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ON DECOMPOSABILITY OF FINITE GROUPS

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Abstract. Let G be a finite group. A normal subgroup N of G is a union of several G-conjugacy classes, and it is called n-decomposable in G if it is a union of n distinct G-conjugacy classes. In this paper, we first classify finite non-perfect groups satisfying the condition that the numbers of conjugacy classes contained in its non-trivial normal subgroups are two consecutive positive integers, and we later prove that there is no non-perfect group such that the numbers of conjugacy classes contained in its non-trivial normal subgroups are 2, 3, 4 and 5.

Keywords: non-perfect group; G-conjugacy class; n-decomposable group

MSC 2010: 20E45, 20D10

1. Introduction

All groups considered in this paper are finite.

Let G be a group. There is close relation between the structure of G and some of its arithmetical conditions, for example, the famous Sylow theorem, Burnside's $p^a q^b$-theorem, and so on. In recent years, some scholars take great interest in investigating the structure of a group by using arithmetical properties of its conjugacy classes. As a normal subgroup N of G is a union of distinct G-conjugacy classes, the number of G-conjugacy classes contained in N has great influence on the structure of the normal subgroup N and the structure of G. Many group researchers have been paying great attention to this topic, and lots of results have been obtained, see [2], [3], [10] and [11] for instance.

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Let N be a normal subgroup of a group G. If N is a union of exactly t distinct G-conjugacy classes for some positive integer t, then we say that N is a t-decomposable normal subgroup of G or N is t-decomposable in G. For convenience, we write $\xi(N) = t$ and set $\mathcal{K}(G) = \{\xi(N)\colon N \trianglelefteq G,\ N \neq G\}$. As the structure of normal subgroups has great influence on the structure of a group G, it is interesting to determine the structure of G by observing the numbers of conjugacy classes contained in its normal subgroups. In 2004, Ashrafi in [3] raised the following question:

Question ([3], Question 2.7)**.** Suppose that X is a finite set of positive integers containing 1. Is there a finite group G such that $\mathcal{K}(G) = X$?

Up to now, the cases when $\mathcal{K}(G) = \{1, n\}$, where n is a positive integer larger than 1, and $\mathcal{K}(G) = \{1, 2, 3\}, \{1, 3, 4\}, \{1, 2, 4\}$, and $\{1, 2, 3, 4\}$ have been investigated in [2], [3], [1], [6], and [5], respectively.

In this paper, we first determine non-perfect groups G with $\mathcal{K}(G) = \{1, m, m+1\}$ for a positive integer m. Notice that the cases $m = 2$ and 3 have been covered in [2] and [3], respectively. So we only concentrate on the case when $m \geqslant 4$ and we have the following theorem.

Theorem A. *Suppose that G is a non-perfect group. Then $\mathcal{K}(G) = \{1, m, m+1\}$ if and only if one of the following holds*

(1) *G is a Frobenius group, G' is the kernel and G' is minimal normal in G, and a complement of G' is cyclic of order* 4.

(2) *$G/N \cong S_3$, the symmetric group on three symbols, where N is the unique minimal normal subgroup of G, and N is a q-group for some prime $q \neq 3$.*

(3) *$|G/G'| = 4$ and G' is the unique minimal normal subgroup of G and G' is non-soluble. Furthermore, for every element x of G of order* 2 *such that $x \notin G'$, $|C_G(x)| = 4$.*

(4) *$G = G' \times Z(G)$, $|Z(G)| = m$ is a prime, G' is a simple group and $\xi(G') = m+1$.*

(5) *G has two non-trivial normal subgroups G'' and G', G'' is non-soluble and $G/G'' \cong \mathbb{Z}_p \ltimes E(2^n)$, $p = 2^n - 1$ is a prime, and for every element $x \in G' - G''$, $|C_G(x)| = 2^n$.*

On the other hand, in a recent paper, we determined the structure of a finite non-perfect group where the numbers of conjugacy classes contained in its non-trivial normal subgroups are three consecutive positive integers. It is natural to ask what can be said about the structure of a finite non-perfect group where the numbers of conjugacy classes contained in its non-trivial normal subgroups are four consecutive positive integers? In fact, we prove the following theorem.

