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A CAUCHY-POMPEIU FORMULA IN SUPER DUNKL-CLIFFORD ANALYSIS

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Abstract. Using a distributional approach to integration in superspace, we investigate a Cauchy-Pompeiu integral formula in super Dunkl-Clifford analysis and several related results, such as Stokes formula, Morera's theorem and Painlevé theorem for super Dunkl-monogenic functions. These results are nice generalizations of well-known facts in complex analysis.

Keywords: super Dunkl-Dirac operator; Stokes formula; Cauchy-Pompeiu integral formula; Morera's theorem; Painlevé theorem

MSC 2010: 30G35, 26B20, 58C50

1. Introduction

Dunkl operators (also called differential-difference operators), introduced by Dunkl (see [7]), are invariant under a finite reflection group and are also pairwise commuting. These operators not only provide a useful tool in the study of special functions with root systems (see [8]), but also they are closely related to some particular representations of degenerated affine Hecke algebras (see [16]) and integrable systems of Calogero-Moser-Sutherland type (see [12]). In 2006, Cerejeiras, Kähler and Ren defined the Dunkl-Dirac operator (see [2]) and constructed the Stokes formula in Clifford analysis by Dunkl transforms (see [15]). The theory of Dunkl-Clifford analysis is further developed in [1], [10], [11], [14], [4] and [17]. In 2013, Fei investigated the fundamental solutions to the Dunkl-Dirac equation, and also obtained the Cauchy integral formula with a Dunkl-Cauchy kernel (see [9]).

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Recently, Sommen, De Bie and others have studied a superspace of dimension $(m, 2n)$ in the frame of Clifford analysis (see [5], [6], [3]). Superspaces are spaces equipped with both a set of commuting variables and a set of anti-commuting variables in order to describe the properties of bosons and fermions in quantum mechanics. In [5], they defined the super Dirac operator (i.e., the Dirac operator in superspace) by the Dirac operator in $\mathbb{R}^m$. In [3], using a distributional approach to integration in superspace, they investigated some properties of the super Dirac operator, such as Stokes formula, Cauchy integral formula and Morera's theorem. Then, we investigated Cauchy-Pompeiu formulas for iterates of Dirac operators and polynomial Dirac operators in superspace (see [18], [19]). Inspired by the above-mentioned results, we want to develop further these ideas for the super Dunkl-Dirac operator.

The paper is organized as follows. In Section 2 we recall the necessary results on the super Dunkl-Clifford analysis (i.e., Dunkl-Clifford analysis in superspace). In Section 3, inspired by De Bie et al., we construct fundamental solutions for the super Dunkl-Laplace and super Dunkl-Dirac operators by the fundamental solutions of the natural powers of the Laplace operator in Dunkl-Clifford analysis. In Section 4, using a distributional approach to integration in superspace, combined with the Stokes formula in Dunkl-Clifford analysis, we consider the Stokes formula in super Dunkl-Clifford analysis. Applying this formula, we get the Cauchy-Pompeiu formula for the super Dunkl-Dirac operator and Morera's theorem for super Dunkl-monogenic functions. Furthermore, using Morera's theorem, we obtain the Painlevé theorem for super Dunkl-monogenic functions.

2. Preliminaries

2.1. Dunkl-Clifford analysis in $\mathbb{R}^m$. Denote by $\langle \cdot, \cdot \rangle$ the standard Euclidean scalar product in $\mathbb{R}^m$ and by $|x| = \langle x, x\rangle^{1/2}$ the associated norm. For $\alpha \in \mathbb{R}^m \setminus \{0\}$, the reflection σ_α in the hyperplane orthogonal to α is given by

$$\sigma_\alpha x = x - 2\frac{\langle \alpha, x\rangle}{|\alpha|^2}\alpha, \quad x \in \mathbb{R}^m.$$

A finite set $\mathsf{R} \subset \mathbb{R}^m \setminus \{0\}$ is called a root system if $\alpha\mathbb{R} \cap \mathsf{R} = \{\alpha, -\alpha\}$ and $\sigma_\alpha \mathsf{R} = \mathsf{R}$ for all $\alpha \in \mathsf{R}$. Each root system can be written as a disjoint union $\mathsf{R} = \mathsf{R}_+ \cup (-\mathsf{R}_+)$, where R_+ and $-\mathsf{R}_+$ are separated by a hyperplane through the origin. The subgroup $G \subset O(m)$ generated by the reflections $\{\sigma_\alpha \colon \alpha \in \mathsf{R}\}$ is called the finite reflection group associated with R. For more information on finite reflection groups we refer the reader to [13].

