

DEPTH AND STANLEY DEPTH OF THE FACET IDEALS
OF SOME CLASSES OF SIMPLICIAL COMPLEXES

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Received April 7, 2016. First published August 8, 2017.

Abstract. Let $\Delta_{n,d}$ (resp. $\Delta'_{n,d}$) be the simplicial complex and the facet ideal $I_{n,d} = (x_1 \dots x_d, x_{d-k+1} \dots x_{2d-k}, \dots, x_{n-d+1} \dots x_n)$ (resp. $J_{n,d} = (x_1 \dots x_d, x_{d-k+1} \dots x_{2d-k}, \dots, x_{n-2d+2k+1} \dots x_{n-d+2k}, x_{n-d+k+1} \dots x_n x_1 \dots x_k)$). When $d \geq 2k + 1$, we give the exact formulas to compute the depth and Stanley depth of quotient rings $S/J_{n,d}$ and $S/I_{n,d}^t$ for all $t \geq 1$. When $d = 2k$, we compute the depth and Stanley depth of quotient rings $S/J_{n,d}$ and $S/I_{n,d}$, and give lower bounds for the depth and Stanley depth of quotient rings $S/I_{n,d}^t$ for all $t \geq 1$.

Keywords: monomial ideal; facet ideal; depth; Stanley depth

MSC 2010: 13C15, 13P10, 13F20, 13F55

1. INTRODUCTION

Let K be a field and $S = K[x_1, \dots, x_n]$ the polynomial ring over K in n variables. Let M be a finitely generated \mathbb{Z}^n -graded S -module. A Stanley decomposition \mathcal{D} of M is a finite direct sum of K -vector spaces

$$\mathcal{D}: M = \bigoplus_{i=1}^r u_i K[Z_i],$$

where $u_i \in M$ is homogeneous and $Z_i \subseteq \{x_1, \dots, x_n\}$, $i = 1, \dots, r$, and its Stanley depth, $\text{sdepth}(\mathcal{D})$, is defined as $\min\{|Z_i|: i = 1, \dots, r\}$. The number

$$\text{sdepth}(M) = \max\{\text{sdepth}(\mathcal{D}): \mathcal{D} \text{ is a Stanley decomposition of } M\}$$

is called the Stanley depth of M .

This work was supported by the Natural Science Foundation of Jiangsu Province (No. BK20140300), the National Natural Science Foundation of China (No. 11501397 and No. 11471234) and the Jiangsu Government Scholarship for Overseas Studies.

Stanley conjectured in [13] that $\text{sdepth}(M) \geq \text{depth}(M)$ for any \mathbb{Z}^n -graded S -module M . There are many researches on this conjecture, especially when M has the form S/I or I with I a squarefree monomial ideal of S , e.g., [1], [8], [11], [12]. In [6], Duval et al. constructed an explicit counterexample to disprove the Stanley conjecture for S/I , where I is a monomial ideal of S . Thus the Stanley conjecture is open for monomial ideals $I \subset S$.

Let $\Delta \subset 2^{[n]}$ be a simplicial complex. Each element of Δ is called a face of Δ , and a face F is called a facet if F is a maximal face with respect to inclusion. Let $\mathcal{F}(\Delta)$ denote the set of facets of Δ . If $F \in \mathcal{F}(\Delta)$, we denote $x_F = \prod_{j \in F} x_j$. Then the facet ideal of Δ is a squarefree monomial ideal $I(\Delta)$ of S , $I(\Delta) = (x_F : F \in \mathcal{F}(\Delta))$. The facet ideal was studied by Faridi in [7] from the depth perspective. In this paper, we consider depth and Stanley depth of $I(\Delta)$ of some classes of simplicial complexes.

A line graph of length n , denoted by L_n , is a graph with the vertex set $V = [n]$ and edge set $E = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}\}$. The depth and Stanley depth of the edge ideal associated to L_n (which is in fact the facet ideal of L_n) were computed by Morey in [9] and Ştefan in [14], respectively. A cyclic graph C_n is a graph with the vertex set $V = [n]$ and edge set $E \cup \{n, 1\}$. The depth and Stanley depth of the edge ideal associated to C_n were computed by Cimpoeaş in [5].

Let $\Delta_{n,d}$ be the simplicial complex with the set of facets $\mathcal{F}(\Delta_{n,d}) = \{\{1, 2, \dots, d\}, \{d-k+1, d-k+2, \dots, 2d-k\}, \dots, \{n-2d+k+1, n-2d+k+2, \dots, n-d+k\}, \{n-d+1, n-d+2, \dots, n\}\}$, where $n \geq d > k \geq 1$. It is easy to see that $d-k \mid n-k$. We denote the facet ideal $I(\Delta_{n,d})$ of $\Delta_{n,d}$ by $I_{n,d}$, where $I_{n,d} = (x_1 \dots x_d, x_{d-k+1} \dots x_{2d-k}, \dots, x_{n-d+1} \dots x_n)$. When $d=2$ and $k=1$, then $I_{n,d} = I(L_n)$.

Let $\Delta'_{n,d}$ be the simplicial complex with the set of facets $\mathcal{F}(\Delta'_{n,d}) = \{\{1, 2, \dots, d\}, \{d-k+1, d-k+2, \dots, 2d-k\}, \dots, \{n-2d+2k+1, n-2d+2k+2, \dots, n-d+2k\}, \{n-d+k+1, \dots, n, 1, \dots, k\}\}$, where $d \geq 2k \geq 2$ and $n \geq 3d-3k$. It is easy to see that $d-k \mid n$. We denote the facet ideal $I(\Delta'_{n,d})$ of $\Delta'_{n,d}$ by $J_{n,d}$, where $J_{n,d} = (x_1 \dots x_d, x_{d-k+1} \dots x_{2d-k}, \dots, x_{n-2d+2k+1} \dots x_{n-d+2k}, x_{n-d+k+1} \dots x_n x_1 \dots x_k)$. If $d=2$ and $k=1$, then $J_{n,d} = I(C_n)$.

The followings are our main results, which generalize some results of [5], [9], [14].

Theorem 1.1. *Let $d \geq 2k+1$. Then*

- (1) $\text{sdepth}(S/I_{n,d}^t) = \text{depth}(S/I_{n,d}^t) = n - \frac{n-k}{d-k}$ for all $t \geq 1$,
- (2) $\text{sdepth}(S/J_{n,d}) = \text{depth}(S/J_{n,d}) = n - \frac{n}{d-k}$.

