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SOME CONCEPTS OF REGULARITY FOR PARAMETRIC MULTIPLE-INTEGRAL PROBLEMS IN THE CALCULUS OF VARIATIONS

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Abstract. We show that asserting the regularity (in the sense of Rund) of a first-order parametric multiple-integral variational problem is equivalent to asserting that the differential of the projection of its Hilbert-Carathéodory form is multisymplectic, and is also equivalent to asserting that Dedecker extremals of the latter $(m+1)$-form are holonomic.

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1. Introduction

In this paper we continue our study of multiple-integral problems in the calculus of variations which are parametric, to use the terminology of Giaquinta and Hildebrandt [7]: these are problems in which the Lagrangian function is homogeneous in an appropriate sense, so that the variational integrals are parameter-independent. Typical single-integral parametric problems are those studied in Finsler geometry.

In previous papers we have shown how to generalize the Hilbert 1-form of Finsler geometry to the first-order multiple-integral case, so as to obtain a decomposable form which we called the Hilbert-Carathéodory form [3]; and we have obtained the conditions on the Lagrangian which result in its Euler-Lagrange equations vanishing identically, that is, the conditions for the Lagrangian to be null [5]. We have also discussed the higher-order case [4]; but here as in [3] and [5] we deal only with first-order problems.

Our purpose in this paper is to investigate what it might be for a first-order parametric multiple-integral problem to be regular. There are in fact (at least) two possible answers to this question to be found in the literature. We wish to propose a third, and we shall show that despite the fact that these three definitions of regularity are conceptually quite different, in practical terms they are equivalent.

A single-integral variational problem which is not of parametric type is regular if the Hessian of the Lagrangian with respect to the velocity variables, considered as a symmetric bilinear form, is non-degenerate. (To be exact, this is the condition for local regularity: there is also a concept of global regularity, in which the Legendre transformation is a global diffeomorphism; local regularity is a necessary but not a sufficient condition for global regularity. Here however we deal only with local issues.) In Finsler geometry this condition can never hold for the Hessian of the Finsler function, because of its homogeneity. For regularity we require the Hessian of the energy, that is, half the square of the Finsler function, to be non-degenerate (indeed, positive-definite).

The first of the definitions of regularity for parametric multiple-integral problems we wish to discuss is based on these observations about Finsler geometry. It was proposed by Rund in the 1960s [9], [10]. Rund's idea was to find a power of the homogeneous Lagrangian which mirrors relevant properties of the Finslerian energy, and to require its Hessian to be non-degenerate for regularity. He was able in this way to develop an extensive theory of first-order parametric multiple-integral problems which generalizes aspects of Finsler geometry.

The second approach to defining regularity for parametric problems is to take advantage of what is known for non-parametric problems, by destroying parameter independence by using special, so-called affine, coordinates. In our discussion of this approach we shall use a formulation of regularity for non-parametric problems given by Krupková [8], which is based on the ideas of Dedecker [6].

The third concept of regularity under consideration is founded on the important role that multisymplectic structures play in first-order field theories [1], [2]. A multisymplectic structure on a manifold consists of a closed form of some order whose characteristic distribution consists just of the zero vector field. The multisymplectic structure for a field theory generalizes the symplectic structure which has such a key function in dynamics. We propose as a third definition of regularity that the exterior derivative of the Hilbert-Carathéodory form should determine a multisymplectic structure.

We shall show that each of these diverse notions of regularity leads to the same basic condition—except, as it happens, for single-integral problems, so the remarks above about analogies between symplectic and multisymplectic structures must be treated with caution. Indeed, several of our results hold only for problems which are strictly of multiple-integral type.

In the next section we shall give the essential background for the study of parametric multiple-integral problems, and in Section 3 we discuss the three definitions of regularity in some detail. In Section 4 we define an object which we call the structural tensor of the Lagrangian, and in Section 5 we show how the structural tensor helps us answer the question of when and where the exterior derivative of the Hilbert-Carathéodory form determines a multisymplectic structure. In the following section we show that the conditions we derive in Section 5 are equivalent to those required for the other concepts of regularity.