A multiplicity function κ on the root system R is a G-invariant function $\kappa\colon\ \mathsf{R}\to\mathbb{C}$, i.e., $\kappa(\alpha)=\kappa(g\alpha)$ for all $g\in G$. We will denote $\kappa(\alpha)$ by κ_α. For abbreviation, we introduce the index

$$\gamma=\gamma_\kappa=\sum_{\alpha\in\mathsf{R}_+}\kappa_\alpha.$$

Moreover, let $h_\kappa(\underline{x})$ denote the weight function

$$h_\kappa(\underline{x})=\prod_{\alpha\in\mathsf{R}_+}|\langle\alpha,\underline{x}\rangle|^{\kappa_\alpha}.$$

In this paper, we will assume that $\kappa_\alpha\geqslant 0$ and $\gamma_\kappa>0$.

For each subsystem R_+ and multiplicity function κ_α we have the Dunkl operators

$$T_if(x)=\frac{\partial f(x)}{\partial x_i}+\sum_{\alpha\in\mathsf{R}_+}\kappa_\alpha\frac{f(x)-f(\sigma_\alpha x)}{\langle x,\alpha\rangle}\alpha_i,\quad i=1,\dots,m,$$

for $f\in C^1(\mathbb{R}^m)$. An important consequence is that the operators T_i are mutually commutating, that is, $T_iT_j=T_jT_i$.

We consider a function $f\colon\ \mathbb{R}^m\to\mathbb{R}_{0,m}$. Hereby $\mathbb{R}_{0,m}$ denotes the Clifford algebra over $\mathbb{R}^m$ generated by $\{e_1,e_2,\dots,e_m\}$ satisfying the anti-commutation relationship $e_ie_j+e_je_i=-2\delta_{ij}$, where δ_{ij} is the Kronecker symbol. By $\underline{x}=\sum_{i=1}^{m}x_ie_i$ we denote the so-called vector variable. A Dunkl-Dirac operator in $\mathbb{R}^m$ for the corresponding reflection group G is defined as $D_h=\sum_{i=1}^{m}e_iT_i$, where T_i are Dunkl operators. Functions belonging to the kernel of the Dunkl-Dirac operator D_h are called Dunkl-monogenic functions.

The classical Dunkl Laplacian is defined as

$$\Delta_h=-D_h^2=\sum_{i=1}^{m}T_i^2.$$

When $\kappa=0$, the Dunkl Laplacian Δ_h is just the ordinary Laplacian. Functions belonging to the kernel of the Dunkl Laplacian Δ_h are called Dunkl-harmonic functions.

2.2. Dunkl-Clifford analysis in $\mathbb{R}^{m|2n}$. On a superspace of dimension $(m,\,2n)$, we have m commuting (or bosonic) variables $x_1,\dots,x_m$ and $2n$ anti-commuting (or fermionic) variables $\grave{x}_1,\dots,\grave{x}_{2n}$ subject to

$$\begin{cases}x_ix_j=x_jx_i,\\ \grave{x}_i\grave{x}_j=-\grave{x}_j\grave{x}_i,\\ x_i\grave{x}_j=\grave{x}_jx_i.\end{cases}$$

Furthermore, we have the Clifford algebra generators $e_1, \ldots, e_m$ and the symplectic Clifford algebra generators $\grave{e}_1, \ldots, \grave{e}_{2n}$. They obey the following rules:

$$\begin{cases} e_j e_k + e_k e_j = -2\delta_{jk}, \\ \grave{e}_{2j}\grave{e}_{2k} - \grave{e}_{2k}\grave{e}_{2j} = 0, \\ \grave{e}_{2j-1}\grave{e}_{2k-1} - \grave{e}_{2k-1}\grave{e}_{2j-1} = 0, \\ \grave{e}_{2j-1}\grave{e}_{2k} - \grave{e}_{2k}\grave{e}_{2j-1} = \delta_{jk}, \\ e_j\grave{e}_k + \grave{e}_k e_j = 0. \end{cases}$$

Taking the above relations into account, we study the superspace by the real algebra

$$\mathrm{Alg}(x_i, e_i; \grave{x}_j, \grave{e}_j) = \mathrm{Alg}(x_i, \grave{x}_j) \otimes \mathrm{Alg}(e_i, \grave{e}_j), \quad i = 1, \ldots, m, \; j = 1, \ldots, 2n,$$

