_{v_1}^\infty} &= \limsup_{j \rightarrow \infty} \frac{\|u\varphi'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v(2+\alpha+p)/p}} \\ &\approx \limsup_{j \rightarrow \infty} j^{(2+\alpha+p)/p} \|u\varphi'\varphi^{j-1}\|_{v_1} \\ &= \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|I_u(\varphi^j)\|_{\mathcal{B}}. \end{aligned}$$

The upper estimate. From the fact  $(uC_\varphi f)'(z) = u'(z)f(\varphi(z)) + u(z) \times \varphi'(z)f'(\varphi(z))$ , it is easy to see that

$$(3.7) \quad \|uC_\varphi\|_{\text{es}, A_\alpha^p \rightarrow \mathcal{B}} \leq \|u'C_\varphi\|_{\text{es}, H_{v(2+\alpha)/p}^\infty \rightarrow H_{v_1}^\infty} + \|u\varphi'C_\varphi\|_{\text{es}, H_{v(2+\alpha+p)/p}^\infty \rightarrow H_{v_1}^\infty}.$$

Then, by (3.5), (3.6) and (3.7) we get

$$\begin{aligned} \|uC_\varphi\|_{\text{es}, A_\alpha^p \rightarrow \mathcal{B}} &\lesssim \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|I_u(\varphi^j)\|_{\mathcal{B}} + \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} \\ &\lesssim \max \left\{ \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|I_u(\varphi^j)\|_{\mathcal{B}}, \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} \right\}. \end{aligned}$$

The lower estimate. From Theorem 2.1 and Lemma 3.1, we have

$$\|uC_\varphi\|_{\text{es}, A_\alpha^p \rightarrow \mathcal{B}} \gtrsim P = \|u'C_\varphi\|_{\text{es}, H_{v(2+\alpha)/p}^\infty \rightarrow H_{v_1}^\infty} \approx \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}}$$

and

$$\|uC_\varphi\|_{\text{es}, A_\alpha^p \rightarrow \mathcal{B}} \gtrsim Q = \|u\varphi' C_\varphi\|_{\text{es}, H_{v(2+\alpha+p)/p}^\infty \rightarrow H_{v_1}^\infty} \approx \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|I_u(\varphi^j)\|_{\mathcal{B}}.$$

Therefore,

$$\|uC_\varphi\|_{\text{es}, A_\alpha^p \rightarrow \mathcal{B}} \gtrsim \max \left\{ \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|I_u(\varphi^j)\|_{\mathcal{B}}, \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} \right\}.$$

This completes the proof.  $\square$

From Theorem 3.2, we immediately get the following result.

**Theorem 3.3.** *Let  $1 \leq p < \infty$ ,  $\alpha > -1$ ,  $u \in H(\mathbb{D})$  and  $\varphi \in S(\mathbb{D})$  such that  $uC_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  is bounded. Then the operator  $uC_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  is compact if and only if*

$$\limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|I_u(\varphi^j)\|_{\mathcal{B}} = 0 \quad \text{and} \quad \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} = 0.$$

We end this section with a new characterization of boundedness, compactness and essential norm of the operator  $uC_\varphi: H^p \rightarrow \mathcal{B}$ . Carefully check the proofs of Theorems 3.1 and 3.2, by replacing  $A_\alpha^p$  and  $\alpha$  by  $H^p$  and  $-1$ , respectively, we get the following result.

**Theorem 3.4.** *Let  $1 \leq p < \infty$ ,  $u \in H(\mathbb{D})$  and  $\varphi \in S(\mathbb{D})$ . Then the following statements hold.*

(a) *The operator  $uC_\varphi: H^p \rightarrow \mathcal{B}$  is bounded if and only if*

$$\sup_{j \geq 1} j^{1/p} \|I_u(\varphi^j)\|_{\mathcal{B}} < \infty \quad \text{and} \quad \sup_{j \geq 1} j^{1/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} < \infty.$$

(b) *If the operator  $uC_\varphi: H^p \rightarrow \mathcal{B}$  is bounded, then  $uC_\varphi: H^p \rightarrow \mathcal{B}$  is compact if and only if*

$$\limsup_{j \rightarrow \infty} j^{1/p} \|I_u(\varphi^j)\|_{\mathcal{B}} = 0 \quad \text{and} \quad \limsup_{j \rightarrow \infty} j^{1/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} = 0.$$

Moreover,

$$\|uC_\varphi\|_{\text{es}, H^p \rightarrow \mathcal{B}} \approx \max \left\{ \limsup_{j \rightarrow \infty} j^{1/p} \|I_u(\varphi^j)\|_{\mathcal{B}}, \limsup_{j \rightarrow \infty} j^{1/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} \right\}.$$

From the above results, we immediately get the following new characterization of the operator  $C_\varphi: A_\alpha^p$  (or  $H^p$ )  $\rightarrow \mathcal{B}$ .

**Corollary 3.1.** Let  $1 \leq p < \infty$ ,  $\alpha > -1$  and  $\varphi \in S(\mathbb{D})$ . Then the following statements hold.

- (a) The operator  $C_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  is bounded if and only if  $\sup_{j \geq 1} j^{(\alpha+2)/p} \|\varphi^j\|_{\mathcal{B}} < \infty$ .
- (b) If the operator  $C_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  is bounded, then  $C_\varphi: A_\alpha^p \rightarrow \mathcal{B}$  is compact if and only if  $\limsup_{j \rightarrow \infty} j^{(\alpha+2)/p} \|\varphi^j\|_{\mathcal{B}} = 0$ . Moreover,

$$\|C_\varphi\|_{\text{es}, A_\alpha^p \rightarrow \mathcal{B}} \approx \limsup_{j \rightarrow \infty} j^{(\alpha+2)/p} \|\varphi^j\|_{\mathcal{B}}.$$

**Corollary 3.2.** Let  $1 \leq p < \infty$  and  $\varphi \in S(\mathbb{D})$ . Then the following statements hold.

- (a) The operator  $C_\varphi: H^p \rightarrow \mathcal{B}$  is bounded if and only if  $\sup_{j \geq 1} j^{1/p} \|\varphi^j\|_{\mathcal{B}} < \infty$ .
- (b) If the operator  $C_\varphi: H^p \rightarrow \mathcal{B}$  is bounded, then  $C_\varphi: H^p \rightarrow \mathcal{B}$  is compact if and only if  $\limsup_{j \rightarrow \infty} j^{1/p} \|\varphi^j\|_{\mathcal{B}} = 0$ . Moreover,

$$\|C_\varphi\|_{\text{es}, H^p \rightarrow \mathcal{B}} \approx \limsup_{j \rightarrow \infty} j^{1/p} \|\varphi^j\|_{\mathcal{B}}.$$

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