Theorem 1.2. *Let $d = 2k$. Then*

- (1) $\text{sdepth}(S/I_{n,d}) = \text{depth}(S/I_{n,d}) = \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$,

- (2) $\text{depth}(S/I_{n,d}^t) \geq \max \left\{ 1, \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil \right\}$ for all $t \geq 1$, $\text{sdepth}(S/I_{n,d}^t) \geq \max \left\{ 1, \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil \right\}$ for all $t \geq 1$,
- (3) $\text{depth}(S/J_{n,d}) = \frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil$,
 $\text{sdepth}(S/J_{n,d}) = \frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil$ for $\frac{n}{k} \equiv 0 \pmod{3}$ and $\frac{n}{k} \equiv 2 \pmod{3}$,
 $\frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil \leq \text{sdepth}(S/J_{n,d}) \leq \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$ for $\frac{n}{k} \equiv 1 \pmod{3}$.

2. DEPTH AND STANLEY DEPTH OF THE FACET IDEALS

First, we recall a well-known result, referred to as the Depth lemma, that will be heavily used in the proofs in this article. Two different versions of the lemma will be used in this article, so both are stated here for ease of reference.

Lemma 2.1 (Depth lemma). *Let S be a local ring or a Noetherian graded ring with S_0 local. If*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is a short exact sequence of finitely generated S -modules, where the maps are all homogeneous, then ([15], Lemma 1.3.9):

- a) *If $\text{depth}(B) < \text{depth}(C)$, then $\text{depth}(A) = \text{depth}(B)$.*
- b) *If $\text{depth}(B) = \text{depth}(C)$, then $\text{depth}(A) \geq \text{depth}(B)$.*
- c) *If $\text{depth}(B) > \text{depth}(C)$, then $\text{depth}(A) = \text{depth}(C) + 1$.*

Also (see [3], Proposition 1.2.9):

- d) $\text{depth}(A) \geq \min\{\text{depth}(B), \text{depth}(C) + 1\}$.
- e) $\text{depth}(B) \geq \min\{\text{depth}(A), \text{depth}(C)\}$.
- f) $\text{depth}(C) \geq \min\{\text{depth}(A) - 1, \text{depth}(B)\}$.

In [12], Rauf proved the analog of Lemma 2.1 (e) for sdepth :

Lemma 2.2. *Let $0 \rightarrow U \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of finitely generated \mathbb{Z}^n -graded S -modules. Then*

$$\text{sdepth}(M) \geq \min\{\text{sdepth}(U), \text{sdepth}(N)\}.$$

Next, we will discuss our main results in two cases.

2.1. The case $d \geq 2k + 1$. Let $I \subset S$ be a monomial ideal. The big height of I , denoted by $\text{bight}(I)$, is the maximum height of the minimal prime ideals of I . The arithmetical rank of I , denoted by $\text{ara}(I)$, is the minimum number r of elements of S such that the ideal (a_1, a_2, \dots, a_r) has the same radical as I . It is well-known that

$$\text{ht}(I) \leq \text{bight}(I) \leq \text{pd}(S/I) \leq \text{ara}(I) \leq |G(I)|,$$

where $\text{pd}(S/I)$ denotes the projective dimension of S/I and $G(I)$ denotes the set of minimal monomial generators of I . We see that $|G(I_{n,d})| = \frac{n-k}{d-k}$ and $|G(J_{n,d})| = \frac{n}{d-k}$.

A prime ideal P is associated to I if $P = (I : c)$ for some monomial $c \in S$. The set of prime ideals associated to I will be denoted by $\text{Ass}(S/I)$. The associated prime ideals of a monomial ideal are monomial prime ideals. The set $\text{Min}(S/I)$ consists of all prime ideals that are minimal over I with respect to inclusion. It is known that $\text{Min}(S/I) \subset \text{Ass}(S/I)$. When I is squarefree, $\text{Ass}(S/I) = \text{Min}(S/I)$.

Our proofs of the main results make heavy use of the following lemma.

Lemma 2.3. $P = (x_{k+1}, x_{k+1+(d-k)}, x_{k+1+2(d-k)}, \dots, x_{n-d+k+1}) \in \text{Min}(S/I_{n,d})$, and $P' = (x_{k+1}, x_{k+1+(d-k)}, x_{k+1+2(d-k)}, \dots, x_{n-d+2k+1}) \in \text{Min}(S/J_{n,d})$.

Proof. Let we have $a_i = x_{1+(i-1)(d-k)}x_{2+(i-1)(d-k)} \dots x_{d+(i-1)(d-k)}$ and $b_j = x_{k+1+(j-1)(d-k)}$, where $i, j = 1, 2, \dots, \frac{n-k}{d-k}$. Then $I_{n,d} = (a_1, a_2, \dots, a_{(n-k)/(d-k)})$ and $P = (b_1, b_2, \dots, b_{(n-k)/(d-k)})$. It is easy to see that b_j divides a_i if and only if $i = j$, so $I_{n,d} \subset P$. We assume that P is not minimal over $I_{n,d}$. Let $P_0 \subsetneq P$ be a minimal prime ideal of $I_{n,d}$. Since $I_{n,d}$ is squarefree, $P_0 \subsetneq P$ is a monomial prime ideal, and there exists a_i such that none of $G(P_0)$ divides a_i . Hence $I_{n,d} \not\subset P_0$, a contradiction. Similarly, P' is a minimal prime ideal of $S/J_{n,d}$. \square

Proposition 2.4. $\text{bight}(I_{n,d}) = \text{pd}(S/I_{n,d}) = \text{ara}(I_{n,d}) = |G(I_{n,d})| = \frac{n-k}{d-k}$.

Proof. Let we have $P = (x_{k+1}, x_{k+1+(d-k)}, x_{k+1+2(d-k)}, \dots, x_{n-d+k+1}) \in \text{Min}(S/I_{n,d})$ and $\text{ht}(P) = \frac{n-k}{d-k}$ by Lemma 2.3. Then $\frac{n-k}{d-k} \leq \text{bight}(I_{n,d}) \leq \text{pd}(S/I_{n,d}) \leq \text{ara}(I_{n,d}) \leq |G(I_{n,d})| = \frac{n-k}{d-k}$. Now the result is clear. \square

Now, we give the exact formulas for $\text{sdepth}(S/I_{n,d})$ and $\text{depth}(S/I_{n,d})$.

Theorem 2.5. $\text{sdepth}(S/I_{n,d}) = \text{depth}(S/I_{n,d}) = n - \frac{n-k}{d-k}$.