which is the tensor product of $\mathrm{Alg}(x_i, \grave{x}_j)$ and $\mathrm{Alg}(e_i, \grave{e}_j)$. The algebra $\mathrm{Alg}(x_i, \grave{x}_j)$ is called a scalar algebra, denoted by $\mathcal{P}$, and the algebra $\mathrm{Alg}(e_i, \grave{e}_j)$ is a Clifford algebra, denoted by $\mathcal{C}_{m|2n}$. Moreover, the elements of both these algebras can commute with each other. When $n = 0$, we have that $\mathcal{P} \otimes \mathcal{C}_{m|0} = \mathbb{R}[x_1, \ldots, x_m] \otimes \mathbb{R}_{0,m}$, where $\mathbb{R}[x_1, \ldots, x_m]$ is generated by the commuting variables x_i. In the case $\mathcal{C}_{m|0} \cong \mathbb{R}_{0,m}$, $\mathbb{R}_{0,m}$ is the standard orthogonal Clifford algebra. When $m = 0$, we have that $\mathcal{P} \otimes \mathcal{C}_{0|2n} = \Lambda_{2n} \otimes \mathcal{W}_{2n}$, with Λ_{2n} being the Grassmann algebra generated by $\grave{x}_j$. In the case $\mathcal{C}_{0|2n} \cong \mathcal{W}_{2n}$, $\mathcal{W}_{2n}$ is the Weyl algebra generated by $\grave{e}_j$.

We define the super vector variable x as follows:

$$x = \underline{x} + \underline{\grave{x}},$$

where $\underline{x} = \sum_{i=1}^{m} x_i e_i$ and $\underline{\grave{x}} = \sum_{j=1}^{2n} \grave{x}_j \grave{e}_j$. By direct calculation, we obtain the square of x:

$$x^2 = \underline{\grave{x}}^2 + \underline{x}^2, \quad \text{where } \underline{\grave{x}}^2 = \sum_{j=1}^{n} \grave{x}_{2j-1}\grave{x}_{2j} \text{ and } \underline{x}^2 = -\sum_{i=1}^{m} x_i^2.$$

Note that $\underline{x}^2 = -\sum_{i=1}^{m} x_i^2$ is the norm squared of a vector in Euclidean space.

Thus, we define a more general function space as

$$C^k(\Omega) \otimes \Lambda_{2n} \otimes \mathcal{C}_{m|2n},$$

where $C^k(\Omega)$ denotes space of k-times continuously differentiable real-valued functions defined in some domain $\Omega \subset \mathbb{R}^m$. We use the notation

$$C^k(\Omega)_{m|2n} = C^k(\Omega) \otimes \Lambda_{2n}.$$

The super Dunkl-Dirac operator is defined to be

$$D = -D_h + D_f = -\sum_{i=1}^{m} e_i T_i + 2\sum_{j=1}^{n} (\grave{e}_{2j}\partial_{\grave{x}_{2j-1}} - \grave{e}_{2j-1}\partial_{\grave{x}_{2j}}),$$

where D_h is the bosonic Dunkl-Dirac operator and D_f is the fermionic Dunkl-Dirac operator.

If we let D act on x, we see that

$$M := \frac{1}{2}Dx = -n + \frac{m}{2} + \gamma_\kappa,$$

where M is the Dunkl version of the super-dimension in contrast to the non-Dunkl case of the super-dimension in [6]. The numerical parameter M is regarded as the ground level energy in physics.

As usual, functions belonging to the kernel of the super Dunkl-Dirac operator are called super Dunkl-monogenic functions.

The square of the left super Dunkl-Dirac operator is the super Dunkl-Laplace operator

$$\Delta = D^2 = -\Delta_h + \Delta_f = -\sum_{i=1}^{m} T_i^2 + 4\sum_{j=1}^{n} \partial_{\grave{x}_{2j-1}}\partial_{\grave{x}_{2j}},$$

where Δ_h is the Dunkl-Laplace operator and Δ_f is the fermionic Dunkl-Laplace operator.

Functions belonging to the kernel of the super Dunkl-Laplace operator are called super Dunkl-harmonic functions.

2.3. Integration in Dunkl superspace. The integration in Dunkl superspace is defined by

$$\int_{\mathbb{R}^{m|2n}} \cdot = \int_{\mathbb{R}^m} h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}) \int_B \cdot = \int_B \int_{\mathbb{R}^m} h_\kappa^2(\underline{x}) \cdot \mathrm{d}V(\underline{x}),$$

where $\mathrm{d}V(\underline{x}) = \mathrm{d}x_1 \ldots \mathrm{d}x_m$ is the usual Lebesgue measure in $\mathbb{R}^m$, and the integration

$$\int_B \cdot = \pi^{-n}\partial_{\grave{x}_{2n}} \ldots \partial_{\grave{x}_1}\cdot$$

used on Λ^{2n} is the so-called Berezin integration.

3. Fundamental solutions for the Dunkl-Laplace and Dunkl-Dirac operators in superspace

We introduce the Mehta-type constant

$$c_h = \left(\int_{\mathbb{R}^m} \exp(-\|\underline{x}\|^2) h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x})\right)^{-1},$$

which is known for all Coxeter groups W (see [8]).

Lemma 3.1 ([9]). *If $0 < s < \gamma + d/2$, then the functions $K_s^{m|0}(\underline{x})$ given by*

$$K_s^{m|0}(\underline{x}) = \frac{(-1)^s c_h \Gamma(\gamma + d/2 - s)}{4^s \Gamma(s)} \frac{1}{\|\underline{x}\|^{2\gamma+d-2s}}$$

are fundamental solutions for the natural powers of the Dunkl-Laplace operator Δ_h.

