CHARACTERIZING PROJECTIVE GENERAL UNITARY GROUPS $\mathrm{PGU}_3(q^2)$ BY THEIR COMPLEX GROUP ALGEBRAS

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Abstract. Let G be a finite group. Let $X_1(G)$ be the first column of the ordinary character table of G. We will show that if $X_1(G) = X_1(\mathrm{PGU}_3(q^2))$, then $G \cong \mathrm{PGU}_3(q^2)$. As a consequence, we show that the projective general unitary groups $\mathrm{PGU}_3(q^2)$ are uniquely determined by the structure of their complex group algebras.

Keywords: character degree; complex group algebra; projective general unitary group

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1. Introduction

All groups in this paper are finite and all characters are complex characters. Let G be a group and $\mathrm{Irr}(G)$ the set of all irreducible characters of G. Denote by $\mathrm{cd}(G)$ the set of all irreducible character degrees of G without multiplicities. Let $X_1(G)$ be the first column of the ordinary character table of G. Note that this means that $X_1(G)$ is the set of all irreducible character degrees of G counting multiplicities. Let $\mathbb{C}G$ denote the group algebra of G over $\mathbb{C}$, the field of complex numbers. By Wedderburn's theorem, $\mathbb{C}G \cong \bigoplus_{i=1}^{n} M_{n_i}(\mathbb{C})$ where the dimensions n_i, $1 \leqslant i \leqslant n$, are exactly the degrees of irreducible characters of G. Therefore, the study of complex group algebras and the relations to their base groups plays an important role in group representation theory.

In [3], Problem 2*, Brauer asked whether two groups G and H are isomorphic given that two group algebras $\mathbb{F}G$ and $\mathbb{F}H$ are isomorphic for all fields $\mathbb{F}$. Kimmerle proved in [9] that if G is a finite group and S is a nonabelian simple group such that $\mathbb{F}G \cong \mathbb{F}S$ for all fields $\mathbb{F}$, then $G \cong S$. Inspired by [9], Tong-Viet proved in [20],

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[22], and [21] that all nonabelian simple groups are determined up to isomorphism by their complex group algebras. Also he posed the following question:

Question. Which groups can be uniquely determined by the structure of their complex group algebras?

Recall that a group G is called quasi-simple if G is a perfect group such that its inner automorphisms group $\mathrm{Inn}(G)$ is a nonabelian simple group. It was shown in [2], [15], and [16] that all quasi-simple groups are uniquely determined by their complex group algebras. In [19], Tong-Viet proved that the symmetric groups are determined up to isomorphism by their complex group algebras. Independently, this result was also proved by Nagl in [14]. Also projective general linear groups of dimension two were shown in [7] to be uniquely determined by their complex group algebras. In [17], the authors proved that projective general linear groups $\mathrm{PGL}_n(q)$, $n \geqslant 3$, are determined up to isomorphism by their complex group algebras, provided that $q-1$ divides neither n nor $n-1$. One of the natural groups to be considered next are almost simple unitary groups. For this purpose, one needs to establish some arithmetic properties of character degrees of simple unitary groups together with determining the behavior of the irreducible characters under the action of outer automorphisms. For unitary groups of dimension 3, the character tables and all conjugacy classes of $\mathrm{SU}_3(q^2)$ and $\mathrm{PSU}_3(q^2)$ have been given in [18]. This together with the results of [8] and [23] enables us to settle the above question for a family of almost simple unitary groups of dimension 3, namely the projective general unitary groups $\mathrm{PGU}_3(q^2)$.

The main result of this paper is the following theorem:

Theorem 1.1. *Let $q = p^f$ be a prime power and let G be a finite group such that $X_1(G) = X_1(\mathrm{PGU}_3(q^2))$. Then $G \cong \mathrm{PGU}_3(q^2)$.*

By Molien's theorem (see [1], Theorem 2.13) we know that the structure of the group algebra $\mathbb{C}G$ is equivalent to knowing the first column of the ordinary character table of G. Therefore, according to Theorems 1.1, we have the following corollary:

Corollary 1.2. *Let $q = p^f$ be a prime power and let G be a finite group such that $\mathbb{C}G = \mathbb{C}\mathrm{PGU}_3(q^2)$. Then $G \cong \mathrm{PGU}_3(q^2)$.*

Note that if $\gcd(3, q+1) = 1$, then $\mathrm{PGU}_3(q^2) = \mathrm{PSU}_3(q^2)$. Suppose that $X_1(G) = X_1(\mathrm{PGU}_3(q^2))$, where $\gcd(3, q+1) = 1$. In these cases, the result of Theorem 1.1 follows from [21], Theorem 1.1. Therefore, it remains to consider the cases in which $\gcd(3, q+1) \neq 1$ to complete the proof of Theorem 1.1.

