THE CLEANNESS OF (SYMBOLIC) POWERS OF STANLEY-REISNER IDEALS

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Abstract. Let Δ be a pure simplicial complex on the vertex set $[n] = \{1,\dots,n\}$ and I_Δ its Stanley-Reisner ideal in the polynomial ring $S = K[x_1,\dots,x_n]$. We show that Δ is a matroid (complete intersection) if and only if $S/I_\Delta^{(m)}$ (S/I_Δ^m) is clean for all $m \in \mathbb{N}$ and this is equivalent to saying that $S/I_\Delta^{(m)}$ (S/I_Δ^m, respectively) is Cohen-Macaulay for all $m \in \mathbb{N}$. By this result, we show that there exists a monomial ideal I with (pretty) cleanness property while S/I^m or $S/I^{(m)}$ is not (pretty) clean for all integer $m \geqslant 3$. If $\dim(\Delta) = 1$, we also prove that $S/I_\Delta^{(2)}$ (S/I_Δ^2) is clean if and only if $S/I_\Delta^{(2)}$ (S/I_Δ^2, respectively) is Cohen-Macaulay.

Keywords: clean; Cohen-Macaulay simplicial complex; complete intersection; matroid; symbolic power

MSC 2010: 13F20, 05E40, 13F55

Introduction

Let Δ be a simplicial complex on the vertex set $[n] = \{1,\dots,n\}$ and $S = k[x_1,\dots,x_n]$ be the polynomial ring in n indeterminates over a field k. The Stanley-Reisner ideal of Δ, I_Δ, is defined by $I_\Delta := \Big(\prod_{i\in F} x_i\colon\ F \notin \Delta\Big)$.

There is a bijection between squarefree monomial ideals I and simplicial complexes. Cohen-Macaulayness (Buchsbaumness, cleanness, generalized Cohen-Macaulayness) of these ideals have been studied by several authors (see [4], [10], [8], [13], [15], [16], [18]). Minh and Trung in [13] and Varbaro in [17] independently proved that Δ is a matroid if and only if $S/I_\Delta^{(m)}$ is Cohen-Macaulay for all $m \in \mathbb{N}$, where $I_\Delta^{(m)}$

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denotes the mth-symbolic power of I_Δ. Later on, Terai and Trung in [16] showed that Δ is a matroid if and only if $S/I_\Delta^{(m)}$ is Cohen-Macaulay for some integer $m \geqslant 3$. The similar characterizations of being Buchsbaum and generalized Cohen-Macaulay were also studied by them. Minh and Trung in [12] proved that for a simplicial complex Δ with $\dim(\Delta) = 1$, $I_\Delta^{(2)}$ is Cohen-Macaulay if and only if $\operatorname{diam}(\Delta) \leqslant 2$, where $\operatorname{diam}(\Delta)$ denotes the diameter of Δ. We pursue this line of research further.

This paper is organized as follows: in Section 1, we collect some preliminaries which will be needed later. In Section 2, we show that if Δ is a matroid, then $S/I_\Delta^{(m)}$ is clean for all $m \in \mathbb{N}$; see Theorem 2.1. Since I_Δ is unmixed, in particular, this shows that $S/I_\Delta^{(m)}$ is Cohen-Macaulay for all $m \in \mathbb{N}$. Therefore this result covers one direction of the result of Minh and Trung in [13] and Varbaro in [17]. Our proof is combinatorial and more elementary than that given in [13]. As our first corollary, by using [16], Theorem 3.6, we show that if Δ is pure and $I = I_\Delta \subset S$, then the following conditions are equivalent:

(a) Δ is a matroid.
(b) $S/I^{(m)}$ is clean for all integers $m > 0$.
(c) $S/I^{(m)}$ is clean for some integer $m \geqslant 3$.
(d) $S/I^{(m)}$ is Cohen-Macaulay for some integer $m \geqslant 3$.
(e) $S/I^{(m)}$ is Cohen-Macaulay for all integers $m > 0$.

Our second corollary asserts that a pure simplicial complex Δ is a complete intersection if and only if S/I_Δ^m is clean for all $m \in \mathbb{N}$ and if and only if S/I_Δ^m is clean for some integer $m \geqslant 3$.

Let $I \subset S$ be a monomial ideal such that S/I is (pretty) clean. It is natural to ask whether S/I^m or $S/I^{(m)}$ is again (pretty) clean for all $m \in \mathbb{N}$? Example 2.5 shows that the answer is negative in general.