Theorem B. *There exists no finite non-perfect group G such that $\mathcal{K}(G) = \{1, 2, 3, 4, 5\}$.*

Let G be a group. Throughout this paper, as usual, G' denotes the derived subgroup of G, $Z(G)$ denotes the center of G and G is said to be perfect if $G' = G$. If x is an element in G, then $x^G = \{x^g : g \in G\}$ is the G-conjugacy class containing x. For a positive integer n, $\mathbb{Z}_n$ denotes the cyclic group of order n, $d(n)$ denotes the set of all positive divisors of n and $E(p^n)$ denotes an elementary abelian group of order p^n for a prime p.

2. Preliminaries

In this section, some fundamental facts are established.

Lemma 2.1 ([7], Lemma 12.3.)**.** *Let G be a soluble group such that G' is the unique minimal normal subgroup of G. Then one of the following holds:*

(i) *G is a p-group, $|G'| = p$ and $Z(G)$ is cyclic.*
(ii) *G is a Frobenius group, G' is the kernel and the complement of G is cyclic.*

Lemma 2.2 ([6], Example 2.1.)**.** *Let G be an abelian group of order n. Then $\mathcal{K}(G) = d(n) - \{n\}$.*

Lemma 2.3. *Let G be a soluble group. Then $G \neq G'T$ for any non-trivial normal subgroup T of G.*

Proof. Suppose to the contrary that T is a non-trivial normal subgroup of G such that $G = G'T$. Then G/T is soluble. However, $(G/T)' = G'T/T = G/T$, which contradicts the fact that G is soluble. $\square$

3. The proof of Theorem A

In this section, we deal with non-perfect groups with $\mathcal{K}(G) = \{1, m, m+1\}$ for some positive integer m. As the cases when $m = 2$ and 3 are covered in [3] and [1], respectively, we concentrate on $m \geqslant 4$, and we always assume that $m \geqslant 4$ in the rest of this section.

To begin with, we list some lemmas which are useful in the sequel.

Lemma 3.1. *Let G be a group with $\mathcal{K}(G) = \{1, m, m+1\}$. Then G is not abelian.*

Proof. Suppose that G is abelian and that $|G| = n$ for some positive integer n. Then by Lemma 2.2, $\mathcal{K}(G) = \{1, m, m+1\} = d(n) - \{n\}$. As 2 divides $m(m+1)$, we have that $m = 2$, which contradicts our assumption. $\square$

Lemma 3.2. *Let G be a group with $\mathcal{K}(G) = \{1, m, m+1\}$. Then G is not of prime power order.*

Proof. Suppose that G is a p-group for some prime p. It is easy to prove that $|G| = p^3$. As G is not abelian by the above lemma, we have that $Z(G)$ is of order p. Therefore, $m = p$. Let M be a normal subgroup of G of order p^2. Then $Z(G) \leqslant M$. Furthermore, $M = Z(G) \cup x^G$ for some element $x \in G$ by the hypothesis. So $|x^G| = p^2 - p = p(p-1)$ divides p^3, which gives that $p = 2$. Whence $m = 2$, and this is a contradiction. $\square$

In the following, we will prove Theorem A and we will distinguish two different cases in which G is soluble or not.

Theorem 3.3. *Suppose that G is a soluble group. Then $\mathcal{K}(G) = \{1, m, m+1\}$ if and only if one of the following holds:*

(1) *G is a Frobenius group, G' is the kernel and G' is minimal normal in G, and a complement of G' is cyclic of order 4.*
(2) *$G/N \cong S_3$, the symmetric group on three symbols, where N is the unique minimal normal subgroup of G, and N is a q-group for some prime $q \neq 3$.*

Proof. We first assume that $\xi(G') = m$. Then G' is the unique minimal normal subgroup of G. In fact, if there exists another minimal normal subgroup N of G, as $\xi(G' \times N) > m+1$, we have that $G = G' \times N$, whence G is abelian, which contradicts Lemma 2.3. Now by Lemma 2.1, G is a Frobenius group, G' is the kernel and the complement of G is cyclic. We may suppose that $G = G'\langle x\rangle$ for some element $x \in G$. Let M be a normal subgroup of G with $\xi(M) = m+1$. Then $G' \leqslant M$ and M/G' is a union of exactly two different G/G'-conjugacy classes. Then G/G' is of order 4 by Theorem 3 of [2], and G has the structure (1) in the theorem. Conversely, if G has the structure described above, it is easy to see that G satisfies the hypothesis of this theorem.