Theorem 1.3. *Let $q = p^f$ be a prime power and let G be a finite group such that $X_1(G) = X_1(\mathrm{PGU}_3(q^2))$, where $q \equiv -1 \pmod 3$. Then $G \cong \mathrm{PGU}_3(q^2)$.*

The rest of this paper is devoted to proving Theorem 1.3. Note that for $q = 2$, $\mathrm{PGU}_3(2^2)$ is a solvable group, and for $q > 2$, $\mathrm{PGU}_3(q^2)$ is nonsolvable. We now outline our argument for the proof of Theorem 1.3. Assume that $X_1(G) = X_1(\mathrm{PGU}_3(q^2))$, where $q \equiv -1 \pmod 3$. For $q > 2$, we prove in Section 3 the following steps:

1. $G' = G''$.
2. If G'/M is a chief factor of G, then $G'/M \cong \mathrm{PSU}_3(q^2)$.
3. $M = 1$ and so $G' \cong \mathrm{PSU}_3(q^2)$.
4. $G/C_G(G') \cong \mathrm{PGU}_3(q^2)$.

Then the result follows since $|G| = |\mathrm{PGU}_3(q^2)|$. This is a modified version of the method used to verify Huppert's conjecture which states that nonabelian simple groups are determined by their sets of character degrees, see [8]. The solvable case, where $X_1(G) = X_1(\mathrm{PGU}_3(2^2))$, is investigated using the degree graph in Section 4.

2. Character degrees of $\mathrm{PGU}_3(q^2)$

In this section, we present and also prove some lemmas concerning the character degrees of $\mathrm{PGU}_3(q^2)$, $q \equiv -1 \pmod 3$. Using the data in [11], the character degrees of $\mathrm{PGU}_3(q^2)$, $q \equiv -1 \pmod 3$, with their multiplicities are given in Table 1.

Degrees	Multiplicity
1	3
$q(q-1)$	3
q^2-q+1	$q-2$
$(q-1)(q^2-q+1)$	$\frac{1}{6}q^2-\frac{1}{6}q+\frac{2}{3}$
$q(q^2-q+1)$	$q-2$
q^3	3
$(q+1)(q^2-q+1)$	$\frac{1}{2}(q+1)(q-2)$
$(q-1)(q+1)^2$	$\frac{1}{3}(q+1)(q-2)$

Table 1. Character degrees of $\mathrm{PGU}_3(q^2)$, $q \equiv -1 \pmod 3$.

Note that $\gcd(q-1, q+1) = 2$ if q is odd, and $\gcd(q^2-q+1, q+1) = 3$ if $q \equiv -1 \pmod 3$. All the other factors of character degrees given in Table 1 are pairwise relatively prime.

Remark 2.1. Compairing the data given in [11] and [18], Table 2, we obtain that $\mathrm{cd}(\mathrm{PGU}_3(q^2)) \subset \mathrm{cd}(\mathrm{PSU}_3(q^2))$, where $q > 2$. In particular, $\mathrm{cd}(\mathrm{PSU}_3(q^2)) = \mathrm{cd}(\mathrm{PGU}_3(q^2)) \cup \{\frac{1}{3}(q-1)(q^2-q+1)\}$, where $q > 2$ and $q \equiv -1 \pmod 3$.

The following lemmas use the Deligne-Lusztig theory of complex characters of finite groups of Lie type, cf. [4]. If $\mathcal{G}$ is a simple algebraic group, let $\pi_1(\mathcal{G})$ denote the fundamental group of $\mathcal{G}$.