In Section 3, we show that if $I \subset S$ is the Stanley-Reisner ideal of a pure simplicial complex Δ with $\dim \Delta = 1$, then for an integer $m > 1$, $S/I^{(m)}$ (S/I^m) is clean if and only if $S/I^{(m)}$ (S/I^m, respectively) is Cohen-Macaulay.

1. Preliminary

A simplicial complex Δ on the vertex set $[n] = \{1, \ldots, n\}$ is a collection of subsets of $[n]$ with the property that if $F \subset G$ and $G \in \Delta$, then $F \in \Delta$. An element of Δ is called a *face*, and the maximal faces of Δ, under inclusion, are called *facets*. We denote by $\mathcal{F}(\Delta)$ the set of facets of Δ. When $\mathcal{F}(\Delta) = \{F_1, \ldots, F_t\}$, we write $\Delta = \langle F_1, \ldots, F_t\rangle$. For each $F \in \Delta$, we set $\dim F := |F| - 1$, and

$$\dim \Delta := \max\{\dim F\colon\ F \in \mathcal{F}(\Delta)\},$$

which is called the dimension of Δ. A simplicial complex Δ is called *pure* if all facets of Δ have the same dimension. According to Björner and Wachs in [3], a simplicial complex Δ is said to be *(non-pure) shellable* if there exists an order $F_1, \dots, F_t$ of the facets of Δ such that for each $2 \leqslant i \leqslant t$, $\langle F_1, \dots, F_{i-1}\rangle \cap \langle F_i\rangle$ is a pure $(\dim F_i - 1)$-dimensional simplicial complex. Such an ordering of facets is called a *shelling*.

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring in n indeterminates over a field K. The Stanley-Reisner ideal of Δ is denoted by I_Δ and defined as $I_\Delta := \Big(\prod_{i\in F} x_i\colon\ F \notin \Delta\Big)$. The facet ideal of Δ is defined as $I(\Delta) := \Big(\prod_{i\in F} x_i\colon\ F \in \mathcal{F}(\Delta)\Big)$.

The *Alexander dual* of Δ is given by $\Delta^\vee := \{F^c\colon\ F \notin \Delta\}$. Let I be a squarefree monomial ideal in S. We denote by $I^\vee$ the squarefree monomial ideal which is minimally generated by all monomials $x_{i_1} \dots x_{i_k}$, where $(x_{i_1}, \dots, x_{i_k})$ is a minimal prime ideal of I. It is easy to see that for any simplicial complex Δ, one has $I_{\Delta^\vee} = (I_\Delta)^\vee$. The complement of a face F is $[n] \setminus F$ and it is denoted by F^c. Also, the complement of a simplicial complex $\Delta = \langle F_1, \dots, F_r\rangle$ is $\Delta^c := \langle F_1^c, \dots, F_r^c\rangle$. It is known that for a simplicial complex Δ one has $I_{\Delta^\vee} = I(\Delta^c)$.

Definition 1.1. A *matroid* Δ is a simplicial complex with the property that for all faces F and G in Δ with $|F| < |G|$, there exists $i \in G \setminus F$ such that $F \cup \{i\} \in \Delta$.

The above definition implies that each matroid is pure. As a consequence of [7], Theorem 12.2.4, a matroid can be characterized by the following exchange property: a pure simplicial complex Δ is a matroid if and only if for any two facets F and G of Δ with $F \neq G$, and for any $i \in F \setminus G$, there exists $j \in G \setminus F$ such that $(F \setminus \{i\}) \cup \{j\} \in \Delta$. A squarefree monomial ideal I in S is called *matroidal* if $I = I(\Delta)$, where Δ is a matroid. On the other hand, by [14], Theorem 2.1.1, Δ is a matroid if and only if Δ^c is a matroid. Altogether, as $I(\Delta^c) = I_{\Delta^\vee}$, we have that Δ is a matroid if and only if $I_{\Delta^\vee}$ is matroidal.

A simplicial complex Δ is called a *complete intersection* if I_Δ is a complete intersection monomial ideal. It is well known that each complete intersection simplicial complex is a matroid.