Proof. Since $|G(I_{n,d})| = \frac{n-k}{d-k}$, by [4], Proposition 1.2, we have $\text{sdepth}(S/I_{n,d}) \geq n - \frac{n-k}{d-k}$. On the other hand, there exists a prime ideal $P \in \text{Ass}(S/I_{n,d})$ such that $\text{ht}(P) = \frac{n-k}{d-k}$ by Lemma 2.3. Then $\text{sdepth}(S/I_{n,d}) \leq n - \frac{n-k}{d-k}$ by [8], Proposition 1.3. By the Auslander-Buchsbaum formula and Proposition 2.4, we get $\text{depth}(S/I_{n,d}) = n - \text{pd}(S/I_{n,d}) = n - \frac{n-k}{d-k}$. \square

The following corollary states that the Stanley inequality holds for $I_{n,d}$.

Corollary 2.6. $\text{sdepth}(I_{n,d}) \geq n - \lfloor \frac{n-k}{2(d-k)} \rfloor \geq \text{depth}(I_{n,d})$.

Proof. Since $|G(I_{n,d})| = \frac{n-k}{d-k}$, it follows that $\text{sdepth}(I_{n,d}) \geq \max\{1, n - \lfloor \frac{1}{2}|G(I_{n,d})| \rfloor\} = n - \lfloor \frac{n-k}{2(d-k)} \rfloor \geq n - \frac{n-k}{d-k} + 1 = \text{depth}(I_{n,d})$ by [10], Theorem 2.3, and Theorem 2.5. \square

Next, we present a main result.

Theorem 2.7. For all $t \geq 1$, $\text{sdepth}(S/I_{n,d}^t) = \text{depth}(S/I_{n,d}^t) = n - \frac{n-k}{d-k}$.

Proof. We use induction on n and t . If $n = d$, then $I_{n,d}^t = (x_1^t \dots x_d^t)$ is principal. It follows that $\text{sdepth}(S/I_{n,d}^t) = \text{depth}(S/I_{n,d}^t) = d - 1 = d - \frac{d-k}{d-k}$ for all $t \geq 1$. Assume that $n \geq 2d - k$ in the following.

If $t = 1$, the result holds for all n by Theorem 2.5. Now let $t \geq 2$. We denote $u := x_{n-d+1} \dots x_{n-d+k}$, $v := x_{n-d+k+1} \dots x_n$ and consider the short exact sequence

$$0 \rightarrow \frac{S}{(I_{n,d}^t : uv)} \rightarrow \frac{S}{(I_{n,d}^t : v)} \rightarrow \frac{S}{((I_{n,d}^t : v), u)} \rightarrow 0.$$

Let $G(I_{n,d}) = \{a_1, a_2, \dots, a_{(n-k)/(d-k)}\}$, the same as in the proof of Lemma 2.3, and $w \in G(I_{n,d}^t)$. If $a_{(n-k)/(d-k)} \mid w$, then $\frac{w}{a_{(n-k)/(d-k)}} \in G(I_{n,d}^t : uv) \cap I_{n,d}^{t-1}$. If $a_{(n-k)/(d-k)} \nmid w$ and $a_{(n-d)/(d-k)} \mid w$, then $\frac{w}{u} \in G(I_{n,d}^t : uv)$ and $\frac{w}{a_{(n-d)/(d-k)}} \mid \frac{w}{u}$, where $\frac{w}{a_{(n-d)/(d-k)}} \in I_{n,d}^{t-1}$. If $a_{(n-k)/(d-k)} \nmid w$ and $a_{(n-d)/(d-k)} \nmid w$, then $\frac{w}{1} \in G(I_{n,d}^t : uv)$ and w must be divisible by some element of $I_{n,d}^{t-1}$. Hence $(I_{n,d}^t : uv) \subseteq I_{n,d}^{t-1}$. It follows that $(I_{n,d}^t : uv) = I_{n,d}^{t-1}$.

We get $\text{sdepth}(S/(I_{n,d}^t : uv)) = \text{sdepth}(S/I_{n,d}^{t-1}) = n - \frac{n-k}{d-k}$ by induction hypothesis on t . Similarly, we prove that $\text{depth}(S/(I_{n,d}^t : uv)) = n - \frac{n-k}{d-k}$.

Since u divides any element of $G(I_{n,d}^t)$ which is divided by $a_{(n-k)/(d-k)}$ or $a_{(n-d)/(d-k)}$, we get $((I_{n,d}^t : v), u) = (I_{n-2d+2k,d}^t S, u)$. Notice that u is regular on $S/I_{n-2d+2k,d}^t S$, hence the induction hypothesis on n and [8], Lemma 3.6, imply that

$$\begin{aligned} \text{sdepth}_S(S/((I_{n,d}^t : v), u)) &= \text{sdepth}_{S_{n-2d+2k}}\left(\frac{S_{n-2d+2k}}{I_{n-2d+2k,d}^t}\right) + (2d - 2k) - 1 \\ &= (n - 2d + 2k) - \frac{(n - 2d + 2k) - k}{d - k} + 2d - 2k - 1 \\ &= n - \frac{n - k}{d - k} + 1, \end{aligned}$$

where $S_{n-2d+2k} = K[x_1, \dots, x_{n-2d+2k}]$. Similarly, $\text{depth}(S/((I_{n,d}^t : v), u)) = n - \frac{n-k}{d-k} + 1$. Then we have $\text{sdepth}(S/(I_{n,d}^t : v)) \geq n - \frac{n-k}{d-k}$ and $\text{depth}(S/(I_{n,d}^t : v)) = n - \frac{n-k}{d-k}$ by Lemma 2.1 and Lemma 2.2.

Since v divides any element of $G(I_{n,d}^t)$ which is divided by $a_{(n-k)/(d-k)}$, $(I_{n,d}^t, v) = (I_{n-d+k,d}^t S, v)$. Noting that v is regular on $S/I_{n-d+k,d}^t S$, by induction hypothesis

on n and [8], Lemma 3.6, we get

$$\begin{aligned} \text{sdepth}_S(S/(I_{n,d}^t, v)) &= \text{sdepth}_{S_{n-d+k}}\left(\frac{S_{n-d+k}}{I_{n-d+k,d}^t}\right) + (d-k) - 1 \\ &= (n-d+k) - \frac{(n-d+k)-k}{d-k} + d-k-1 \\ &= n - \frac{n-k}{d-k}, \end{aligned}$$

where $S_{n-d+k} = K[x_1, \dots, x_{n-d+k}]$. Similarly, we get $\text{depth}(S/(I_{n,d}^t, v)) = n - \frac{n-k}{d-k}$. By applying Lemmas 2.1 and 2.2 to the short exact sequence

$$0 \rightarrow \frac{S}{(I_{n,d}^t : v)} \rightarrow \frac{S}{I_{n,d}^t} \rightarrow \frac{S}{(I_{n,d}^t, v)} \rightarrow 0,$$

we obtain $\text{sdepth}(S/I_{n,d}^t) \geq n - \frac{n-k}{d-k}$ and $\text{depth}(S/I_{n,d}^t) = n - \frac{n-k}{d-k}$.