2. Background

We work on a configuration manifold E of dimension $N = m + n$. An m-frame at a point $u \in E$ is an ordered linearly-independent set $\xi = (\xi_1, \xi_2, \dots, \xi_m)$ of elements of $T_u E$, and the collection of all m-frames at all points of E is a fibre bundle over E which we denote by $\mathscr{F}^{(m)}E$ and call the m-frame bundle. Thus $\mathscr{F}^{(m)}E$ is an open submanifold of the Whitney sum of m copies of TE. We write (u^A) for coordinates on E and (u^A, u^A_i) for coordinates on $\mathscr{F}^{(m)}E$; we emphasize that, whereas the superscript A is a genuine coordinate index, the subscript i simply identifies the i-th vector ξ_i in the set. By linear independence, the fibre coordinates are such that the matrix (u^A_i) has rank m.

The i-th component of the identity map $\mathscr{F}^{(m)}E \to \mathscr{F}^{(m)}E$ is a map $X_i\colon \mathscr{F}^{(m)}E \to TE$ fibred over the projection $\mathscr{F}^{(m)}E \to E$, and is therefore a vector field along that projection called the *i-th total derivative*. The coordinate expression of X_i is thus

$$X_i = u^A_i \frac{\partial}{\partial u^A}.$$

The group $GL(m)^+$ of $m \times m$ matrices of positive determinant acts on $\mathscr{F}^{(m)}E$, where the action of $a = (a^j_i) \in GL(m)^+$ on $\mathscr{F}^{(m)}E$ is given by $(u^A, u^A_i) \mapsto (u^A, a^j_i u^A_j)$. This action makes $\mathscr{F}^{(m)}E$ into a principal fibre bundle with group $GL(m)^+$; we denote the base by $\mathscr{S}^{(m)}E$ and call it the m-sphere bundle, since it generalizes the sphere bundle to which it reduces in the case $m = 1$. A point of $\mathscr{S}^{(m)}E$ is an oriented m-dimensional contact element at a point of E, or an oriented m-dimensional subspace of a tangent space to E, of which any corresponding frame is a consistently oriented basis.

We sometimes relate our constructions to those on the jet bundle of a fibration. If $\pi\colon E \to M$ is a fibration where $\dim M = m$ then the first jet bundle $J^1\pi$ may be identified with an open submanifold of $\mathscr{S}^{(m)}E$; the points of $\mathscr{S}^{(m)}E$ which are excluded from $J^1\pi$ are those where the oriented m-dimensional subspace is tangent to the fibration. In these circumstances, we would work on the corresponding open submanifold of $\mathscr{F}^{(m)}E$. When considering such fibrations, we shall assume that the base manifold M is orientable.

A function L on $\mathscr{F}^{(m)}E$ is said to be a homogeneous Lagrangian function if, for every $a \in GL(m)^+$,

$$L(u^A, a_i^j u_j^A) = (\det a) L(u^A, u_i^A).$$

Lagrangian functions which are homogeneous in this sense give rise to parametric variational problems. By differentiating the condition above with respect to a_i^j and evaluating at the identity matrix we find that a homogeneous Lagrangian must satisfy

$$u_i^A \frac{\partial L}{\partial u_j^A} = \delta_i^j L.$$

The vector fields Δ_i^j on $\mathscr{F}^{(m)}E$ specified in coordinates by

$$\Delta_i^j = u_i^A \frac{\partial}{\partial u_j^A}$$

and vertical over E are defined globally, and are in fact the fundamental vector fields corresponding to the $GL(m)^+$-action; it follows that the condition $\Delta_i^j(L) = \delta_i^j L$ is sufficient, as well as necessary, for L to be homogeneous.

If λ is an m-form on the sphere bundle semi-basic over E, we may use it to define a homogeneous Lagrangian function L in the following way. The semi-basic property allows us to take the contraction of λ with vectors at points of E, and so we define the value of L at a point $(\xi_1, \xi_2, \ldots, \xi_m)$ of $\mathscr{F}^{(m)}E$ to be given by the contraction of λ with the m component vectors ξ_i of that point. In terms of the total derivatives X_i we have

$$L = \lambda(X_1, X_2, \ldots, X_m).$$

Any Lagrangian m-form on the jet bundle $J^1\pi$ of a fibration $\pi\colon E \to M$ is semi-basic over M and so may be used in this way to define a Lagrangian function on the corresponding open subset of $\mathscr{F}^{(m)}E$.