If $F \subseteq [n]$, then we put $P_F := (x_i\colon\ i \in F)$. We have $I_\Delta = \bigcap_{F\in\mathcal{F}(\Delta^c)} P_F$, hence for each $m \in \mathbb{N}$, the mth-symbolic power of I_Δ is the ideal

$$I_\Delta^{(m)} = \bigcap_{F\in\mathcal{F}(\Delta^c)} P_F^m.$$

An ideal $I \subset S$ is called *normally torsionfree* if $\mathrm{Ass}(S/I^m) \subseteq \mathrm{Ass}(S/I)$ for all $m \in \mathbb{N}$. If I is a squarefree monomial ideal, then I is normally torsionfree if and only if $I^{(m)} = I^m$ for all m; see [7], Theorem 1.4.6.

Let $I \subset S$ be a monomial ideal. A chain of monomial ideals

$$\mathcal{F}\colon\ I = I_0 \subset I_1 \subset \ldots \subset I_r = S$$

is called a *prime filtration* of S/I if for each $i = 1, \ldots, r$ there exists a monomial prime ideal $\mathfrak{p}_i$ of S such that $I_i/I_{i-1} \cong S/\mathfrak{p}_i$. The set of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ which define the cyclic quotients of $\mathcal{F}$ will be denoted by $\operatorname{Supp}\mathcal{F}$. It is known (and easy to see) that

$$\operatorname{Ass} S/I \subseteq \operatorname{Supp}\mathcal{F} \subseteq \operatorname{Supp} S/I.$$

Let $\operatorname{Min} I$ denote the set of minimal prime ideals of $\operatorname{Supp} S/I$. Dress in [5] called a prime filtration $\mathcal{F}$ of S/I *clean* if $\operatorname{Supp}\mathcal{F} = \operatorname{Min} I$ and in [5], Theorem on page 53, proved that a simplicial complex Δ is (non-pure) shellable if and only if $K[\Delta]$ is a clean ring. Pretty clean filtrations were defined as a generalization of clean filtrations by Herzog and Popescu in [8]. A prime filtration $\mathcal{F}$ is called *pretty clean* if for all $i < j$ for which $\mathfrak{p}_i \subseteq \mathfrak{p}_j$, it follows that $\mathfrak{p}_i = \mathfrak{p}_j$. If $\mathcal{F}$ is a pretty clean filtration of S/I, then $\operatorname{Supp}\mathcal{F} = \operatorname{Ass} S/I$; see [8], Corollary 3.4. S/I is called *clean* (*pretty clean*) if it admits a clean (pretty clean) filtration. Obviously, cleanness implies pretty cleanness.

Let $I \subset S$ be a monomial ideal. Then S/I is *sequentially Cohen-Macaulay* if there exist a chain of monomial ideals

$$I = I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_r = S$$

such that each quotient I_i/I_{i-1} is Cohen-Macaulay and

$$\dim(I_1/I_0) < \dim(I_2/I_1) < \ldots < \dim(I_r/I_{r-1}).$$

Clearly, if S/I is Cohen-Macaulay, then it is sequentially Cohen-Macaulay. Also, if S/I is pretty clean, then by [8] it is sequentially Cohen-Macaulay.

The monomial ideal I has *linear quotients* if one can order the set of minimal monomial generators of I, $\mathrm{G}(I) = \{u_1, \ldots, u_m\}$, so that the colon ideal $(u_1, \ldots, u_{i-1}) : u_i$ is generated by a subset of the variables for all $i = 2, \ldots, m$. This means for each $j < i$ there exists a $k < i$ such that $u_k : u_i = x_t$ and $x_t \mid u_j : u_i$, where $t \in [n]$ and $u_k : u_i = u_k/\gcd(u_k, u_i)$. In the case I is squarefree, it is enough to show that for each $j < i$ there exists a $k < i$ such that $u_k : u_i = x_t$ and $x_t \mid u_j$ for some $t \in [n]$.

Let $u = \prod_{i=1}^{n} x_i^{a_i}$ be a monomial in S. Then

$$u^p := \prod_{i=1}^{n}\prod_{j=1}^{a_i} x_{i,j} \in K[x_{1,1}, \ldots, x_{1,a_1}, \ldots, x_{n,1}, \ldots, x_{n,a_n}]$$

is called the *polarization* of u. Let I be a monomial ideal of S with the unique set of minimal monomial generators $\mathrm{G}(I) = \{u_1, \ldots, u_m\}$. Then the ideal $I^p := (u_1^p, \ldots, u_m^p)$ of

$$T := K[x_{i,j} \colon\ i = 1, \ldots, n,\ j = 1, \ldots, a_i]$$

is called the *polarization* of I.

2. Matroids and complete intersection simplicial complexes

We will characterize matroids (complete intersection simplicial complexes) Δ in terms of the cleanness of the symbolic (ordinary) powers of I_Δ.