From Lemma 2.3, $P = (x_{k+1}, x_{k+1+(d-k)}, \dots, x_{n-d+k+1}) \in \text{Min}(S/I_{n,d}) = \text{Min}(S/I_{n,d}^t) \subseteq \text{Ass}(S/I_{n,d}^t)$ for all $t \geq 1$, and $\text{ht}(P) = \frac{n-k}{d-k}$. Then $\text{sdepth}(S/I_{n,d}^t) \leq \dim(S/P) = n - \frac{n-k}{d-k}$ by [8], Proposition 1.3. This completes the proof. \square

Proposition 2.8. $\text{bight}(J_{n,d}) = \text{pd}(S/J_{n,d}) = \text{ara}(J_{n,d}) = |G(J_{n,d})| = \frac{n}{d-k}$.

Proof. We have $P = (x_{k+1}, x_{k+1+(d-k)}, x_{k+1+2(d-k)}, \dots, x_{n-d+2k+1}) \in \text{Min}(S/J_{n,d})$ and $\text{ht}(P) = \frac{n}{d-k}$ by Lemma 2.3. Then $\frac{n}{d-k} \leq \text{bight}(J_{n,d}) \leq \text{pd}(S/J_{n,d}) \leq \text{ara}(J_{n,d}) \leq |G(J_{n,d})| = \frac{n}{d-k}$. The proof is completed. \square

Theorem 2.9. $\text{sdepth}(S/J_{n,d}) = \text{depth}(S/J_{n,d}) = n - \frac{n}{d-k}$.

Proof. Since $|G(J_{n,d})| = \frac{n}{d-k}$, by [4], Proposition 1.2, we have $\text{sdepth}(S/J_{n,d}) \geq n - \frac{n}{d-k}$. On the other hand, there exists a prime ideal $P \in \text{Ass}(S/J_{n,d})$ such that $\text{ht}(P) = \frac{n}{d-k}$ by Lemma 2.3. Then $\text{sdepth}(S/J_{n,d}) \leq n - \frac{n}{d-k}$ by [8], Proposition 1.3. By the Auslander-Buchsbaum formula and Proposition 2.8, we obtain $\text{depth}(S/J_{n,d}) = n - \text{pd}(S/J_{n,d}) = n - \frac{n}{d-k}$. \square

The following corollary implies that the Stanley inequality holds for $J_{n,d}$.

Corollary 2.10. $\text{sdepth}(J_{n,d}) \geq n - \lfloor \frac{n}{2(d-k)} \rfloor > \text{depth}(J_{n,d})$.

Proof. Since $|G(J_{n,d})| = \frac{n}{d-k}$, we have $\text{sdepth}(J_{n,d}) \geq \max\{1, n - \lfloor \frac{1}{2}|G(J_{n,d})| \rfloor\} = n - \lfloor \frac{n}{2(d-k)} \rfloor > n - \frac{n}{d-k} + 1 = \text{depth}(J_{n,d})$ by [10], Theorem 2.3, and Theorem 2.9. \square

2.2. The case $d = 2k$. Let $\mathcal{P} \subset 2^{[n]}$ be a poset. If $F, G \subset [n]$, the interval $[F, G]$ consists of all subsets X of $[n]$ such that $F \subset X \subset G$. Let $\mathbf{P} : \mathcal{P} = \bigcup_{i=1}^r [F_i, G_i]$ be a partition of \mathcal{P} , i.e. $[F_i, G_i] \cap [F_j, G_j] = \emptyset$ for all $i \neq j$. We denote $\text{sdepth}(\mathbf{P}) = \min_{i \in [r]} \{|G_i|\}$. Also, we define the Stanley depth of \mathcal{P} to be the number

$$\text{sdepth}(\mathcal{P}) = \max\{\text{sdepth}(\mathbf{P}) : \mathbf{P} \text{ is a partition of } \mathcal{P}\}.$$

For $\alpha \in \mathbb{N}$ and $\sigma \in \mathcal{P}$, we put

$$\mathcal{P}_\alpha = \{\tau \in \mathcal{P} : |\tau| = \alpha\}, \quad \mathcal{P}_{\alpha, \sigma} = \{\tau \in \mathcal{P}_\alpha : \sigma \subset \tau\}.$$

From the proof of [5], Theorem 1.9, we see that if $\sigma \in \mathcal{P}$ is such that $\mathcal{P}_{\alpha, \sigma} = \emptyset$, then $\text{sdepth}(\mathcal{P}) < \alpha$. We recall the method of Herzog, Vladioiu and Zheng in [8] for computing the Stanley depth of S/I and I , where I is a squarefree monomial ideal. Let $G(I) = \{u_1, \dots, u_s\}$ be the set of minimal monomial generators of I . We define the following two posets:

$$\mathcal{P}_I = \left\{ \sigma \subset [n] : u_i \mid x_\sigma = \prod_{j \in \sigma} x_j \text{ for some } i \right\} \quad \text{and} \quad \mathcal{P}_{S/I} = 2^{[n]} \setminus \mathcal{P}_I.$$

From [8], Corollary 2.2, it follows that $\text{sdepth}(I) = \text{sdepth}(\mathcal{P}_I)$ and $\text{sdepth}(S/I) = \text{sdepth}(\mathcal{P}_{S/I})$.

Now, we give another main result of this article.

Theorem 2.11. $\text{sdepth}(S/I_{n,d}) = \text{depth}(S/I_{n,d}) = \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$.

Proof. First, we show that $\text{sdepth}(S/I_{n,d}) \geq \text{depth}(S/I_{n,d}) = \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$ by induction on n . If $n = d$, then $I_{n,d} = (x_1 \dots x_d)$ is principal. Thus $\text{sdepth}(S/I_{n,d}) = \text{depth}(S/I_{n,d}) = d - 1$. If $n = d + k$, we denote $u := x_{k+1} \dots x_d$ and consider the short exact sequence

$$0 \rightarrow S/(I_{n,d} : u) \rightarrow S/I_{n,d} \rightarrow S/(I_{n,d}, u) \rightarrow 0.$$

Note that $(I_{n,d} : u) = (x_1 \dots x_k, x_{d+1} \dots x_{d+k})$, $(I_{n,d}, u) = (u)$, and they both are complete intersections. Thus $\text{sdepth}(S/(I_{n,d} : u)) = \text{depth}(S/(I_{n,d} : u)) = n - 2$ and $\text{sdepth}(S/(I_{n,d}, u)) = \text{depth}(S/(I_{n,d}, u)) = n - 1$. Then we get $\text{sdepth}(S/I_{n,d}) \geq \text{depth}(S/I_{n,d}) = n - 2$ by Lemmas 2.1 and 2.2.