Concerning the refinement to Clifford analysis, we clearly have that $D_h K_s^{m|0}(\underline{x})$ are fundamental solutions for the natural powers of the Dunkl-Dirac operator D_h.

Lemma 3.2 ([9]). *For $l \in \mathbb{N}$, we denote by $K_l^{m|0}(\underline{x})$ the fundamental solutions for the natural powers of the Dunkl-Dirac operator D_h.*

For $2\gamma + m$ odd,

$$K_l^{m|0}(\underline{x}) = \begin{cases} c_{\kappa,m,l} \dfrac{\underline{x}}{\|\underline{x}\|^{2\gamma+m-l+1}}, & l \text{ odd}, \\[2ex] c_{\kappa,m,l} \dfrac{x}{\|\underline{x}\|^{2\gamma+m-l}}, & l \text{ even}. \end{cases}$$

For $2\gamma + m$ even,

$$K_l^{m|0}(\underline{x}) = \begin{cases} c_{\kappa,m,l} \dfrac{\underline{x}}{\|\underline{x}\|^{2\gamma+m-l+1}}, & l \text{ odd and } l < 2\gamma + m - 1, \\[2ex] c_{\kappa,m,l} \dfrac{x}{\|\underline{x}\|^{2\gamma+m-l}}, & l \text{ even and } l < 2\gamma + m, \\[2ex] (c_{\kappa,m,l} \log \|\underline{x}\| + c'_{\kappa,m,l}) \dfrac{\underline{x}}{\|\underline{x}\|^{2\gamma+m-l+1}}, & l \text{ odd and } l \geqslant 2\gamma + m - 1, \\[2ex] (c_{\kappa,m,l} \log \|\underline{x}\| + c'_{\kappa,m,l}) \dfrac{x}{\|\underline{x}\|^{2\gamma+m-l}}, & l \text{ even and } l \geqslant 2\gamma + m. \end{cases}$$

From the above lemmas, we have the fundamental solution for the super Dunkl-Laplace operator as follows.

Theorem 3.3. *The function* $K_2^{m|2n}(x)$ *given by*

$$K_2^{m|2n}(x) = \pi^n \sum_{k=0}^{n} \frac{4^k k!}{(n-k)!} K_{2k+2}^{m|0} \underline{\grave{x}}^{2n-2k},$$

with $K_{2k+2}^{m|0}$ *as in Lemma* 3.1, *is a fundamental solution for the operator* Δ.

Proof. From the definition of the super Dunkl-Laplace operator, we have

$$\begin{aligned}
\Delta \pi^n \sum_{k=0}^{n} \frac{4^k k!}{(n-k)!} K_{2k+2}^{m|0} \underline{\grave{x}}^{2n-2k} &= (-\Delta_h + \Delta_f)\pi^n \sum_{k=0}^{n} \frac{4^k k!}{(n-k)!} K_{2k+2}^{m|0} \underline{\grave{x}}^{2n-2k} \\
&= \pi^n \sum_{k=0}^{n} \frac{4^k k!}{(n-k)!} (-\Delta_h) K_{2k+2}^{m|0} \underline{\grave{x}}^{2n-2k} \\
&\quad + \pi^n \sum_{k=0}^{n-1} \frac{4^k k!}{(n-k)!} K_{2k+2}^{m|0} (2n-2k)(-2k-2) \underline{\grave{x}}^{2n-2k-2} \\
&= \delta(\underline{x}) \frac{\pi^n}{n!} (\underline{\grave{x}})^{2n} + \pi^n \sum_{k=1}^{n} \frac{4^k k!}{(n-k)!} K_{2k}^{m|0} \underline{\grave{x}}^{2n-2k} \\
&\quad + \pi^n \sum_{k=1}^{n} \frac{4^{k-1}(k-1)!}{(n-k+1)!} K_{2k}^{m|0} (2n-2k+2)(-2k) \underline{\grave{x}}^{2n-2k} \\
&= \delta(\underline{x}) \frac{\pi^n}{n!} (\underline{\grave{x}})^{2n} + \pi^n \sum_{k=1}^{n} \Big(\frac{4^k k!}{(n-k)!} + \frac{4^{k-1}(k-1)!}{(n-k+1)!} (2n-2k+2)(-2k) \Big) K_{2k}^{m|0} \underline{\grave{x}}^{2n-2k} \\
&= \delta(x),
\end{aligned}$$

where $\delta(x) = \delta(\underline{x})\pi^n n!^{-1} \underline{\grave{x}}^{2n}$ is the super distribution in $\mathbb{R}^{m|2n}$. Thus, we completed the proof. □

Note that $\Delta K_2^{m|2n}(x) = \delta(x)$. It follows that a fundamental solution for the super Dunkl-Dirac operator D is given by $DK_2^{m|2n}(x)$. This leads to the following statement.