Now assume that $\xi(G') = m+1$. Then G' is the unique maximal normal subgroup of G by Lemma 2.3. Furthermore, if M and N are two distinct minimal normal subgroups of G, then both M and N are contained in G' and $\xi(M) = \xi(N) = m$. It follows that $\xi(M \times N) > m+1$, hence $M \times N = G$, which contradicts the fact that $MN \leqslant G'$. Therefore, G has a unique minimal normal subgroup, say N, which is a q-group for some prime q. In the following, we denote by $\overline{G} = G/N$. Then

$\overline{G}$ has a unique non-trivial normal subgroup G'/N, and G'/N is a union of exactly two $\overline{G}$-conjugacy classes. Now by Theorem 3 of [2], $\overline{G} \cong S_3$. Let $G' = N \cup x^G$ for some element $x \in G$. Then $|x^G| = |G'| - |N| = 2|N|$, so $|C_G(x)| = 3$, which shows that $q \neq 3$. Conversely, if G has the structure described above, we can see that N and G' are non-trivial normal subgroups of G, and G satisfies the hypothesis of this theorem. □

Theorem 3.4. *Suppose that G is a non-soluble non-perfect group. Then $\mathcal{K}(G) = \{1, m, m+1\}$ if and only if one of the following holds:*

(1) $|G/G'| = 4$ *and G' is the unique minimal normal subgroup of G and G' is non-soluble. Furthermore, for every element x of G of order 2 such that $x \notin G'$,* $|C_G(x)| = 4$.

(2) $G = G' \times Z(G)$, $|Z(G)| = m$ *is a prime, G' is a simple group and* $\xi(G') = m+1$.

(3) *G has two non-trivial normal subgroups G'' and G', G'' is non-soluble and $G/G'' \cong \mathbb{Z}_p \ltimes E(2^n)$, $p = 2^n - 1$ is a prime, and for every element $x \in G' - G''$,* $|C_G(x)| = 2^n$.

Proof. First suppose that $\xi(G') = m$. Then G' is a minimal normal subgroup of G. If G has another minimal normal subgroup $N \neq G'$, then $G = N \times G'$ as $\xi(G'N) > m+1$. So $N \cong G/G'$ is abelian. Suppose that $|N| = p^s$ for some prime p and some positive integer t. Then $s = 2$ as $\mathcal{K}(G) = \{1, m, m+1\}$. Let x be an element of N of order p. Then $M = G'\langle x \rangle$ is a normal subgroup of G, and $|M| = p|G'|$. Then $|x^G| = (p-1)|G'|$, so $p-1$ divides p^2. Therefore, $p = 2$. So $|C_G(x)| = 4$. As $N \leqslant C_G(x)$, we conclude that $C_{G'}(x) = 1$. Then by Theorem 10.1.4 of [4], G' is abelian, which shows that G is soluble, a contradiction. Therefore, G' is the unique minimal normal subgroup of G. Now for every normal subgroup K of G such that $\xi(K) = m+1$, we have that $G' \leqslant K$ and K/G' is a union of two G/G'-conjugacy classes. Then by Theorem 3 of [2], G/G' is of order 4. Let y be an arbitrary element of G of order 2 which is not in G'. Then $K = G' \cup y^G$ is a normal subgroup of G. It follows that $|y^G| = |G'|$ and $|C_G(y)| = 4$, we can see that G has the structure described in (1) of this theorem.

Now suppose that $\xi(G') = m+1$. If there exists some normal subgroup N of G such that $\xi(N) = m$ and $N \nleqslant G'$, then $G = G' \times N$. Hence, $N = Z(G)$. Furthermore, $|N| = |G/G'| = p$ for some prime p as G' is a maximal normal subgroup of G. So, $m = p$. If there exists a normal subgroup T of G and $T < G'$, then $\xi(T \times Z(G)) > m+1$, and thus $G = T \times Z(G) < G' \times Z(G) = G$, which is a contradiction. Therefore, G' is minimal normal in G. It is easy to see that every minimal normal subgroup of G is equal to G' or $Z(G)$. Now as $1 < G' < G$ is a chief series of G and $1 < Z(G) < G$ is a normal series of G, by Jordan-Hölder theorem, $Z(G)$ is a maximal subgroup of G. Therefore, G' and $Z(G)$ are all non-trivial normal subgroups of G. Since $G' \cong G/Z(G)$, and $Z(G)$ is maximal normal in G, G' is a simple group, and this is case (2) in this theorem.