Suppose that $n \geq d + 2k$ and consider the short exact sequence

$$0 \rightarrow \frac{S}{(I_{n,d} : x_{n-d+1} \dots x_{n-d+k})} \rightarrow \frac{S}{I_{n,d}} \rightarrow \frac{S}{(I_{n,d}, x_{n-d+1} \dots x_{n-d+k})} \rightarrow 0.$$

Then $(I_{n,d} : x_{n-d+1} \dots x_{n-d+k}) = (I_{n-3k,d}S, x_{n-3k+1} \dots x_{n-d}, x_{n-d+k+1} \dots x_n) := I'$. Since $x_{n-3k+1} \dots x_{n-d}, x_{n-d+k+1} \dots x_n$ is a regular sequence on $S/I_{n-3k,d}S$, by induction hypothesis and [8], Lemma 3.6 we get

$$\begin{aligned} \text{depth}_S(S/I') &= \text{depth}_{S_{n-3k}}(S_{n-3k}/I_{n-3k,d}) + 3k - 2 \\ &= \frac{(d-2)(n-3k)}{d} + \left\lceil \frac{2(n-3k)}{3d} \right\rceil + 3k - 2 \\ &= \frac{(d-2)n}{d} + \left\lceil \frac{2n}{3d} \right\rceil, \end{aligned}$$

where $S_{n-3k} = K[x_1, \dots, x_{n-3k}]$. Similarly, $\text{sdepth}(S/I') \geq \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$.

Also, we have $(I_{n,d}, x_{n-d+1} \dots x_{n-d+k}) = (I_{n-d,d}S, x_{n-d+1} \dots x_{n-d+k}) := I''$, and $x_{n-d+1} \dots x_{n-d+k}$ is regular on $S/I_{n-d,d}S$. We deduce that

$$\begin{aligned} \text{depth}_S(S/I'') &= \text{depth}_{S_{n-d}}(S_{n-d}/I_{n-d,d}) + d - 1 \\ &= \frac{(d-2)(n-d)}{d} + \left\lceil \frac{2(n-d)}{3d} \right\rceil + d - 1 \\ &= \frac{(d-2)n}{d} + \left\lceil \frac{2n+d}{3d} \right\rceil \end{aligned}$$

by induction hypothesis and [8], Lemma 3.6, where $S_{n-d} = K[x_1, \dots, x_{n-d}]$. Similarly, $\text{sdepth}(S/I'') \geq \frac{(d-2)n}{d} + \lceil \frac{2n+d}{3d} \rceil$. Then $\text{sdepth}(S/I_{n,d}) \geq \text{depth}(S/I_{n,d}) = \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$ by Lemmas 2.1 and 2.2.

Next we only need to show that $\text{sdepth}(S/I_{n,d}) \leq \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$.

Let $\mathcal{P} = \mathcal{P}_{S/I_{n,d}}$, $t = \lceil \frac{n}{3k} \rceil$ and $\alpha = \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$. We consider the following two cases.

1. If $\frac{n}{k} \equiv 1 \pmod{3}$ and $\sigma = \{k+1, \dots, 2k, k+1+3k, \dots, 2k+3k, \dots, k+1+(t-2)3k, \dots, 2k+(t-2)3k, k+1+(t-2)3k+2k, \dots, 2k+(t-2)3k+2k\}$, then $\mathcal{P}_{\alpha+1, \sigma} = \emptyset$. Thus $\text{sdepth}(S/I_{n,d}) = \text{sdepth}(\mathcal{P}) \leq \alpha$.

2. If $\frac{n}{k} \equiv 0 \pmod{3}$ or $\frac{n}{k} \equiv 2 \pmod{3}$, and $\sigma = \{k+1, \dots, 2k, k+1+3k, \dots, 2k+3k, \dots, k+1+(t-1)3k, \dots, 2k+(t-1)3k\}$, then $\mathcal{P}_{\alpha+1, \sigma} = \emptyset$. Thus $\text{sdepth}(S/I_{n,d}) = \text{sdepth}(\mathcal{P}) \leq \alpha$.

It follows that $\text{sdepth}(S/I_{n,d}) \leq \alpha = \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$. □

Remark 2.12. Set $d = 2, k = 1$ in Theorem 2.11. Then we get $\text{depth}(S/I_{n,2}) = \text{sdepth}(S/I_{n,2}) = \lceil \frac{1}{3}n \rceil$, so our results generalize [9], Lemma 2.8, and [14], Lemma 4. On the other hand, by the Auslander-Buchsbaum formula, we have $\text{pd}(S/I_{n,2}) = n - \lceil \frac{1}{3}n \rceil$, which coincides with [2], Proposition 3.1.1 (1).

As a consequence of Theorem 2.11, we get the following corollary.

Corollary 2.13. $\text{sdepth}(I_{n,d}) \geq n - \lfloor \frac{n-k}{2k} \rfloor \geq \text{depth}(I_{n,d})$.

Proof. Note that $|G(I_{n,d})| = \frac{n}{k} - 1$, which implies that $\text{sdepth}(I_{n,d}) \geq \max\{1, n - \lfloor \frac{1}{2}|G(I_{n,d})| \rfloor\} = n - \lfloor \frac{n-k}{2k} \rfloor \geq \text{depth}(I_{n,d})$ by [10], Theorem 2.3 and Theorem 2.11. \square

The following proposition generalizes [9], Proposition 3.2, (where $d = 2$).

Proposition 2.14. $\text{depth}(S/I_{n,d}^t) \geq \max\{1, \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil\}$ for all $t \geq 1$, and $\text{sdepth}(S/I_{n,d}^t) \geq \max\{1, \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil\}$ for all $t \geq 1$.

Proof. Notice that $(x_1, \dots, x_n) \notin \text{Ass}(S/I_{n,d}^t)$, hence $\text{depth}(S/I_{n,d}^t) \geq 1$ for all $t \geq 1$. By [4], Theorem 2.1, we also get $\text{sdepth}(S/I_{n,d}^t) \geq 1$ for all $t \geq 1$. Then it remains to show that $\text{depth}(S/I_{n,d}^t) \geq \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil$ and $\text{sdepth}(S/I_{n,d}^t) \geq \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil$.

The proof is by induction on n and t . If $n = d$, then $I_{n,d}^t = (x_1^t \dots x_d^t)$ is principal. Thus $\text{sdepth}(S/I_{n,d}^t) = \text{depth}(S/I_{n,d}^t) = d - 1 \geq \frac{(d-2)d}{d} + \lceil \frac{2d-dt+d}{3d} \rceil$ for all $t \geq 1$. If $n = d + k$, then $I_{n,d} = (x_1 \dots x_d, x_{k+1} \dots x_{3k})$. Next we use induction on j to show that $\text{depth}(S/I_{n,d}^j) \geq 3(k-1) + \lceil \frac{4-j}{3} \rceil$ and $\text{sdepth}(S/I_{n,d}^j) \geq 3(k-1) + \lceil \frac{4-j}{3} \rceil$ for all $j \geq 1$.