Lemma 2.2 ([5], Lemma 2.5). *Let $\mathcal{G}$ be a simple algebraic group in characteristic p, F a Frobenius map on $\mathcal{G}$, and let $G := \mathcal{G}^F$. Let the pair $(\mathcal{G}^*, F^*)$ be dual to $(\mathcal{G}, F)$ and $G^* := \mathcal{G}^{*F^*}$. Assume $s \in G^*$ is a semisimple element of order r which is coprime to both $|\pi_1(\mathcal{G}^*)|$ and $|Z(G)|$.*

(a) *Then G has an irreducible character χ_s of degree $[G^* : C_{G^*}(s)]_{p'}$ which is trivial at $Z(G)$.*

(b) *Let σ and σ^* be automorphisms of (abstract) groups $\mathcal{G}$ and $\mathcal{G}^*$, respectively, induced by a field automorphism $x \mapsto x^q$ for some power q of p and such that $\sigma \circ F = F \circ \sigma$. Assume in addition that $r \nmid |(\mathcal{G}^*)^{\sigma^*}|$. Then χ_s is not σ-invariant.*

Lemma 2.3. *For $q = p^f > 2$, the group $\mathrm{PGU}_3(q^2)$ has a so-called semisimple irreducible character χ_s of degree $(q-1)(q+1)^2$, where $s \in \mathrm{SU}_3(q^2)$ is a semisimple element. Moreover, χ_s restricts irreducibly to $\mathrm{PSU}_3(q^2)$.*

Proof. Let $\mathcal{G}^* := SL_3(\overline{\mathbb{F}_p})$. It is known, cf. [12], Example 21.2, that $(\mathcal{G}^*)^{F^*} = \mathrm{SU}_3(q^2)$ for the Frobenius map F^* given by $(a_{ij}) \mapsto ({a_{ij}}^q)^{-tr}$. Since $q > 2$, by a classical result of Zsigmondy, see [24], there exists a primitive prime divisor r of $p^{6f} - 1 = q^6 - 1$; that is, a prime divisor of $p^{6f} - 1$ which does not divide $\prod_{i=1}^{6f-1} (p^i - 1)$. Let $\omega \in \overline{\mathbb{F}_p}$ be of order r. Let s be a semisimple element of $\mathrm{SU}_3(q^2)$ with eigenvalues ω, ω^{q^2}, and $\omega^{q^4} = \omega^{-q}$. From the data in [18], Table 1a, we obtain that $|C_{\mathrm{SU}_3(q^2)}(s)| = q^2 - q + 1$. Therefore, by Lemma 2.2 (a), χ_s is a semisimple character of $\mathcal{G}^F = \mathrm{PGU}_3(q^2)$ of degree $(q^2-1)(q^3+1)/q^2 - q + 1$. Moreover, χ_s restricts irreducibly to $\mathrm{PSU}_3(q^2)$, since $\mathrm{PSU}_3(q^2)$ has no irreducible character of degree $\frac{1}{3}(q-1)(q+1)^2$, see [18], Table 2. □

3. Proof of Theorem 1.3 for $\mathrm{PGU}_3(q^2)$, $q > 2$

In this section, we will prove Theorem 1.3 for the family of groups $\mathrm{PGU}_3(q^2)$, $q > 2$, through verifying steps 1–4. Assume throughout this section that $X_1(G) = X_1(\mathrm{PGU}_3(q^2))$, where $q > 2$ and $q \equiv -1 \pmod 3$. Therefore, $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{PGU}_3(q^2))$ and $|G| = |\mathrm{PGU}_3(q^2)|$. By Remark 2.1, $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{PSU}_3(q^2)) \setminus \{\frac{1}{3}(q-1)(q^2-q+1)\}$. Hence, arguing as in step 1 in [8], we deduce that $G' = G''$. Let $G'/M \cong S \times \ldots \times S$, the direct product of k copies of a nonabelian simple group S, be a chief factor of G. Following the arguments in [8] and [23] we show that $k = 1$ and $S \cong \mathrm{PSU}_3(q^2)$.