In the following, we assume that all minimal normal subgroups of G are contained in G'. Now let T be a normal subgroup of G and $\xi(T) = m$. Then it is easy to see that T is the unique minimal normal subgroup of G and by Theorem 3 of [2], $G/T \cong S_3$ or $G/T \cong \mathbb{Z}_p \ltimes E(2^n)$, where n is a positive integer and $p = 2^n - 1$ is a prime. If $G/T \cong S_3$, then $|G'/T| = 3$. Suppose that $G' = T \cup z^G$ for some element $z \in G$. Then $|C_G(z)| = 3$, so $G' = T\langle z \rangle$ is a Frobenius group, whence N is nilpotent and G is soluble, which is a contradiction. Therefore, the only possibility is $G/T \cong \mathbb{Z}_p \ltimes E(2^n)$. As G'/T is abelian and G' is non-soluble, $T = G''$. By the hypothesis of this theorem, we see that G'/G'' is the unique non-trivial normal subgroup of G/G''. For every element $w \in G' - G''$, we see that $G' = G'' \cup w^G$. It is easy to show that $|C_G(w)| = 2^n$, and this is case (3) of this theorem.

Conversely, if G has the structure described in the above three paragraphs, it is easy to see that G satisfies the hypothesis of this theorem. □

4. The proof of Theorem B

In this section, we attempt to obtain the structure of a non-perfect group G with $\mathcal{K}(G) = \{1, 2, 3, 4, 5\}$.

First, some basic lemmas are needed.

Lemma 4.1. *If G is a group with $\mathcal{K}(G) = \{1, 2, 3, 4, 5\}$, then G is soluble.*

Proof. Suppose that G is non-soluble. Let N_1, N_2 be normal subgroups of G such that $\xi(N_1) = 2$ and $\xi(N_2) = 3$. We show that $N_1 < N_2$. For otherwise, as $N_1 \cap N_2 = 1$, $\xi(N_1 \times N_2) > 5$. It follows that $G = N_1 \times N_2$. However, we see that both N_1 and N_2 are soluble by [10] and [11], and thus G is soluble, which is a contradiction. If $\xi(G') < 4$, then again by [10] and [11], G' is soluble, so G is soluble, which is a contradiction. Therefore, $\xi(G') \geqslant 4$. If there exists a non-trivial normal subgroup N of G with $\xi(N) < 4$ and $N \leqslant G'$, then G/N is non-perfect, and $\mathcal{K}(G/N) \subseteq \{1, 2, 3, 4\}$. By [5], we see that G/N is soluble, and G is soluble too, which contradicts our assumption. So no 2- or 3-decomposable normal subgroup of G is contained in G'. Now let N_1, N_2 be normal subgroups of G and $\xi(N_1) = 2$, $\xi(N_2) = 3$. As $G' \cap N_1 = 1$, $G' \cap N_2 = 1$, we have that $G = N_1 \times G' = N_2 \times G'$. So $N_1 \cong G/G' \cong N_2$, which is a contradiction as we have proved that $N_1 < N_2$. □

Lemma 4.2. *If G is a group with $\mathcal{K}(G) = \{1, 2, 3, 4, 5\}$, then G is not abelian.*

P r o o f. Suppose that G is an abelian group of order n for some positive integer n. Then by Lemma 2.2, $\mathcal{K}(G) = d(n) - \{n\}$. So $d(n) - \{n\} = \{1, 2, 3, 4, 5\}$. As both 3 and 4 divide n, we have that 12 divides n. However, $12 \notin \{1, 2, 3, 4, 5\}$, which is a contradiction. $\square$

Lemma 4.3. *If G is a group with $\mathcal{K}(G) = \{1, 2, 3, 4, 5\}$, then G is not of prime power order.*

P r o o f. Suppose that G is a p-group for some prime p. Let N be a minimal normal subgroup of G. Then $\mathcal{K}(G/N) = \{1, 2, 3, 4\}$. However, by Lemma 2.4 of [5], there is no $\{1, 2, 3, 4\}$-decomposable group of prime power order, which is a contradiction. $\square$

We now come to the proof of Theorem B and we divide it into the following four theorems, in which $\xi(G') = 2, 3, 4$, and 5, respectively.