Theorem 2.1. *Let $I \subset S$ be the Stanley-Reisner ideal of a matroid Δ. Then $S/I^{(m)}$ is clean for all $m \in \mathbb{N}$.*

Proof. Let $I = I_\Delta = \bigcap_{i=1}^{t} P_{F_i}$ be the irredundant irreducible primary decomposition of I, where $\Delta^c = \langle F_1, \ldots, F_t \rangle$ and $r = |F_i|$ for all $i = 1, \ldots, t$. Then $I^{(m)} = \bigcap_{i=1}^{t} P_{F_i}^m$. By [11], Theorem 3.10, it is enough to show that $T/(I^{(m)})^p$ is clean.

One can see by [6], Proposition 2.3 (3), that $((I^{(m)})^p)^\vee = \sum_{i=1}^{r} ((P_{F_i}^m)^p)^\vee$. If $F_i = \{s_1, \ldots, s_r\}$, then by [6], Proposition 2.5 (2), $(P_{F_i}^m)^p$ has the irredundant irreducible primary decomposition

$$(P_{F_i}^m)^p = \bigcap_{\substack{1 \leqslant t_j \leqslant m \\ \sum t_j \leqslant m+r-1}} (x_{s_1,t_1}, \ldots, x_{s_r,t_r}).$$

It follows that the ideal $J := ((I^{(m)})^p)^\vee$ is generated by the monomials

$$x_{i_1,a_1} x_{i_2,a_2} \ldots x_{i_r,a_r} \quad \text{with } \{i_1, \ldots, i_r\} \in \mathcal{F}(\Delta^c),$$

where a_j are positive integers satisfying $1 \leqslant a_j \leqslant m$ and $\sum_{j=1}^{r} a_j \leqslant m + r - 1$. For showing that $T/(I^{(m)})^p$ is clean, it is enough to show that J has linear quotients; see for example [2], Lemma 2.1.

Now, we order the variables in T as follows:

$x_{i,a} > x_{j,b} \Leftrightarrow (i,a) < (j,b)$, and $(i,a) < (j,b)$ if $a < b$, or $a = b$ and $i < j$. Then we show that J has linear quotients with respect to the reverse lexicographical order of its generators induced from the above ordering. Indeed, let $u = x_{i_1,a_1} x_{i_2,a_2} \ldots x_{i_r,a_r}$ and $v = x_{j_1,b_1} x_{j_2,b_2} \ldots x_{j_r,b_r}$ be two monomials in $G(J)$ with $u > v$. We have to show that there exists $w \in G(J)$ with $w > v$ such that $w : v = x_{i_l,a_l}$ and $x_{i_l,a_l} \mid u$.

Since $u > v$, there exists an integer t such that $x_{i_t,a_t} > x_{j_t,b_t}$ and $x_{i_k,a_k} = x_{j_k,b_k}$ for all $k > t$. In particular, we have $a_t < b_t$, or $a_t = b_t$ and $i_t < j_t$. We first claim that there exists $1 \leqslant l \leqslant t$ such that

$$x_{j_1} \dots x_{j_{t-1}} x_{i_l} x_{j_{t+1}} \dots x_{j_r} \in G(I_{\Delta^\vee}) = G(I(\Delta^c)).$$

This is obvious, if $x_{j_t} \mid x_{i_1} x_{i_2} \dots x_{i_t}$, and if $x_{j_t} \nmid x_{i_1} x_{i_2} \dots x_{i_t}$, then, as $I^\vee$ is matroidal, it follows that there exists $1 \leqslant l \leqslant t$ such that $x_{j_1} \dots x_{j_{t-1}} x_{i_l} x_{j_{t+1}} \dots x_{j_r} \in G(I^\vee)$. Here, we used the fact that $i_k = j_k$ for $k = t+1, \dots, r$. Then

$$w := x_{j_1,b_1} x_{j_2,b_2} \dots x_{j_{t-1},b_{t-1}} x_{i_l,a_l} x_{j_{t+1},b_{t+1}} \dots x_{j_r,b_r} \in G(J),$$

because $a_l \leqslant b_t$. Moreover, we have $w : v = x_{i_l,a_l}$ and $x_{i_l,a_l} \mid u$.