If $j = 1$, then the results hold by Theorem 2.11. Let $j \geq 2$. We denote $w_1 := x_{k+1} \dots x_d$, $w_2 := x_{d+1} \dots x_{3k}$ and consider the short exact sequence

$$0 \rightarrow \frac{S}{(I_{n,d}^j : w_1 w_2)} \rightarrow \frac{S}{(I_{n,d}^j : w_2)} \rightarrow \frac{S}{((I_{n,d}^j : w_2), w_1)} \rightarrow 0.$$

Note that $(I_{n,d}^j : w_1 w_2) = I_{n,d}^{j-1}$ from the proof of Theorem 2.7. We get $\text{depth}(S/(I_{n,d}^j : w_1 w_2)) \geq 3(k-1) + \lceil \frac{4-j}{3} \rceil$ and $\text{sdepth}(S/(I_{n,d}^j : w_1 w_2)) \geq 3(k-1) + \lceil \frac{4-j}{3} \rceil$ by induction hypothesis. Also, $((I_{n,d}^j : w_2), w_1) = (w_1)$ is principal. Thus $\text{depth}(S/((I_{n,d}^j : w_2), w_1)) = \text{sdepth}(S/((I_{n,d}^j : w_2), w_1)) = 3k - 1$. Then $\text{depth}(S/(I_{n,d}^j : w_2)) \geq 3(k-1) + \lceil \frac{4-j}{3} \rceil$ and $\text{sdepth}(S/(I_{n,d}^j : w_2)) \geq 3(k-1) + \lceil \frac{4-j}{3} \rceil$ by Lemmas 2.1 and 2.2. We consider another short exact sequence

$$0 \rightarrow \frac{S}{(I_{n,d}^j : w_2)} \rightarrow \frac{S}{I_{n,d}^j} \rightarrow \frac{S}{(I_{n,d}^j, w_2)} \rightarrow 0.$$

Since $(I_{n,d}^j, w_2) = (x_1^j \dots x_d^j, w_2)$ is a complete intersection, $\text{depth}(S/(I_{n,d}^j, w_2)) = \text{sdepth}(S/(I_{n,d}^j, w_2)) = 3k - 2$. Then we get $\text{depth}(S/I_{n,d}^j) \geq 3(k-1) + \lceil \frac{4-j}{3} \rceil$ and $\text{sdepth}(S/I_{n,d}^j) \geq 3(k-1) + \lceil \frac{4-j}{3} \rceil$ by Lemmas 2.1 and 2.2.

Assume that $n \geq d + 2k$. If $t = 1$, by Theorem 2.11 the results hold for all n . Now let $t \geq 2$. We denote $u := x_{n-d+1} \dots x_{n-d+k}$, $v := x_{n-d+k+1} \dots x_n$ and consider the short exact sequence

$$(*) \quad 0 \rightarrow \frac{S}{(I_{n,d}^t : uv)} \rightarrow \frac{S}{(I_{n,d}^t : u)} \rightarrow \frac{S}{((I_{n,d}^t : u), v)} \rightarrow 0.$$

Let $G(I_{n,d}) = \{a_1, a_2, \dots, a_{(n-k)/(d-k)}\}$, the same as in the proof of Lemma 2.3. Note that $(I_{n,d}^t : uv) = I_{n,d}^{t-1}$ from the proof of Theorem 2.7. By induction hypothesis on t ,

$$\begin{aligned} \text{depth}(S/(I_{n,d}^t : uv)) &= \text{depth}(S/I_{n,d}^{t-1}) \\ &\geq \frac{(d-2)n}{d} + \left\lceil \frac{2n-d(t-1)+d}{3d} \right\rceil \\ &= \frac{(d-2)n}{d} + \left\lceil \frac{2n-dt+2d}{3d} \right\rceil. \end{aligned}$$

Similarly, $\text{sdepth}(S/(I_{n,d}^t : uv)) \geq \frac{(d-2)n}{d} + \lceil \frac{2n-dt+2d}{3d} \rceil$.

Since v divides any element of $G(I_{n,d}^t)$ which is divided by $a_{(n-k)/(d-k)}$, we have $((I_{n,d}^t : u), v) = ((I_{n-k,d}^t S : u), v)$. Noting that v is regular on $S/(I_{n-k,d}^t S : u)$, by [8], Lemma 3.6, we get $\text{depth}_S(S/((I_{n,d}^t : u), v)) = \text{depth}_{S_{n-k}}(S_{n-k}/(I_{n-k,d}^t : u)) + k - 1$ and $\text{sdepth}_S(S/((I_{n,d}^t : u), v)) = \text{sdepth}_{S_{n-k}}(S_{n-k}/(I_{n-k,d}^t : u)) + k - 1$, where $S_{n-k} = K[x_1, \dots, x_{n-k}]$. We denote $w := x_{n-3k+1} \dots x_{n-d}$ and consider another short exact sequence

$$0 \rightarrow \frac{S_{n-k}}{(I_{n-k,d}^t : wu)} \rightarrow \frac{S_{n-k}}{(I_{n-k,d}^t : u)} \rightarrow \frac{S_{n-k}}{((I_{n-k,d}^t : u), w)} \rightarrow 0.$$

From the proof of Theorem 2.7, $(I_{n-k,d}^t : wu) = I_{n-k,d}^{t-1}$. By induction hypothesis,

$$\begin{aligned} \text{depth}(S_{n-k}/(I_{n-k,d}^t : wu)) &= \text{depth}(S_{n-k}/I_{n-k,d}^{t-1}) \\ &\geq \frac{(d-2)(n-k)}{d} + \left\lceil \frac{2(n-k)-d(t-1)+d}{3d} \right\rceil \\ &= \frac{(d-2)n}{d} + \left\lceil \frac{2n-dt+d}{3d} \right\rceil - (k-1). \end{aligned}$$

Similarly, $\text{sdepth}(S_{n-k}/(I_{n-k,d}^t : wu)) \geq \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil - (k-1)$.