Theorem 3.4. *The function* $K_1^{m|2n}(x)$ *given by*

$$K_1^{m|2n}(x) = \pi^n \sum_{k=0}^{n-1} \frac{2 \cdot 4^k k!}{(n-k-1)!} K_{2k+2}^{m|0} \underline{\grave{x}}^{2n-2k-1} - \pi^n \sum_{k=0}^{n-1} \frac{4^k k!}{(n-k-1)!} K_{2k+1}^{m|0} \underline{\grave{x}}^{2n-2k},$$

with $K_{2k+2}^{m|0}$ *as in Lemma* 3.1 *and* $K_{2k+1}^{m|0} = D_h K_{2k+2}^{m|0}$ *as in Lemma* 3.2, *is a fundamental solution for the super Dunkl-Dirac operator* D.

4. Fundamental theorems in super Dunkl-Clifford analysis

4.1. Stokes formula in super Dunkl-Clifford analysis. In [2], we see that the Stokes formula in Dunkl-Clifford analysis reads as follows.

Lemma 4.1 ([2]). *For* $\varphi(\underline{x}), \psi(\underline{x}) \in C^\infty(\Omega) \otimes \mathbb{R}_{0,m}$,

$$\int_\Omega [(\varphi(\underline{x})D_h)\psi(\underline{x}) + \varphi(\underline{x})(D_h\psi(\underline{x}))]h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}) = \int_{\partial\Omega} \varphi(\underline{x})h_\kappa^2(\underline{x})\,\mathrm{d}\sigma(\underline{x})\psi(\underline{x}), \tag{4.1}$$

with the vector-valued surface element $\mathrm{d}\sigma_{\underline{x}} = \sum\limits_{i=1}^{m}(-1)^i e_i\,\mathrm{d}x_1 \ldots \widehat{\mathrm{d}x_i} \ldots \mathrm{d}x_m$ *and the volume element* $\mathrm{d}V(\underline{x}) = \mathrm{d}x_1 \ldots \mathrm{d}x_m$.

If we consider a distribution α with compact support and if $f(\underline{x})$, $g(\underline{x}) \in C^\infty(\mathbb{R}^m) \otimes \mathbb{R}_{0,m}$, then

$$\int_{\mathbb{R}^m} [(fD_h)\alpha g + fD_h(\alpha)g + f\alpha(D_hg)]h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}) = 0. \tag{4.2}$$

Thus, we have

$$\int_{\mathbb{R}^m} [(fD_h)\alpha g + f\alpha(D_hg)]h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}) = -\int_{\mathbb{R}^m} fD_h(\alpha)gh_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}), \tag{4.3}$$

which is the most general form of the Stokes formula in Dunkl-Clifford analysis.

Lemma 4.2 (Fermionic Stokes formula, [3]). *For* $f, g \in \Lambda_{2n} \otimes \mathcal{W}_{2n}$ *and* $\alpha \in \Lambda_{2n}$, *the following holds:*

$$-\int_B (f\widehat{\alpha}\partial_{\underline{\grave{x}}})g + \int_B f\alpha(\partial_{\underline{\grave{x}}}g) = \int_B f(\alpha\partial_{\underline{\grave{x}}})g. \tag{4.4}$$

Using Lemmas 4.1 and 4.2, we obtain the Stokes formula in super Dunkl-Clifford analysis as follows.

Theorem 4.3. *Let* $\Omega \subset \mathbb{R}^m$. *If* $f, g \in C^\infty(\Omega)_{m|2n} \otimes \mathcal{C}_{m|2n}$, *then*

$$\int_{\mathbb{R}^{m|2n}} [(f\widehat{\alpha}D)g + f\alpha(Dg)]h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}) = -\int_{\mathbb{R}^{m|2n}} f(\alpha D)gh_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}) \tag{4.5}$$

for $\alpha \in R[x_1, \ldots, x_m] \otimes \Lambda_{2n}$ *a distribution with compact support* $\Sigma \subset \Omega$.

P r o o f. For $\alpha = \beta\gamma$ with $\beta \in R[x_1, \ldots, x_m]$ and $\gamma \in \Lambda_{2n}$, we have (4.5) from (4.3) and Lemma 4.2. □

Corollary 4.4. *Let Σ be a compact oriented differentiable m-dimensional manifold with smooth boundary $\partial\Sigma$. If $f, g \in C^1(\Sigma)_{m|2n} \otimes \mathcal{C}_{m|2n}$, then*

$$\int_\Sigma \int_B [(f\widehat{\beta}D)g + f\beta(Dg)]h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}) = -\int_{\partial\Sigma}\int_B f\beta h_k^2(\underline{x})\,\mathrm{d}\sigma_{\underline{x}} g + \int_\Sigma\int_B f(\beta D_f)gh_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}), \tag{4.6}$$

where $\beta \in \Lambda_{2n}$.