The Hilbert-Carathéodory form Θ of a nowhere-vanishing homogeneous Lagrangian L is the m-form

$$\Theta = L \bigwedge_{i=1}^{m} \frac{1}{L} \frac{\partial L}{\partial u_i^A} \, \mathrm{d}u^A;$$

this definition was given, slightly differently, in [3]. The Hilbert-Carathéodory form is clearly decomposable and semi-basic over E. Furthermore, Θ is easily seen to be invariant under the $GL(m)^+$-action, and so defines a semi-basic m-form $\widetilde{\Theta}$ on $\mathscr{S}^{(m)}E$. Evidently, from the differential homogeneity condition,

$$\Theta(X_1, X_2, \ldots, X_m) = \widetilde{\Theta}(X_1, X_2, \ldots, X_m) = L;$$

the similarity with the formula for constructing L from a semi-basic form on the sphere bundle or a Lagrangian form on a jet bundle is, of course, no accident.

We can use the Hilbert-Carathéodory form to represent the Euler-Lagrange equations for L in terms of a field of m-frames $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_m)$ defined on $\mathscr{F}^{(m)}E$ (that is, as a section of the m-frame bundle of $\mathscr{F}^{(m)}E$ rather than of E), so that each component Γ_k is a vector field on $\mathscr{F}^{(m)}E$. We require Γ to satisfy the second-order condition

$$\Gamma_k = u_k^A \frac{\partial}{\partial u^A} + \Gamma_{jk}^A \frac{\partial}{\partial u_j^A} \quad \text{with } \Gamma_{kj}^A = \Gamma_{jk}^A,$$

and a straightforward calculation gives

$$\Gamma \lrcorner \, \mathrm{d}\Theta = (-1)^m \Big(\frac{\partial L}{\partial u^A} - \Gamma_i \Big(\frac{\partial L}{\partial u_i^A} \Big) \Big) \, \mathrm{d}u^A,$$

which indeed will also be evident from the formula for $d\Theta$ given in Lemma 1 below. Thus we say that Γ satisfies the Euler-Lagrange equations for L if $\Gamma \lrcorner \, \mathrm{d}\Theta = 0$.

Suppose further that $\widehat{\Gamma}$ is a second field of frames satisfying $\widehat{\Gamma}_k = \Gamma_k + A_{jk}^i \Delta_i^j$ for some functions A_{jk}^i symmetric in their lower indices; then $\widehat{\Gamma}$ also satisfies the second-order condition, and another simple calculation shows that $\widehat{\Gamma}$ satisfies the Euler-Lagrange equations exactly when Γ does. In view of the significance of the Δ_i^j explained above, it should be clear that this degree of indeterminacy is just what is to be expected in a parametric problem.

3. Concepts of regularity

In this section we review in more detail the three concepts of regularity whose equivalence we demonstrate later.

The first of these concepts applies to a homogeneous Lagrangian L defined on $\mathscr{F}^{(m)}E$, and is described by Rund in [9, Chapter 4 Section 5]. The idea is to define, for each such Lagrangian, a suitable metric tensor g as a section of the bundle $\mathscr{V}^* \odot \mathscr{V}^* \to \mathscr{F}^{(m)}E$, where $\mathscr{V} \subset T\mathscr{F}^{(m)}E$ is the bundle of tangent vectors vertical over E. Thus g may be thought of as specifying, at each point ξ of $\mathscr{F}^{(m)}E$, a symmetric bilinear form on $\mathscr{V}_\xi$, the subspace of $T_\xi \mathscr{F}^{(m)}E$ consisting of vectors annihilated by the projection onto E. The metric is specified by its formula in coordinates,

$$g_{AB}^{ij} = \frac{m}{2} \frac{\partial^2 (L^{2/m})}{\partial u_i^A \partial u_j^B}.$$

Here (g_{AB}^{ij}) is to be regarded as an $m(m+n) \times m(m+n)$ matrix, symmetric for the interchange of i, A with j, B. It may be checked that the construction is tensorial.

In the single-integral case such a formula describes the Hessian of the Finslerian energy. Rund demonstrates that taking the particular power $2/m$ gives the metric a homogeneity property analogous to that enjoyed by the Finslerian energy, so that in the general case we may consider $\frac{1}{2}mL^{2/m}$ as the 'energy' of the Lagrangian. Given such a metric g, the Lagrangian may be recovered as

$$L = \Big(\frac{1}{m}u_i^A u_j^B g_{AB}^{ij}\Big)^{m/2};$$

again this generalizes the way that a Finsler function can be recovered from the Hessian of its energy.