Lemma 3.1 ([8], Step 2)**.** *Let H be a finite group such that $\mathrm{cd}(H) = \mathrm{cd}(\mathrm{PSU}_3(q^2))$, where $2 < q \leqslant 8$. If $H'/M \cong S \times \ldots \times S$, the direct product of k copies of a nonabelian simple group S, is a chief factor of H, then $k = 1$ and $S \cong \mathrm{PSU}_3(q^2)$.*

Lemma 3.2 ([23], Propositions 4.4–4.6 and Section 4.4)**.** *Let H be a finite group such that $\mathrm{cd}(H) = \mathrm{cd}(\mathrm{PSU}_3(q^2))$, where $q > 9$. If $H'/M \cong S \times \ldots \times S$, the direct product of k copies of a nonabelian simple group S, is a chief factor of H, then $k = 1$ and $S \cong \mathrm{PSU}_3(q^2)$.*

According to the proofs of Lemma 3.1 and Lemma 3.2, the results of these lemmas would remain still valid if we assume that H is a nonsolvable group such that $H' = H''$ and $\mathrm{cd}(H) \subseteq \mathrm{cd}(\mathrm{PSU}_3(q^2))$. Since $[G : G'] = 3$, $G' \neq 1$. In addition, step 1 verifies that $G' = G''$. Also, by Remark 2.1, we have $\mathrm{cd}(G) \subseteq \mathrm{cd}(\mathrm{PSU}_3(q^2))$. Therefore, arguing exactly as in the proofs of Lemma 3.1 and Lemma 3.2 one could prove the following result:

Proposition 3.3. *Let G be a finite group such that $X_1(G) = X_1(\mathrm{PGU}_3(q^2))$, where $q > 2$ and $q \equiv -1 \pmod 3$. Let $G'/M \cong S \times \ldots \times S$, the direct product of k copies of a nonabelian simple group S, be a chief factor of G. Then $k = 1$ and $S \cong \mathrm{PSU}_3(q^2)$.*

Proposition 3.4. *In the contex of Proposition 3.3, we have*
(a) *$G' \cong \mathrm{PSU}_3(q^2)$;*
(b) *$C_G(G')$ is abelian.*

Proof. (a) We have shown in Proposition 3.3 that $G'/M \cong \mathrm{PSU}_3(q^2)$. So $|G'| = |M||\mathrm{PSU}_3(q^2)|$ divides $|G| = 3|\mathrm{PSU}_3(q^2)|$. Therefore, $|M| \mid 3$. If $|M| = 3$, then $G' = G$, which is impossible since according to Table 1, G has three linear characters. Hence, $M = 1$ and $G' \cong \mathrm{PSU}_3(q^2)$.

(b) By (a), G' is a nonabelian simple group and so $Z(G') = 1$. Let $x, y \in C_G(G')$. Then, $xyx^{-1}y^{-1} \in C_G(G') \cap G' = Z(G')$. So $xy = yx$. □

Proposition 3.5. *If G is a finite group such that $X_1(G) = X_1(\mathrm{PGU}_3(q^2))$, where $q > 2$ and $q \equiv -1 \pmod 3$, then $G/C_G(G') \cong \mathrm{PGU}_3(q^2)$.*

Proof. By Proposition 3.4, G' is a nonabelian simple group. Since $G' \cap C_G(G') = 1$, it follows that $G' \cong G'C_G(G')/C_G(G') \trianglerighteq G/C_G(G') \leqslant \mathrm{Aut}(G')$. Therefore, $G/C_G(G')$ is an almost simple group and it induces on G' some outer automorphism α. Let $q = p^f$. It is well known, cf. [6], Theorem 2.5.12, that

$$\mathrm{Out}(G') \cong \langle d\rangle : \langle \sigma\rangle,$$

where d is a diagonal automorphism of degree $\gcd(3, q+1) = 3$, σ is the automorphism of G' of order $2f$ induced by the field automorphism $x \mapsto x^p$ on the finite field $\mathbb{F}_{q^2}$. We consider the following cases and show that assuming as in Theorem 1.3, only the case $G/C_G(G') \cong \mathrm{PGU}_3(q^2)$ may occur, as desired.