We see that $((I_{n-k,d}^t : u), w) = (I_{n-3k,d}^t S_{n-k}, w) := I'$, since w divides any element of $G(I_{n-k,d}^t)$ which is divided by $a_{(n-d)/(d-k)}$ or $a_{(n-3k)/(d-k)}$. Noticing that w is regular on $S_{n-k}/I_{n-3k,d}^t S_{n-k}$, by induction hypothesis on n and [8], Lemma 3.6, we get

$$\begin{aligned} \text{depth}_{S_{n-k}}(S_{n-k}/I') &= \text{depth}_{S_{n-3k}}(S_{n-3k}/I_{n-3k,d}^t) + 2k - 1 \\ &\geq \frac{(d-2)(n-3k)}{d} + \left\lceil \frac{2(n-3k)-dt+d}{3d} \right\rceil + 2k - 1 \\ &= \frac{(d-2)n}{d} + \left\lceil \frac{2n-dt+d}{3d} \right\rceil - (k-1), \end{aligned}$$

where $S_{n-3k} = K[x_1, \dots, x_{n-3k}]$. Similarly, $\text{sdepth}(S_{n-k}/I') \geq \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil - (k-1)$. Thus we have $\text{depth}(S_{n-k}/(I_{n-k,d}^t : u)) \geq \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil - (k-1)$ and

$\text{sdepth}(S_{n-k}/(I_{n-k,d}^t : u)) \geq \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil - (k-1)$ by Lemmas 2.1 and 2.2. It follows that $\text{depth}(S/((I_{n,d}^t : u), v)) \geq \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil$ and

$$\text{sdepth}(S/((I_{n,d}^t : u), v)) \geq \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil.$$

By applying Lemmas 2.1 and 2.2 to the sequence (*), we get $\text{depth}(S/(I_{n,d}^t : u)) \geq \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil$ and $\text{sdepth}(S/(I_{n,d}^t : u)) \geq \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil$.

Finally, we consider the following short exact sequence

$$0 \rightarrow \frac{S}{(I_{n,d}^t : u)} \rightarrow \frac{S}{I_{n,d}^t} \rightarrow \frac{S}{(I_{n,d}^t, u)} \rightarrow 0.$$

Since u divides any element of $G(I_{n,d}^t)$ which is divided by element $a_{(n-k)/(d-k)}$ or $a_{(n-d)/(d-k)}$, we get $(I_{n,d}^t, u) = (I_{n-d,d}^t S, u)$. Noting that u is regular on $S/I_{n-d,d}^t S$, by induction hypothesis on n and [8], Lemma 3.6, we obtain

$$\begin{aligned} \text{depth}_S(S/(I_{n,d}^t, u)) &= \text{depth}_{S_{n-d}}(S_{n-d}/I_{n-d,d}^t) + d - 1 \\ &\geq \frac{(d-2)(n-d)}{d} + \lceil \frac{2(n-d)-dt+d}{3d} \rceil + d - 1 \\ &= \frac{(d-2)n}{d} + \lceil \frac{2n-dt+2d}{3d} \rceil, \end{aligned}$$

where $S_{n-d} = K[x_1, \dots, x_{n-d}]$. Similarly, $\text{sdepth}(S/(I_{n,d}^t, u)) \geq \frac{(d-2)n}{d} + \lceil \frac{2n-dt+2d}{3d} \rceil$. Now the results follow from Lemmas 2.1 and 2.2. \square

Example 2.15. Let $\Delta_{6,4}$ be the simplicial complex with the set of facets $\mathcal{F}(\Delta_{6,4}) = \{\{1, 2, 3, 4\}, \{3, 4, 5, 6\}\}$. Then $I_{6,4} = (x_1 x_2 x_3 x_4, x_3 x_4 x_5 x_6)$ and we compute that $\text{depth}(S/I_{6,4}^4) = 4$, while in this case $\frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil = 3$. So the bound for the depth given by Proposition 2.14 is not necessarily sharp.

Now we consider the depth and Stanley depth of $S/J_{n,d}$. The next proposition generalizes [5], Propositions 1.3 and 1.8 (where $d = 2, k = 1$).

Proposition 2.16. $\text{sdepth}(S/J_{n,d}) \geq \text{depth}(S/J_{n,d}) = \frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil$.

Proof. We use induction on n . If $n = 3k$, then we have $J_{n,d} = (x_1 \dots x_{2k}, x_{k+1} \dots x_{3k}, x_{2k+1} \dots x_{3k} x_1 \dots x_k)$. We denote $u := x_{2k+1} \dots x_{3k}$ and consider the short exact sequence

$$0 \rightarrow S/(J_{n,d} : u) \rightarrow S/J_{n,d} \rightarrow S/(J_{n,d}, u) \rightarrow 0.$$

Note that $(J_{n,d} : u) = (x_1 \dots x_k, x_{k+1} \dots x_{2k})$, $(J_{n,d}, u) = (x_1 \dots x_{2k}, u)$ and they are both complete intersections. Thus $\text{sdepth}(S/(J_{n,d} : u)) = \text{depth}(S/(J_{n,d} : u)) =$

$3k-2$ and $\text{sdepth}(S/(J_{n,d}, u)) = \text{depth}(S/(J_{n,d}, u)) = 3k-2$. Hence $\text{sdepth}(S/J_{n,d}) \geq \text{depth}(S/J_{n,d}) = 3k-2$ by Lemmas 2.1 and 2.2.

Suppose that $n \geq 4k$ and consider the short exact sequence

$$0 \rightarrow \frac{S}{(J_{n,d} : x_{n-k+1} \dots x_n)} \rightarrow \frac{S}{J_{n,d}} \rightarrow \frac{S}{(J_{n,d}, x_{n-k+1} \dots x_n)} \rightarrow 0.$$

We denote $w := x_{n-k+1} \dots x_n$. Then $(J_{n,d} : w) = (x_1 \dots x_k, x_{n-2k+1} \dots x_{n-k}, I'S)$, where $I' := (x_{k+1} \dots x_{3k}, \dots, x_{n-4k+1} \dots x_{n-2k}) \subset S' := K[x_{k+1}, \dots, x_{n-2k}]$, and $x_1 \dots x_k, x_{n-2k+1} \dots x_{n-k}$ is a regular sequence on $S/I'S$. By [8], Lemma 3.6, and Theorem 2.11, we have

$$\begin{aligned} \text{depth}_S(S/(J_{n,d} : w)) &= \text{depth}_{S'}(S'/I') + 3k - 2 \\ &= \text{depth}_{S_{n-3k}}(S_{n-3k}/I_{n-3k,d}) + 3k - 2 \\ &= \frac{(d-2)(n-3k)}{d} + \left\lceil \frac{2(n-3k)}{3d} \right\rceil + 3k - 2 \\ &= \frac{(d-2)n}{d} + \left\lceil \frac{2n}{3d} \right\rceil, \end{aligned}$$

where $S_{n-3k} = K[x_1, \dots, x_{n-3k}]$. Similarly, $\text{sdepth}(S/(J_{n,d} : w)) = \frac{(d-2)n}{d} + \left\lceil \frac{2n}{3d} \right\rceil \geq \frac{(d-2)n}{d} + \left\lceil \frac{2n-d}{3d} \right\rceil$.