Next, we will show that $w > v$. If $x_{i_l,a_l} > x_{j_{t-1},b_{t-1}}$, then $w > v$ because $x_{j_{t-1},b_{t-1}} > x_{j_t,b_t}$. Otherwise, one has $x_{i_l,a_l} < x_{j_{t-1},b_{t-1}}$. We know that $a_t < b_t$, or $a_t = b_t$ and $i_t < j_t$. Since $a_l \leqslant a_t$, if $a_t < b_t$, then $w > v$. Now, assume that $a_t = b_t$ and $i_t < j_t$. Since $a_l < a_t$ or $a_l = a_t$, and $i_l < i_t < j_t$, one has $x_{i_l,a_l} > x_{j_t,b_t}$ and $w > v$. □

We shall use the following lemma.

Lemma 2.2. *Let $I \subset S$ be a monomial ideal. Then S/I is Cohen-Macaulay if and only if S/I is sequentially Cohen-Macaulay and I is unmixed.*

P r o o f. If S/I is Cohen-Macaulay, then it is obvious that S/I is sequentially Cohen-Macaulay and I is unmixed. Conversely, assume that S/I is sequentially Cohen-Macaulay and I is unmixed. Then there exists a chain of monomial ideals

$$I = I_0 \subset I_1 \subset I_2 \subset \dots \subset I_r = S$$

such that each quotient I_i/I_{i-1} is Cohen-Macaulay and

$$\dim(I_1/I_0) < \dim(I_2/I_1) < \dots < \dim(I_r/I_{r-1}).$$

By [9], Lemma 1.2, $\operatorname{depth}(S/I) = \dim(I_1/I_0)$. On the other hand, by [8], Proposition 2.5, $\operatorname{Ass}(S/I) = \bigcup_{i=1}^{r} \operatorname{Ass}(I_i/I_{i-1})$. Since I is unmixed, it follows that $\dim(S/I) = \dim(I_i/I_{i-1})$ for all i. Hence $\operatorname{depth}(S/I) = \dim(I_1/I_0) = \dim(S/I)$, and so S/I is Cohen-Macaulay. □

If we combine our results with [16], Theorem 3.6, we get the following characterization of matroids.

Corollary 2.3. *Let Δ be a pure simplicial complex and $I = I_\Delta \subset S$. Then the following conditions are equivalent:*

(a) *Δ is a matroid.*
(b) *$S/I^{(m)}$ is clean for all integers $m > 0$.*
(c) *$S/I^{(m)}$ is clean for some integer $m \geqslant 3$.*
(d) *$S/I^{(m)}$ is Cohen-Macaulay for some integer $m \geqslant 3$.*
(e) *$S/I^{(m)}$ is Cohen-Macaulay for all integers $m > 0$.*

P r o o f. In view of Theorem 2.1, (a) $\Rightarrow$ (b) holds. The implications (a) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) follow from [16], Theorem 3.6. The implication (b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (d) Suppose that for an integer $m \geqslant 3$, $S/I^{(m)}$ is clean. Then by [8], Corollary 4.3, $S/I^{(m)}$ is sequentially Cohen-Macaulay. On the other hand, $I^{(m)}$ is an unmixed monomial ideal for all m, because I is unmixed and $\mathrm{Ass}(S/I^{(m)}) = \mathrm{Ass}(S/I)$. Hence by Lemma 2.2, $S/I^{(m)}$ is Cohen-Macaulay. □

It is known [1] that a simplicial complex Δ is a complete intersection if and only if S/I_Δ^m is Cohen-Macaulay for all $m \in \mathbb{N}$. Since for a complete intersection monomial ideal I_Δ the symbolic powers coincide with its ordinary powers, we have:

Corollary 2.4. *Let Δ be a pure simplicial complex and $I = I_\Delta \subset S$. Then the following conditions are equivalent:*

(a) *Δ is a complete intersection.*
(b) *S/I^m is clean for all integers $m > 0$.*
(c) *S/I^m is clean for some integer $m \geqslant 3$.*
(d) *S/I^m is Cohen-Macaulay for some integer $m \geqslant 3$.*
(e) *S/I^m is Cohen-Macaulay for all integers $m > 0$.*

P r o o f. The equivalences (a) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) follow from [16], Theorem 4.3. The implication (b) $\Rightarrow$ (c) is obvious. The proof of (c) $\Rightarrow$ (d) is similar to that of the same case in Corollary 2.3. Note that, as S/I^m is clean for some integer $m \geqslant 3$, it follows that

$$\mathrm{Ass}(S/I^m) = \mathrm{Min}(I^m) = \mathrm{Min}(I) = \mathrm{Ass}(S/I).$$