When the metric g is everywhere non-degenerate (as a symmetric bilinear form) we shall say that L is *Rund regular*, and then we see from the recovery formula that L must be non-vanishing. In such a case, the fibre coordinates u_i^A on $\mathscr{F}^{(m)}E$ may be replaced by 'momentum' coordinates $p_B^j = g_{AB}^{ij}u_i^A$; this replacement represents a local identification of the frame bundle with its dual coframe bundle, and specifies a corresponding Hamiltonian system. An explicit statement of the Rund regularity of L is that

$$\det\Big(\frac{\partial^2 L}{\partial u_i^A \partial u_j^B} + \frac{(2-m)}{m}\frac{1}{L}\frac{\partial L}{\partial u_i^A}\frac{\partial L}{\partial u_j^B}\Big) \neq 0;$$

the expression inside the bracket is just $L^{1-2/m}g_{AB}^{ij}$.

The other two concepts of regularity are concerned with a certain m-form and its exterior derivative, defined on the sphere bundle (or perhaps on a suitable open subset thereof).

The second concept of regularity appears in the work of Dedecker [6], who considered a first-order variational problem on the bundle of contact elements, where extremals are submanifolds of E. Given such a problem, Dedecker studied certain related 'zeroth-order variational problems', where the extremals are submanifolds of the contact bundle itself; he defined a problem of the latter kind to be 'equivalent' to the original problem if its extremals are always prolongations of those of the original one. A weaker version of this property arises when a certain well-defined subset of the extremals consists of prolongations.

The sphere bundle is a double cover of the contact bundle, and similar considerations apply in our case. We shall say that a semi-basic m-form θ on $\mathscr{S}^{(m)}E$ is a *Lepage form* if $Z \lrcorner \, \mathrm{d}\theta$ is a contact form whenever Z is a vector field on $\mathscr{S}^{(m)}E$ vertical over E; if λ is some other semi-basic m-form on $\mathscr{S}^{(m)}E$ then we say that θ is a *Lepage equivalent of* λ if it is a Lepage form and if $\theta - \lambda$ is a contact form. An oriented m-dimensional submanifold $U \subset E$ is an extremal of the first-order variational problem defined by λ if, for any vector field X on E, the restriction of $X^1 \lrcorner \, \mathrm{d}\lambda$ to the prolonged submanifold $U^1 \subset \mathscr{S}^{(m)}E$ vanishes, where X^1 denotes the prolongation of X to a vector field on $\mathscr{S}^{(m)}E$. On the other hand, an oriented m-dimensional submanifold $W \subset \mathscr{S}^{(m)}E$ is an extremal of the zeroth-order variational problem defined by θ if, for any vector field Y on $\mathscr{S}^{(m)}E$, $Y \lrcorner \, \mathrm{d}\theta$ vanishes when restricted to W. If θ is a Lepage equivalent of λ and U is an extremal of λ then the prolongation U^1 is an extremal of θ.

Our concern now is with a weak version of the converse. We say that an oriented m-dimensional submanifold $W \subset \mathscr{S}^{(m)}E$ is a *Dedecker submanifold* if every 2-contact differential form on $\mathscr{S}^{(m)}E$ vanishes when restricted to W. Given a non-vanishing homogeneous Lagrangian function L on $\mathscr{F}^{(m)}E$, the projection $\widetilde{\Theta}$ of its Hilbert-Carathéodory form is always a Lepage form on $\mathscr{S}^{(m)}E$, as we shall see shortly; we shall say that L is *Dedecker regular* if every extremal W of $\widetilde{\Theta}$ (as a zeroth-order problem) which is a Dedecker submanifold is then necessarily a prolongation U^1.

To confirm that $\widetilde{\Theta}$ is a Lepage form on $\mathscr{S}^{(m)}E$, and to establish a coordinate formula for Dedecker regularity, we use the observation made earlier that for a fibration $\pi\colon E \to M$, $J^1\pi$ may be identified with an open submanifold of $\mathscr{S}^{(m)}E$, to introduce local coordinates on $\mathscr{S}^{(m)}E$. We choose 'split' coordinates (x^i, y^α) on E where $i = 1, \ldots, m$ and $\alpha = m+1, \ldots, m+n$, thereby fibring E locally, with the x^i coordinates on the (notional) base and the y^α coordinates on the (notional) fibre. The fibre coordinates on $\mathscr{S}^{(m)}E$ corresponding to the (notional) fibre coordinates on $J^1\pi \to E$ are denoted by y^α_i. So $(x^i, y^\alpha, y^\alpha_i)$ are local coordinates on $\mathscr{S}^{(m)}E$; it is coordinates of this type that we meant when we referred to affine coordinates earlier. In effect, we are identifying a suitable open subset of $\mathscr{S}^{(m)}E$ with those m-frames on E for which

$$\xi_i = \frac{\partial}{\partial x^i} + y^\alpha_i \frac{\partial}{\partial y^\alpha}.$$