Proof. This is a special case of Theorem 4.3 for $\alpha = H(\nu)\beta$, with $\nu(\underline{x}) > 0$ if $x \in \Sigma$, $\nu(\underline{x}) < 0$ if $\underline{x} \in \mathbb{R}^m \setminus \Sigma$. It is easy to see that (4.6) holds by Lemmas 4.1 and 4.2. □

4.2. A Cauchy-Pompeiu formula for the super Dunkl-Dirac operator. First we introduce the translation operator (see [15])

$$\tau_y f(x) = (V_h)_y(V_h)_x[(V_h)^{-1}(f)(x+y)], \quad x, y \in \mathbb{R}^m, \tag{4.7}$$

where V_h denotes the Dunkl-intertwining operator, i.e.,

$$D_jV_h = V_h\frac{\partial}{\partial x_j}$$

and $V_h(1) = 1$. Then, using this translation operator we have the Dunkl-convolution defined by

$$f *_D g(y) = \int_{\mathbb{R}^m} \tau_y f(-x)g(x)h_\kappa^2(x)\,\mathrm{d}x. \tag{4.8}$$

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^m$ and let $\overline{\Omega}$ be a compact oriented differentiable m-dimensional manifold with smooth boundary $\partial\Omega$. Let $f(x) \in C^\infty(\Omega)_{m|2n} \otimes \mathcal{C}_{m|2n}$ and let the function $K_1^{m|2n}(x)$ be the fundamental solution for the super Dunkl-Dirac operator D. Then*

$$\int_{\partial\Omega}\int_B \tau_y K_1^{m|2n}(-x)h_k^2(\underline{x})\,\mathrm{d}\sigma_{\underline{x}} f(x) + \int_\Omega\int_B \tau_y K_1^{m|2n}(-x)[Df(x)]h_k^2(\underline{x})\,\mathrm{d}V(\underline{x}) = \begin{cases} 0, & \underline{y} \in \mathbb{R}^m \setminus \overline{\Omega}, \\ -f(y), & \underline{y} \in \Omega. \end{cases} \tag{4.9}$$

P r o o f. For $\underline{y} \in \mathbb{R}^m \setminus \overline{\Omega}$, it follows by Corollary 4.4 for $\beta = 1$ that

$$\begin{aligned}\int_{\partial\Omega}\int_B \tau_y K_1^{m|2n}(-x)h_k^2(\underline{x})\,\mathrm{d}\sigma_{\underline{x}} f(x) \\ = -\Big[\int_\Omega\int_B [\tau_y K_1^{m|2n}(-x)D]f(x)h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}) \\ + \int_\Omega\int_B \tau_y K_1^{m|2n}(-x)[Df(x)]h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x})\Big] \\ = -\int_\Omega\int_B \tau_y K_1^{m|2n}(-x)[Df(x)]h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}).\end{aligned}$$

Thus, we have (4.9) for $\underline{y} \in \mathbb{R}^m \setminus \overline{\Omega}$. For $\underline{y} \in \Omega$,

$$\begin{aligned}&\int_{\partial\Omega}\int_B \tau_y K_1^{m|2n}(-x)h_k^2(\underline{x})\,\mathrm{d}\sigma_{\underline{x}} f(x) \\ &= -\Big[\int_\Omega\int_B [\tau_y K_1^{m|2n}(-x)D]f(x)h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}) \\ &\qquad + \int_\Omega\int_B \tau_y K_1^{m|2n}(-x)[Df(x)]h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x})\Big] \\ &= -\int_\Omega\int_B [\tau_y \delta(-x)]f(x)h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}) - \int_\Omega\int_B \tau_y K_1^{m|2n}(-x)[Df(x)]h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}) \\ &= -f(y) - \int_\Omega\int_B \tau_y K_1^{m|2n}(-x)[Df(x)]h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}).\end{aligned}$$

This implies that (4.9) holds for $\underline{y} \in \Omega$. □

4.3. Morera's theorem for super Dunkl-monogenic functions. Applying the Stokes formula in Dunkl-Clifford analysis, we obtain Morera's theorem for Dunkl-monogenic functions as follows.

Lemma 4.6. *A function f is left Dunkl-monogenic in the open set $\Omega \subset \mathbb{R}^m$ if and only if f is continuous in Ω and*

$$\int_{\partial I} h_\kappa^2(\underline{x})\,\mathrm{d}\sigma_{\underline{x}} f = 0 \tag{4.10}$$

for all intervals $I \subset \Omega$.