It remains to show (a) $\Rightarrow$ (b). Since I is complete intersection, for any $m > 0$, one has $\mathrm{Ass}(S/I^m) = \mathrm{Min}(I^m) = \mathrm{Min}(I)$. Hence by the definition of symbolic powers (see [18], Definition 3.3.22), $I^m = I^{(m)}$ for all $m > 0$. Since any complete intersection complex is a matroid, therefore by Theorem 2.1, S/I^m is clean for all $m > 0$. □

Example 2.5. Let $I := (x_1x_2, x_2x_3, x_3x_4)$. Obviously, I is an unmixed squarefree monomial ideal. Since $|\operatorname{G}(I)| \leqslant 3$, it follows by [2], Corollary 2.6, that S/I is clean. On the other hand, $I^\vee = (x_1x_3, x_2x_3, x_2x_4)$ is not matroidal. Hence, I is not the Stanley-Reisner ideal of a matroid. So by Corollary 2.3, $S/I^{(m)}$ is not clean for all integers $m \geqslant 3$. Also, S/I is not complete intersection, so by Corollary 2.4 S/I^m is not clean for all integers $m \geqslant 3$. Now, consider the ideal I as the edge ideal of a graph G. Obviously, G is a bipartite graph, so by [7], Corollary 10.3.17, I is normally torsionfree. Therefore for any m,

$$\operatorname{Ass}(S/I^m) = \operatorname{Ass}(S/I) = \operatorname{Min}(I) = \operatorname{Min}(I^m).$$

It follows by [8], Corollary 3.5, that S/I^m is not pretty clean for all integers $m \geqslant 3$.

We note that the above example shows that, if $I \subset S$ is a pretty clean monomial ideal, then necessarily $S/I^{(m)}$ cannot be pretty clean for all integers $m > 0$.

3. Second symbolic power and cleanness

Let Δ be a 1-dimensional simplicial complex and $I = I_\Delta \subset S$. Minh and Trung in [12] studied under which conditions $S/I^{(2)}$ and S/I^2 are Cohen-Macaulay. In this section we will give a characterization for the Cohen-Macaulayness of $S/I^{(2)}$ and S/I^2 in terms of the cleanness property.

Let $G = (V, E)$ be a simple graph. In graph theory, the distance between two vertices u and v of G is the minimal length of paths from u to v and is denoted by $d(u, v)$. This length is infinite if there is no path connecting them. The diameter of G, $\operatorname{diam}(G)$, is defined by $\operatorname{diam}(G) := \max\{d(u, v)\colon u, v \in V\}$.

Theorem 3.1. *Let Δ be a pure simplicial complex on $[n]$ with $\dim \Delta = 1$ and $I = I_\Delta \subset S$. Then the following conditions are equivalent:*

(a) *$S/I^{(2)}$ is clean.*
(b) *$S/I^{(2)}$ is Cohen-Macaulay.*
(c) $\operatorname{diam} \Delta \leqslant 2$.

Proof. (a) $\Rightarrow$ (b) Since $S/I^{(2)}$ is sequentially Cohen-Macaulay and $I^{(2)}$ is unmixed, the desired conclusion follows from Lemma 2.2.

(b) $\Rightarrow$ (c) follows from [12], Theorem 2.3.

(c) $\Rightarrow$ (a) By [11], Theorem 3.10, it is enough to show that $S/(I^{(2)})^p$ is clean. Let $I = I_\Delta = \bigcap\limits_{i=1}^{t} P_{F_i}$ be a primary decomposition of I. Then $\Delta^c = \langle F_1, \ldots, F_t \rangle$ with

$|F_i| = n-2$ for all $i = 1,\dots,t$. We know that

$$(I^{(2)})^p = \bigcap_{i=1}^{t} (P_{F_i}^2)^p.$$

If $F \subset [n]$, then by [6], Proposition 2.5 (2),

$$(P_F^2)^p = \bigcap_{1\leqslant j\leqslant n-2} P_{(F,2_j)} \cap P_{(F,1)},$$

where if $F = \{r_1,\dots,r_{n-2}\}$ with $r_1 < r_2 < \dots < r_{n-2}$, then we set $(F,1) := \{(r_i,1)\colon r_i \in F\}$ and $(F,2_j) := \{(r_j,2)\} \cup \{(r_i,1)\colon 1 \leqslant i \leqslant n-2,\ i \neq j\}$. Note that $(I^{(2)})^p$ is a monomial ideal in a polynomial ring $T = K[x_{(1,1)},\dots,x_{(n,1)},x_{(1,2)},\dots,x_{(n,2)}]$. Since $(I^{(2)})^p$ is the Stanley-Reisner ideal of the simplicial complex

$$\Gamma = \langle (F_i,1)^c, (F_i,2_j)^c\colon 1 \leqslant i \leqslant t,\ 1 \leqslant j \leqslant n-2\rangle,$$

by a result of Dress in [5] it is enough to prove that Γ is shellable.