Theorem 4.4. *There is no non-perfect group G such that $\mathcal{K}(G) = \{1, 2, 3, 4, 5\}$ and $\xi(G') = 2$.*

P r o o f. Let N be a normal subgroup of G with $\xi(N) \geqslant 3$. Then $G' \leqslant N$. In fact, if $G' \nleqslant N$, then $\xi(G'N) > 5$, and thus $G = G'N$, which contradicts Lemma 2.3.

Now let N be a normal subgroup of G with $\xi(N) = 3$. Then N/G' is a union of two G/G'-conjugacy classes. We denote by $\overline{G} = G/G'$. Then $\{1, 2\} \subseteq \mathcal{K}(G/G') \subseteq \{1, 2, 3, 4\}$. As G/G' is abelian and G/G' has at least three non-trivial normal subgroups, by Theorem 3 of [2], Theorem of [3], Main theorem of [5] and Theorems 3.2 and 3.3 of [6], the only possibility for the structure of G/G' is that $G/G' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

By Theorem 1 of [10], we may assume that $|G'| = p^n$ for some prime p and some positive integer n. Then $|G| = 4p^n$. So $p \neq 2$ by Lemma 4.3. Let $x \in G$ such that $G' = 1 \cup x^G$. Then $|x^G| = p^n - 1$. As $|x^G|$ divides $|G|$, we have that $p^n - 1$ divides $4p^n$. Since $(p^n - 1, p^n) = 1$, $p^n - 1$ divides 4. It follows that $p^n = 3$ or 5. Let $N = G' \cup y^G \cup z^G$ be a 4-decomposable normal subgroup of G. Then $2p^n = |N| = |G'| + |y^G| + |z^G|$. It follows that $p^n = |y^G| + |z^G|$. In both cases, we have that $|y^G| = 1$ or $|z^G| = 1$. So $Z(G) \neq 1$. Then $G' \nleqslant Z(G)$. By the first paragraph of the proof, we conclude that $|Z(G)| = 2$. If $p^n = 5$, then $|x^G| = 4$ and hence $|C_G(x)| = 5$, which contradicts the fact that $Z(G) \leqslant C_G(x)$. Therefore, $p^n = 3$. Now let $K = G' \cup u^G \cup v^G \cup w^G$ be a 5-decomposable normal subgroup of G, where u, v and w are elements of G. It follows that $2p^n = |K| = |G'| + |u^G| + |v^G| + |w^G|$. Therefore, $p^n = 3 = |u^G| + |v^G| + |w^G|$, and thus $|u^G| = |v^G| = |w^G| = 1$, whence $|Z(G)| \geqslant 4$, which is a contradiction. $\square$

Theorem 4.5. *There is no non-perfect group G such that $\mathcal{K}(G) = \{1, 2, 3, 4, 5\}$ and $\xi(G') = 3$.*

P r o o f. Suppose that G is a non-perfect group with $\mathcal{K}(G) = \{1, 2, 3, 4, 5\}$ and $\xi(G') = 3$. We will show that G' is contained in every normal subgroup K of G with $\xi(K) \geqslant 4$, G' contains every normal subgroup N of G with $\xi(N) = 2$ and G' is the unique 3-decomposable normal subgroup of G. In fact, if $G' \nleqslant K$, then $G'K \trianglelefteq G$ and $\xi(G'K) > 5$. It follows that $G = G'N$, which contradicts Lemma 2.3. The latter two conclusions can be obtained similarly.