Furthermore, we have the following lemma, which is an extension of Lemma 4.6.

Lemma 4.7. *Let $I \subset \Omega \subset \mathbb{R}^m$. If $f, g \in C^1(\Omega) \otimes \mathbb{R}_{0,m}$ and*

$$\int_{\partial I} h_\kappa^2(\underline{x})\,\mathrm{d}\sigma_{\underline{x}} f = \int_I g h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}), \tag{4.11}$$

then $D_h f = g$ in Ω.

Proof. As $g \in C^1(\Omega) \otimes \mathbb{R}_{0,m}$, there exists $\varphi \in C^1(\Omega) \otimes \mathbb{R}_{0,m}$ such that $g = D_h\varphi$. Applying Lemma 4.1 and (4.11), we obtain

$$\int_{\partial I} h_\kappa^2(\underline{x})\,\mathrm{d}\sigma_{\underline{x}}[f-\varphi] = \int_{\partial I} h_\kappa^2(\underline{x})\,\mathrm{d}\sigma_{\underline{x}} f - \int_I D_h\varphi h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}) = 0.$$

It follows by Lemma 4.6 that $f - \varphi$ is left Dunkl-monogenic. Thus we have $D_h f = D_h\varphi$. □

In order to obtain our main result in this section, we need the following lemma.

Lemma 4.8 ([3])**.** *Let $p \in \Lambda_{2n}$. If*

$$\int_B pq = 0 \tag{4.12}$$

for any $q \in \Lambda_{2n}$, then $p = 0$.

Theorem 4.9. *Let $\Omega \subset \mathbb{R}^m$. A function $f \in C^0(\Omega)_{m|2n} \otimes \mathcal{C}_{m|2n}$ is super Dunkl-monogenic in Ω if and only if*

$$\int_{\partial I}\int_B \alpha h_\kappa^2(\underline{x})\,\mathrm{d}\sigma_{\underline{x}} f - \int_I\int_B (\alpha D_f) f h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}) = 0 \tag{4.13}$$

for all intervals $I \subset \Omega$ and $\alpha \in \Lambda_{2n}$.

Proof. Suppose that f is super Dunkl-monogenic in Ω. Then (4.13) holds by Corollary 4.4. To the contrary, we suppose that $f \in C^0(\Omega)_{m|2n} \otimes \mathcal{C}_{m|2n}$. Then

$$\int_{\partial I}\int_B \alpha h_k^2(\underline{x})\,\mathrm{d}\sigma_{\underline{x}} f = \int_I\int_B (\alpha D_f) f h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x})$$

for all intervals $I \subset \Omega$ and $\alpha \in \Lambda_{2n}$. Using Lemma 4.2, we get

$$\int_I\int_B (\alpha D_f) f h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}) = \int_I\int_B \alpha(D_f f) h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}).$$

Thus, we have

$$\int_{\partial I}\int_B \alpha h_k^2(\underline{x})\,\mathrm{d}\sigma_{\underline{x}} f = \int_I\int_B \alpha(D_f f) h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}). \tag{4.14}$$

If (4.14) holds for every α, then it follows by Lemma 4.8 that

$$\int_{\partial I} h_k^2(\underline{x})\,\mathrm{d}\sigma_{\underline{x}} f = \int_I D_f f h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}). \tag{4.15}$$

Inspired by De Bie ([6]), we have the full decomposition

$$f = \sum_{k=0}^{n} \sum_{j=0}^{2n-2k} \sum_{l} f_{j,k,l}\, \grave{\underline{x}} M_k^{l,j},$$

where $M_k^{l,j}$ is the space of spherical monogenics of degree k depending on the constants l, j. Thus, (4.15) can be rewritten as

$$\int_{\partial I} h_\kappa^2(\underline{x})\,\mathrm{d}\sigma_{\underline{x}} f_{j-1,k,l} = \int_I f_{j,k,l} h_\kappa^2(\underline{x})\,\mathrm{d}V(\underline{x}), \quad j = 1, \ldots, 2n-2k,\ \forall\, I, \tag{4.16}$$

and

$$\int_{\partial I} h_\kappa^2(\underline{x})\,\mathrm{d}\sigma_{\underline{x}} f_{2n-2k,k,l} = 0, \quad \forall\, I. \tag{4.17}$$