Also, $(J_{n,d}, w) = (I_{n-k,d}S, w)$ and w is regular on $S/I_{n-k,d}S$. By [8], Lemma 3.6, and Theorem 2.11, we get

$$\begin{aligned} \text{depth}_S(S/(J_{n,d}, w)) &= \text{depth}_{S_{n-k}}(S_{n-k}/I_{n-k,d}) + k - 1 \\ &= \frac{(d-2)(n-k)}{d} + \left\lceil \frac{2(n-k)}{3d} \right\rceil + k - 1 \\ &= \frac{(d-2)n}{d} + \left\lceil \frac{2n-d}{3d} \right\rceil, \end{aligned}$$

where $S_{n-k} = K[x_1, \dots, x_{n-k}]$. Similarly, $\text{sdepth}(S/(J_{n,d}, w)) = \frac{(d-2)n}{d} + \left\lceil \frac{2n-d}{3d} \right\rceil$, thus $\text{sdepth}(S/J_{n,d}) \geq \frac{(d-2)n}{d} + \left\lceil \frac{2n-d}{3d} \right\rceil$ by Lemma 2.2.

Now we consider the depth. If $\frac{n}{k} \equiv 0 \pmod{3}$ or $\frac{n}{k} \equiv 2 \pmod{3}$, then $\frac{(d-2)n}{d} + \left\lceil \frac{2n}{3d} \right\rceil = \frac{(d-2)n}{d} + \left\lceil \frac{2n-d}{3d} \right\rceil$, so $\text{depth}(S/J_{n,d}) = \frac{(d-2)n}{d} + \left\lceil \frac{2n-d}{3d} \right\rceil$ by Lemma 2.1.

Assume that $\frac{n}{k} \equiv 1 \pmod{3}$. We have

$$\begin{aligned} \frac{(J_{n,d} : x_{n-k+1} \dots x_n)}{J_{n,d}} &\cong x_{n-2k+1} \dots x_{n-k}(R/Q)[x_{n-2k+1}, \dots, x_{n-k}] \\ &\bigoplus_{i=1}^k x_1 \dots x_k \cdot x_{n-k+2-i} \dots x_{n-k}(R_i/Q_1)[x_1, \dots, x_k, x_{n-k+2-i}, \dots, x_{n-k}], \end{aligned}$$

where $R = K[x_1, \dots, x_{n-2k}, x_{n-k+1}, \dots, x_n]$, $Q = (x_{n-3k+1} \dots x_{n-2k}, x_{n-k+1} \dots x_n, I_{n-3k,d}S)$, $R_i = K[x_{k+1}, \dots, x_{n-2k}, x_{n-2k+1}, \dots, x_{n-k-i}, x_{n-k+1}, \dots, x_n]$, $1 \leq i \leq k$, and $Q_1 = (x_{k+1} \dots x_{2k}, x_{n-k+1} \dots x_n, x_{2k+1} \dots x_{4k}, \dots, x_{n-4k+1} \dots x_{n-2k})$. Using the isomorphism and Theorem 2.11, we obtain that

$$\begin{aligned} \text{depth}\left(\frac{(J_{n,d} : w)}{J_{n,d}}\right) &= \min_i \{ \text{depth}_R(R/Q) + k, \text{depth}_{R_i}(R_i/Q_1) + (k + i - 1) \} \\ &= \frac{(d-2)(n-4k)}{d} + \left\lceil \frac{2(n-4k)}{3d} \right\rceil + (3k - i - 2) + (k + i - 1) \\ &= \frac{(d-2)n}{d} + \left\lceil \frac{2n-d}{3d} \right\rceil. \end{aligned}$$

Now, applying Lemma 2.1 to the short exact sequence

$$0 \rightarrow \frac{(J_{n,d} : w)}{J_{n,d}} \rightarrow \frac{S}{J_{n,d}} \rightarrow \frac{S}{(J_{n,d} : w)} \rightarrow 0,$$

the proof is completed. \square

Corollary 2.17. $\text{sdepth}(J_{n,d}) \geq n - \lfloor \frac{n}{2k} \rfloor \geq \text{depth}(J_{n,d})$.

Proof. Since $|G(J_{n,d})| = \frac{n}{k}$, we have $\text{sdepth}(J_{n,d}) \geq \max\{1, n - \lfloor \frac{1}{2} |G(J_{n,d})| \rfloor\} = n - \lfloor \frac{n}{2k} \rfloor \geq \text{depth}(J_{n,d})$ by [10], Theorem 2.3, and Proposition 2.16. \square

The next theorem generalizes [5], Theorem 1.9.

Theorem 2.18. (1) $\text{sdepth}(S/J_{n,d}) = \frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil$ for $\frac{n}{k} \equiv 0 \pmod{3}$ and $\frac{n}{k} \equiv 2 \pmod{3}$. (2) $\text{sdepth}(S/J_{n,d}) \leq \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$ for $\frac{n}{k} \equiv 1 \pmod{3}$.

Proof. Let $\mathcal{P} = \mathcal{P}_{S/J_{n,d}}$, $t = \lceil \frac{n}{3k} \rceil$ and $\alpha = \frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil$. We consider the following two cases.

1. If $\frac{n}{k} \equiv 0 \pmod{3}$ or $\frac{n}{k} \equiv 2 \pmod{3}$, and $\sigma = \{1, \dots, k, 1+3k, \dots, k+3k, \dots, 1+(t-1)3k, \dots, k+(t-1)3k\}$, then $\mathcal{P}_{\alpha+1, \sigma} = \emptyset$. Thus $\text{sdepth}(S/J_{n,d}) = \text{sdepth}(\mathcal{P}) \leq \alpha = \frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil$.

2. If $\frac{n}{k} \equiv 1 \pmod{3}$ and $\sigma = \{1, \dots, k, 1+3k, \dots, k+3k, \dots, 1+(t-2)3k, \dots, k+(t-2)3k, 1+(t-2)3k+2k, \dots, k+(t-2)3k+2k\}$, then $\mathcal{P}_{\alpha+2, \sigma} = \emptyset$. Thus $\text{sdepth}(S/J_{n,d}) = \text{sdepth}(\mathcal{P}) \leq \alpha + 1 = \frac{(d-2)n}{d} + \lceil \frac{2n+2d}{3d} \rceil = \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$.

Then the results follow from Proposition 2.16. \square

Acknowledgement. The authors thank the referee for his or her carefully reading this manuscript. Also we would like to thank Professor Zhongming Tang and Dr. Cheng Gong for their helpful discussions.

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