Now let N be a normal subgroup of G with $\xi(N) = 2$ and write $\overline{G} = G/N$. By Theorem 1 of [10], we may assume that $|N| = p^n$ for some prime p and some positive integer n. Then G'/N is a union of two $\overline{G}$-conjugacy classes. And if K is a normal subgroup of G with $\xi(K) = 4$, then K/N is a union of three $\overline{G}$-conjugacy classes. Therefore, $\{1, 2, 3\} \subseteq \mathcal{K}(\overline{G}) \subseteq \{1, 2, 3, 4\}$. Since $\overline{G}$ has at least three non-trivial normal subgroups, by Theorem of [3] and Main theorem of [5], $\overline{G} \cong Q_8, D_8, D_{12}$ or H, where $H = \langle a, b\colon\ a^7 = b^6 = 1,\ b^{-1}ab = a^5\rangle$.

First suppose that $\overline{G} \cong Q_8$ or D_8. Then $|\overline{G}| = 8$. It follows that $|G| = 8p^n$. So $p \neq 2$ by Lemma 4.3. In both cases, we have $|G'| = 2p^n$. Let $N = 1 \cup x^G$ for some element $x \in G$. Then $|x^G| = p^n - 1$. As $|x^G|$ divides $|G|$ and $(p^n - 1, p^n) = 1$, we have that $p^n - 1$ divides 8. It follows that $p^n = 3, 5$ or 9. If $p^n = 3$, then $|C_G(x)| = 12$ and $C_G(x) \trianglelefteq G$. As $N = \langle x\rangle \leqslant Z(C_G(x))$, we have that $C_G(x) = N \times T$, with $|T| = 4$. It follows that $T \trianglelefteq G$. However, we have shown that every normal subgroup of G contains or is contained in G', and that $|G'| = 2 \cdot 3 = 6$, which is a contradiction. If $p^n = 5$, then $|x^G| = 4$ and thus $|C_G(x)| = 10$. As $N = \langle x\rangle \leqslant Z(C_G(x))$, we have that $C_G(x) = C_G(N) = N \times T$, with $|T| = 2$. So $T \trianglelefteq G$. It follows that $T \leqslant Z(G)$. On the other hand, $G/C_G(N)$ is of order 4, which is abelian, so $G' \leqslant C_G(N)$. Since $|G'| = 10 = |C_G(N)|$, $G' = C_G(N)$. Suppose that $G' = N \cup y^G$ for some element $y \in G$. Then $|y^G| = 5$, and thus $|C_G(y)| = 8$, which contradicts the fact that $N \leqslant C_G(y)$. If $p^n = 9$, then we can take U to be a normal subgroup of G with $\xi(U) = 5$. We may assume that $U = N \cup u^G \cup v^G \cup w^G$ for some elements $u, v, w \in G$. Then $|U/N| = 4$ and $|u^G| + |v^G| + |w^G| = 27$. As $|u^G|, |v^G|, |w^G|$ divides $|G| = 72$, we have that $|u^G| = |v^G| = |w^G| = 9$, and thus $|C_G(u)| = |C_G(v)| = |C_G(w)| = 8$. Therefore, u, v, w are contained in the center of some Sylow 3-subgroup of G, and thus all of them are in the same conjugacy class of G, which is a contradiction.

Now suppose that $\overline{G} \cong D_{12}$. Then $|G'/N| = 3$ and $|G'| = 3p^n$. Let T/N be a normal subgroup of $\overline{G}$ of order 2. Then $|T| = 2p^n$. However, we have shown that every non-trivial normal subgroup of G contains or is contained in G'. So $T \leqslant G'$ or $G' \leqslant T$, which is a contradiction by order consideration.

Finally suppose that $\overline{G} \cong H$, where $H = \langle a, b\colon\ a^7 = b^6 = 1,\ b^{-1}ab = a^5\rangle$. Then $|G| = 2 \cdot 3 \cdot 7 \cdot p^n$ and $|G'| = 7 \cdot p^n$. As $\xi(G') = 3$, by Theorem 1 of [11], $|G'| = p^{n+l}$ for some positive integer l or $|G'| = p^n q$ for some prime $q \neq p$. We now distinguish the two cases. If $|G'| = p^{n+l}$, then $p = 7$ and $l = 1$ as $|G'| = 7 \cdot p^n$.