Formula (4.17) implies that $f_{2n-2k,k,l}$ is Dunkl-monogenic in Ω, and also implies that $f_{2n-2k,k,l} \in C^\infty(\Omega) \otimes \mathbb{R}_{0,m}$. Now we proceed by induction (from $j = 2n-2k-1$ to $j = 0$). Suppose that $D_h f_{j,k,l} = f_{j+1,k,l}$ and $f_{j,k,l}$ is Dunkl-polyharmonic in Ω. Thus, using Lemma 4.7 and (4.16), we have $D_h f_{j-1,k,l} = f_{j,k,l}$. It follows that $f_{j-1,k,l}$ is Dunkl-polyharmonic in Ω. Therefore, we obtain that f is differentiable and that

$$Df = -\sum_{k=0}^{n} \sum_{j=0}^{2n-2k-1} \grave{\underline{x}}^j \sum_l M_k^{l,j} D_h f_{j,k,l} + \sum_{k=0}^{n} \sum_{j=1}^{2n-2k} \sum_l \grave{\underline{x}}^{j-1} M_k^{l,j-1} f_{j,k,l} = 0,$$

which implies that f is super Dunkl-monogenic in Ω. □

4.4. Painlevé theorem for super Dunkl-monogenic functions.

Theorem 4.10. *Let Ω be open in $\mathbb{R}^m$ and Ω' be open in $\mathbb{R}^{m-1}$ such that $\Omega \cap \mathbb{R}^m = \Omega'$. Let $f \in C^0(\Omega)_{m|2n} \otimes \mathcal{C}_{m|2n}$. If $f(x)$ is super Dunkl-monogenic in $\Omega \setminus \Omega'$ and moreover continuous in Ω, then $f(x)$ is super Dunkl-monogenic in Ω.*

P r o o f. Since $f(x)$ is super Dunkl-monogenic in $\Omega \setminus \Omega'$, it follows by Theorem 4.9 that

$$\int_{\partial I}\int_B \alpha h_k^2(\underline{x})\,\mathrm{d}\sigma_{\underline{x}} f - \int_I\int_B (\alpha D_f) f h_k^2(\underline{x})\,\mathrm{d}V(\underline{x}) = 0 \tag{4.18}$$

for any closed interval $I \subset \Omega \setminus \Omega'$. Suppose that a closed interval I has the following form: $I = I' \times [0, a_0]$, where I' is a closed interval contained in Ω'.

For $\varepsilon \in [0, a_0]$, we put $I_\varepsilon = I' \times [0, \varepsilon]$. Then we have

$$\int_{\partial I_\varepsilon} \int_B \alpha h_k^2(\underline{x})\, \mathrm{d}\sigma_{\underline{x}} f - \int_{I_\varepsilon} \int_B (\alpha D_f) f h_k^2(\underline{x})\, \mathrm{d}V(\underline{x}) = 0. \tag{4.19}$$

Due to linearity it suffices to prove this theorem for $f(x) = f_1(\underline{x}) f_2(\underline{\grave{x}})$, where f_1 contains only commuting variables and f_2 contains only anti-commuting variables.

Then by the continuity of f, we have

$$\begin{aligned}\int_{\partial I_\varepsilon} \int_B \alpha h_\kappa^2(\underline{x})\, \mathrm{d}\sigma_{\underline{x}} f &= \int_B \alpha \int_{\partial I_\varepsilon} h_\kappa^2(\underline{x})\, \mathrm{d}\sigma_{\underline{x}} f_1(\underline{x}) f_2(\underline{\grave{x}}) \\ &= \int_B \alpha \int_{I'} [f_1(\varepsilon + \underline{x}') - f_1(0 + \underline{x}')] h_\kappa^2(\underline{x})\, \mathrm{d}s f_2(\underline{\grave{x}}) \\ &\quad + \int_{\partial I' \times [0,\varepsilon]} \int_B (\alpha D_f) f h_\kappa^2(\underline{x})\, \mathrm{d}V(\underline{x}),\end{aligned}$$

where $\mathrm{d}s = (-1)^{i-1} e_i\, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}\hat{x}_i \ldots \wedge \mathrm{d}x_m$, $i = 1, 2, \ldots, m$. It follows that

$$\lim_{\varepsilon \to 0^+} \int_{\partial I_\varepsilon} \int_B \alpha h_\kappa^2(\underline{x})\, \mathrm{d}\sigma_{\underline{x}} f = \int_{\partial I'} \int_B \alpha h_k^2(\underline{x})\, \mathrm{d}\sigma_{\underline{x}} f,$$

and

$$\lim_{\varepsilon \to 0^+} \int_{I_\varepsilon} \int_B (\alpha D_f) f h_k^2(\underline{x})\, \mathrm{d}V(\underline{x}) = \int_{I'} \int_B (\alpha D_f) f h_k^2(\underline{x})\, \mathrm{d}V(\underline{x}).$$

Thus, we have

$$\int_{\partial I'} \int_B \alpha h_k^2(\underline{x})\, \mathrm{d}\sigma_{\underline{x}} f - \int_{I'} \int_B (\alpha D_f) f h_k^2(\underline{x})\, \mathrm{d}V(\underline{x}) = 0. \tag{4.20}$$

It is easy to see that (4.20) holds for all $I' \subset \Omega'$. Therefore, we have the result from Theorem 4.9. □

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