Case 1. $G/C_G(G')$ possesses only inner automorphisms. In this case, $G/C_G(G') \cong \mathrm{PSU}_3(q^2)$. Therefore $\frac{1}{3}(q-1)(q^2-q+1) \in \mathrm{cd}(G/C_G(G')) \subset \mathrm{cd}(G)$. This leads to contradiction since according to Table 1, G has no irreducible character of degree $\frac{1}{3}(q-1)(q^2-q+1)$.

Case 2. $G/C_G(G')$ possesses only inner and diagonal automorphisms. In this case, $\alpha = d$ is a diagonal automorphism of G' of degree 3, and we have $G/C_G(G') \cong \mathrm{PSU}_3(q^2) : \langle d\rangle \cong \mathrm{PGU}_3(q^2)$, as desired.

Case 3. $\alpha = d^a\sigma^b$ where $1 \leqslant a \leqslant 3$, $1 \leqslant b \leqslant 2f-1$. By Lemma 2.3, $\mathrm{PGU}_3(q^2)$ has a semisimple irreducible character χ_s of degree $(q-1)(q+1)^2$, where s is a semisimple element of $\mathrm{SU}_3(q^2)$ whose order is a primitive prime divisor of $p^{6f}-1$. So $|s| \nmid \prod\limits_{i=1}^{6f-1}(p^i-1)$. Moreover, χ_s restricts irreducibly to $\mathrm{PSU}_3(q^2)$. Hence, $\chi_s \in \mathrm{Irr}(\mathrm{PSU}_3(q^2))$. By Table 1, $\mathrm{PGU}_3(q^2) \cong \mathrm{PSU}_3(q^2) : \langle d\rangle$ has no degree which is a proper multiple of $(q-1)(q+1)^2$. Therefore, χ_s is d-invariant. Also, using Lemma 2.3 (b) we deduce that χ_s is not σ^b-invariant since $|s|$ does not divide $|SL_3(p^b)|$. We have shown that χ_s is not α-invariant. Hence, G has an irreducible character whose degree is a proper multiple of $(q-1)(q+1)^2$, which is impossible according to Table 1. $\square$

4. Proof of Theorem 1.3 for $\mathrm{PGU}_3(2^2)$

In this section, we will prove Theorem 1.3 for the solvable group $\mathrm{PGU}_3(2^2)$. Our proof is based on a result in [13] about finite groups whose degree graph is empty. The degree graph $\Delta(G)$ of the group G has a vertex set consisting of the primes that divide degrees in $\mathrm{cd}(G)$; there is an edge between p and q if pq divides some degree $a \in \mathrm{cd}(G)$.

Proposition 4.1. *Let G be a finite group such that $X_1(G) = X_1(\mathrm{PGU}_3(2^2))$. Then $G \cong \mathrm{PGU}_3(2^2)$.*

P r o o f. Using Table 1, we have $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{PGU}_3(q^2)) = \{1, 2, 3, 8\}$. Also $|G| = |\mathrm{PGU}_3(2^2)| = 2^3 \times 3^3$. So by Burnside's $\{p, q\}$-theorem, G is also a solvable group. The degree graph of both G and $\mathrm{PGU}_3(2^2)$ is the empty graph with two connected component, $\{2\}$, and $\{3\}$. Using [13], Main theorem, and [10], Lemmas 3.2–3.5, we obtain that both G and $\mathrm{PGU}_3(2^2)$ are isomorphic to the semidirect product of a subgroup H acting on a subgroup P, where P is an elementary abelian group of order 9 and $H/C_H(P) \cong SL_2(3)$, where $C_H(P) \leqslant Z(H)$ and the action of H on P is the natural action of $SL_2(3)$ on P. The order of this semidirect product is $9|H|$, where $|SL_2(3)| \mid |H|$. Since $|G| = |\mathrm{PGU}_3(2^2)| = 9|SL_2(3)|$, we must have that $|C_H(P)| = 1$ and $H \cong SL_2(3)$. Hence, both G and $\mathrm{PGU}_3(2^2)$ are isomorphic to the semidirect product of $SL_2(3)$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$, with the natural action of $SL_2(3)$ on $\mathbb{Z}_3 \times \mathbb{Z}_3$. Therefore, $G \cong \mathrm{PGU}_3(2^2)$. □

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