Assume that $N = 1 \cup x^G$ for some element $x \in G$. Then $|x^G| = 7^n - 1$ divides $2 \cdot 3 \cdot 7^{n+1}$. As $(7^n - 1, 7^{n+1}) = 1$, $7^n - 1$ divides $2 \cdot 3$. It follows that $7^n = 7$, whence $|G| = 2 \cdot 3 \cdot 7^2$. Suppose that $G' = N \cup y^G$ for some element $y \in G$. Then $|y^G| = |G'| - |N| = 7^2 - 7 = 7 \cdot 6$. It follows that $|C_G(y)| = 7$, which is a contradiction as $|G'|$ is abelian of order 7^2. If $|G'| = p^n q$, then $q = 7 \neq p$. Suppose that $N = 1 \cup u^G$ for some element $u \in G$. Then $|u^G| = p^n - 1$ divides $2 \cdot 3 \cdot 7 \cdot p^n$. As $(p^n - 1, p^n) = 1$, $p^n - 1$ divides $2 \cdot 3 \cdot 7$. It follows that $p^n = 2, 3, 4, 8$ or 43. If $p^n = 2$, then $N \leqslant Z(G)$. Let $G' = N \cup v^G$ for some $v \in G$. Then $|v^G| = 12$, whence $|C_G(v)| = 7$, which is a contradiction. If $p^n = 3$, then $|G| = 2 \cdot 3^2 \cdot 7$. Let $N = 1 \cup w^G$ for some element $w \in G$. Then $|w^G| = 2$. It follows that $C_G(N) = C_G(w)$ is a normal subgroup of G of index 2. So $G' \leqslant C_G(N)$. Suppose that $G' = N \cup t^G$ for some element $t \in G$. Then $|t^G| = 18$, and thus $|C_G(t)| = 7$, which contradicts the fact that $N \leqslant C_G(G') \leqslant C_G(t)$. If $p^n = 4$, then $|G| = 2^3 \cdot 3 \cdot 7$. Let $N = 1 \cup \alpha^G$ and $G' = N \cup \beta^G$ for elements $\alpha, \beta \in G$. Then $|C_G(\alpha)| = 2^3 \cdot 7$ and $|C_G(\beta)| = 7$. As G' contains all Sylow 7-subgroups of G, we see a contradiction. If $p^n = 2^3$, we may let T/N be a normal subgroup of $\overline{G}$ of order $2 \cdot 7$, and let $T = G' \cup z^G$ for some element $z \in G$. Then $|z^G| = 2^3 \cdot 7$ and $|C_G(z)| = 6$. However, as z is a 2-element, 4 must divide $|C_G(z)|$, which is a contradiction. If $p^n = 43$, then $|G| = 2 \cdot 3 \cdot 7 \cdot 43$. Let $N = 1 \cup \varepsilon^G$ and $G' = N \cup \xi^G$ for elements $\varepsilon, \xi \in G$. Then $|C_G(\varepsilon)| = 43$ and $|C_G(\xi)| = 7$. As all Sylow subgroups of G are cyclic of prime order, by Theorem 6.18 of [9], $G = \langle a, b\colon\ a^m = b^n = 1,\ b^{-1}ab = a^r,\ ((r-1)n, m) = 1,\ r^n \equiv 1 \pmod m,\ |G| = mn \rangle$. Therefore, $|C_G(\varepsilon)| > 43$ or $|C_G(\xi)| > 7$, which is a contradiction. □

Theorem 4.6. *There is no non-perfect group G such that $\mathcal{K}(G) = \{1, 2, 3, 4, 5\}$ and $\xi(G') = 4$.*

P r o o f. Let K be a normal subgroup of G with $\xi(K) = 5$. Then $G' \leqslant K$. In fact, if $G' \nleqslant K$, then $\xi(G'K) > 5$. So $G'K = G$, which contradicts Lemma 2.3. Similarly, we can prove that every normal subgroup N of G with $\xi(N) \leqslant 3$ is contained in G'. Let N and T be normal subgroups of G with $\xi(N) = 2$ and $\xi(T) = 3$. If $N \nleqslant T$, then $\xi(N \times T) > 4$, which contradicts $N, T \leqslant G'$. Therefore, there is a series of normal subgroups of G as follows:

$$1 < N < T < G' < K < G.$$

Let $\overline{G} = G/N$. Then $\mathcal{K}(\overline{G}) = \{1, 2, 3, 4\}$ and G'/N is a union of four conjugacy classes of $\overline{G}$. However, by Theorem 3.2 of [5], there is no such group. So, the proof is complete. □