We set $A_0 := \emptyset$ and $A_i := \left\{F_j^c \in \mathcal{F}(\Delta)\colon i \in F_j^c \text{ and } F_j^c \notin \bigcup_{s=1}^{i-1} A_s\right\}$ for all $i = 1,\dots,n$. Note that $\mathcal{F}(\Delta) = \bigcup_{i=1}^{n} A_i$. We order the facets of Γ by the following process and show that the given order is a shelling order. For the convenience we can assume that $A_1 = \{F_1^c,\dots,F_{s_1}^c\}$ for some $1 \leqslant s_1 \leqslant t$. Let the initial part of our order be

$$(*)\qquad (F_1,1)^c,(F_1,2_1)^c,\dots,(F_1,2_{n-2})^c,(F_2,1)^c,(F_2,2_1)^c,\dots,(F_2,2_{n-2})^c,\dots,$$
$$(F_{s_1},1)^c,(F_{s_1},2_1)^c,\dots,(F_{s_1},2_{n-2})^c.$$

Then the following inequalities hold:

$$\begin{aligned} n &= |(F_1,1)^c \cap (F_1,2_j)^c| - 1 = \dim(\langle (F_1,1)^c\rangle \cap \langle (F_1,2_j)^c\rangle) \\ &\leqslant \dim(\langle (F_1,1)^c,(F_1,2_1)^c,\dots,(F_1,2_{j-1})^c\rangle \cap \langle (F_1,2_j)^c\rangle) \\ &\leqslant \dim\langle (F_1,2_j)^c\rangle - 1 = |(F_1,2_j)^c| - 2 = n. \end{aligned}$$

Now, let $2 \leqslant d \leqslant s_1$. Then

$$\begin{aligned} n &= \dim(\langle (F_1,1)^c\rangle \cap \langle (F_d,1)^c\rangle) \\ &\leqslant \dim(\langle (F_1,1)^c,(F_1,2_1)^c,\dots,(F_{d-1},2_{n-2})^c\rangle \cap \langle (F_d,1)^c\rangle) \\ &\leqslant \dim\langle (F_d,1)^c\rangle - 1 = n. \end{aligned}$$

Also, for any $1 \leqslant j \leqslant n-2$, we have

$$\begin{aligned} n &= \dim(\langle (F_d,1)^c\rangle \cap \langle (F_d,2_j)^c\rangle) \\ &\leqslant \dim(\langle (F_1,1)^c,(F_1,2_1)^c,\ldots,(F_d,1)^c,\ldots,(F_d,2_{j-1})^c\rangle \cap \langle (F_d,2_j)^c\rangle) \\ &\leqslant \dim\langle (F_d,2_j)^c\rangle - 1 = n. \end{aligned}$$

Suppose that Γ_1 is a simplicial complex whose facets are all of the sets belonging to $(*)$. If we rename the facets of Γ_1 in the same order as above by $G_1,\ldots,G_{s_1(n-1)}$, then it is easy to see that $\langle G_1,\ldots,G_{i-1}\rangle \cap \langle G_i\rangle$ is a pure simplicial complex for all $i = 1,\ldots,s_1(n-1)$. Therefore, Γ_1 is shellable.

Assume that $A_i = \{F^c_{s_{i-1}+1},\ldots,F^c_{s_i}\}$ for $1 \leqslant i \leqslant h-1 < n$, where $s_0 = 0$ and $s_{i-1} < s_i$. Then we may assume by induction process that the following order is a shelling order for the simplicial complex with the set of facets

$$\begin{gathered} (F_1,1)^c,(F_1,2_1)^c,\ldots,(F_1,2_{n-2})^c,\ldots,(F_j,1)^c,(F_j,2_1)^c,\ldots,(F_j,2_{n-2})^c, \\ (F_{j+1},1)^c,(F_{j+1},2_1)^c,\ldots,(F_{j+1},2_{n-2})^c,\ldots, \\ (F_{s_{h-1}},1)^c,(F_{s_{h-1}},2_1)^c,\ldots,(F_{s_{h-1}},2_{n-2})^c, \end{gathered}$$

where $1 < j < s_{h-1}$.

Now, let $1 < h \leqslant n$. If there exists $F^c \in \bigcup_{i=1}^{h-1} A_i$ such that $h \in F^c$, then we take an arbitrary element G of A_h and set $F^c_{s_{h-1}+1} := G$. In this case, we have

$$\begin{aligned} n &= \dim(\langle (F,1)^c\rangle \cap \langle (F_{s_{h-1}+1},1)^c\rangle) \\ &\leqslant \dim(\langle (F_1,1)^c,(F_1,2_1)^c,\ldots,(F_{s_{h-1}},2_{n-2})^c\rangle \cap \langle (F_{s_{h-1}+1},1)^c\rangle) \\ &\leqslant \dim\langle (F_{s_{h-1}+1},1)^c)\rangle - 1 = n. \end{aligned}$$

Otherwise, for any $F^c \in \bigcup_{i=1}^{h-1} A_i$, $h \notin F^c$. Hence $\{1,h\} \notin \mathcal{F}(\Delta)$. Since $\operatorname{diam}(\Delta) \leqslant 2$, it follows that there exists $m \in [n]$ such that $m \neq 1$, $m \neq h$ and $\{m,h\} \in A_h$, and $F^c := \{1,m\} \in A_1$. In this case we set $F^c_{s_{h-1}+1} := \{m,h\}$.

Now, the following inequalities hold:

$$\begin{aligned} n &= \dim(\langle (F,1)^c\rangle \cap \langle (F_{s_{h-1}+1},1)^c\rangle) \\ &\leqslant \dim(\langle (F_1,1)^c,(F_1,2_1)^c,\ldots,(F_{s_{h-1}},2_{n-2})^c\rangle \cap \langle (F_{s_{h-1}+1},1)^c\rangle) \\ &\leqslant \dim\langle (F_{s_{h-1}+1},1)^c\rangle - 1 = n. \end{aligned}$$

We order all the other facets of Γ which correspond to A_h as

$$(F_{s_{h-1}+1},2_1)^c,\ldots,(F_{s_{h-1}+1},2_{n-2})^c,\ldots,(F_{s_h},1)^c,(F_{s_h},2_1)^c,\ldots,(F_{s_h},2_{n-2})^c,$$

where $s_{h-1} < s_h$.

In the same way as previously, we can easily check that the given order is a shelling order. □

A 1-dimensional simplicial complex Δ on the vertex set $[n]$ is called a cycle of length n if the facets of Δ are $\{1,n\}$ and $\{i,i+1\}$ for all $i=1,\dots,n-1$.

Corollary 3.2. *Let Δ be a pure simplicial complex on $[n]$ with $\dim\Delta=1$ and $I=I_\Delta\subset S$. Then the following conditions are equivalent:*

(a) *S/I^2 is clean.*
(b) *S/I^2 is Cohen-Macaulay.*
(c) *Δ is a path of length $1,2$ or a cycle of length $3,4,5$.*

Proof. (a) $\Rightarrow$ (b) Since S/I^2 is sequentially Cohen-Macaulay and I^2 is unmixed, the desired conclusion follows from Lemma 2.2.

(b) $\Rightarrow$ (c) If $n=2$, then Δ is a path of length 1. If $n=3$, then Δ is either a path of length 2 or a triangle (a cycle of length 3). Finally, if $n\geqslant 4$, then by [12], Corollary 3.4, Δ is a cycle of length 4 or 5.

(c) $\Rightarrow$ (a) It is easy to see that in each case, we have $\operatorname{diam}\Delta\leqslant 2$ and $I^{(2)}=I^2$. Hence the desired conclusion follows by Theorem 3.1. □

It is known that if I is a monomial ideal and S/I is clean, then S/I is sequentially Cohen-Macaulay. In particular when I is unmixed, then S/I is Cohen-Macaulay. But the converse is not true in general. In some special cases, like edge ideals of unmixed bipartite graphs, it is known that Cohen-Macaulayness and cleanness are equivalent. As another corollary of our results we get the following:

Corollary 3.3. *Let $m>1$ be an integer, Δ a pure simplicial complex with $\dim\Delta=1$, and $I=I_\Delta\subset S$. Then $S/I^{(m)}$ (S/I^m) is clean if and only if $S/I^{(m)}$ (S/I^m, respectively) is Cohen-Macaulay.*

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