Theorem 4.7. *There is no non-perfect group G such that $\mathcal{K}(G) = \{1, 2, 3, 4, 5\}$ and $\xi(G') = 5$.*

P r o o f. In this case, by Lemma 2.3, G' contains all non-trivial normal subgroups of G. It is easy to see that G has a series of normal subgroups

$$1 < N < M < T < G' < G,$$

with $\xi(N) = 2, \xi(M) = 3, \xi(T) = 4$ and $\xi(G') = 5$. Let $\overline{G} = G/N$. Then $\mathcal{K}(\overline{G}) = \{1, 2, 3, 4\}$ and G'/N is a union of four conjugacy classes of $\overline{G}$. By Theorem 3.3 of [5], $|\overline{G}| = 2^3 \cdot 3^3$ or $2^3 \cdot 3 \cdot 5^2$. By Theorem 1 of [10], $|N| = p^n$ for some prime p and some positive integer n. Furthermore, $|M| = p^{n+l}$ for some positive integer l or $|M| = p^n q$ for some prime $q \neq p$.

First suppose that $|\overline{G}| = 2^3 \cdot 3^3$. In this case, all non-trivial normal subgroups of $\overline{G}$ are of order $3^2, 2 \cdot 3^2, 2^3 \cdot 3^2$. Therefore, $|M| = 3^2 p^n$. If $|M| = p^n q$, then $q = 3^2$, which is a contradiction. Therefore, $|M| = p^{n+l} = 3^2 p^n$, so $p = 2$ and $l = 2$. Let $N = 1 \cup w^G$ for some element $w \in G$. Then $|N| = 3^n - 1$ divides $2^3 \cdot 3^{3+n}$. It follows that $3^n - 1$ divides 2^3. So $3^n = 3$ or 9. If $3^n = 1$, then $|w^G| = 2$, and $C_G(w) = C_G(N)$ is a normal subgroup of G of index 2. However, $\overline{G}$ has no normal subgroup of index 2. If $3^n = 3^2$, then let $N = 1 \cup x^G$, $M = N \cup y^G$ for some elements $x, y \in G$. It follows that $|C_G(x)| = 3^5$ and $|C_G(y)| = 3^3$. Therefore, $N = Z(M)$ and M is non-abelian. By Theorem 2 of [8], T is a Frobenius group with kernel M. As $|T| = 2|M|$, M has a fixed point free automorphism of order 2. Then M is abelian by Theorem 10.1.4 of [4], which is a contradiction.

Now suppose that $|\overline{G}| = 2^3 \cdot 3 \cdot 5^2$. In this case, if $|M| = p^n q$, then all non-trivial normal subgroups of $\overline{G}$ are of orders 5^2, $2 \cdot 5^2$, $2^3 \cdot 5^2$. Therefore, $|M| = 5^2 p^n$. If $|M| = p^n q$, then $q = 5^2$, a contradiction. Therefore, $|M| = p^{n+l} = 5^2 p^n$. It follows that $p = 5$ and $l = 2$. Let $N = 1 \cup v^G$ for some element $v \in G$. Then $|v^G| = 5^n - 1$ divides $2^3 \cdot 3 \cdot 5^{n+2}$. Therefore, $5^n - 1$ divides $2^3 \cdot 3$. So, $5^n = 5$ or 25. If $5^n = 5$, then $|v^G| = 4$, whence $C_G(v) = C_G(N)$ is a normal subgroup of G of index 4, which contradicts the above claim. If $5^n = 5^2$, we may suppose that $M = N \cup w^G$ for some element $w \in G$. Then $|w^G| = 5^4 - 5^2 = 5^2 \cdot 2^3 \cdot 3$. So $|C_G(w)| = 5^2$, whence M is not abelian. However, by Theorem 2 of [8], T is a Frobenius group of order $2 \cdot 5^4$. Therefore, M has a fixed point free automorphism of index 2, and thus M is abelian by Theorem 10.1.4 of [4], which is a contradiction. □

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