We set $\mathscr{L}(x^i, y^\alpha, y^\alpha_i) = L(x^i, y^\alpha, \delta^j_i, y^\alpha_i)$. In these coordinates $\widetilde{\Theta}$ may be expressed as

$$\widetilde{\Theta} = \mathscr{L} \bigwedge_{i=1}^{m} \Big(\mathrm{d}x^i + \frac{1}{\mathscr{L}} \frac{\partial \mathscr{L}}{\partial y^\alpha_i} \omega^\alpha \Big)$$

where $\omega^\alpha = \mathrm{d}y^\alpha - y^\alpha_j \mathrm{d}x^j$ (see [3], where we have again used slightly different notation; also, these calculations are similar to those in [8]). This m-form is just the Carathéodory form of the local Lagrangian $\mathscr{L}\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m$ and is well known to be a Lepage form. Expanding the formula for $\widetilde{\Theta}$, and writing

$$\mathscr{L}^i_\alpha = \frac{\partial \mathscr{L}}{\partial y^\alpha_i}, \qquad \mathscr{L}^{ij}_{\alpha\beta} = \frac{1}{\mathscr{L}} \Big(\frac{\partial \mathscr{L}}{\partial y^\alpha_i} \frac{\partial \mathscr{L}}{\partial y^\beta_j} - \frac{\partial \mathscr{L}}{\partial y^\beta_i} \frac{\partial \mathscr{L}}{\partial y^\alpha_j} \Big),$$

we have

$$\widetilde{\Theta} = \mathscr{L}\mathrm{d}^m x + \mathscr{L}^i_\alpha \omega^\alpha \wedge \mathrm{d}^{m-1}x_i + \tfrac{1}{4} \mathscr{L}^{ij}_{\alpha\beta} \omega^\alpha \wedge \omega^\beta \wedge \mathrm{d}^{m-2}x_{ij} \qquad \mathrm{mod}\, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$$

where

$$\mathrm{d}^m x = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m, \quad \mathrm{d}^{m-1}x_i = \frac{\partial}{\partial x^i} \lrcorner\, \mathrm{d}^m x, \quad \mathrm{d}^{m-2}x_{ij} = \frac{\partial}{\partial x^j} \lrcorner\, \mathrm{d}^{m-1}x_i$$

and where the factor of a quarter in the final term arises because the implied sum is over all $1 \leqslant i, j \leqslant m$ and $m+1 \leqslant \alpha, \beta \leqslant m+n$ rather than over terms where $i < j$ and $\alpha < \beta$. Thus we get

$$\mathrm{d}\widetilde{\Theta} = \Big(\frac{\partial \mathscr{L}}{\partial y^\alpha} - X_i \mathscr{L}^i_\alpha\Big)\omega^\alpha \wedge \mathrm{d}^m x + \Big(\frac{\partial \mathscr{L}^i_\alpha}{\partial y^\gamma_k} - \mathscr{L}^{ik}_{\alpha\gamma}\Big)\mathrm{d}y^\gamma_k \wedge \omega^\alpha \wedge \mathrm{d}^{m-1}x_i \quad \mod \omega^\alpha \wedge \omega^\beta.$$

Taking the contraction with $Y = \partial/\partial y^\beta_j$ we obtain

$$Y \lrcorner\, \mathrm{d}\widetilde{\Theta} = \Big(\frac{\partial \mathscr{L}^i_\alpha}{\partial y^\beta_j} - \mathscr{L}^{ij}_{\alpha\beta}\Big) \wedge \omega^\alpha \wedge \mathrm{d}^{m-1}x_i$$

and so we see that the condition for Dedecker regularity of L is that

$$\det\Big(\frac{\partial^2 \mathscr{L}}{\partial y^\alpha_i \partial y^\beta_j} - \frac{1}{\mathscr{L}}\Big(\frac{\partial \mathscr{L}}{\partial y^\alpha_i}\frac{\partial \mathscr{L}}{\partial y^\beta_j} - \frac{\partial \mathscr{L}}{\partial y^\beta_i}\frac{\partial \mathscr{L}}{\partial y^\alpha_j}\Big)\Big) \neq 0.$$

We should perhaps mention that Dedecker's analysis of regularity is quite general, and gives different explicit criteria for different Lepage equivalents; the property we have been discussing should strictly speaking be called Dedecker regularity for the Carathéodory form.

Our third and final concept of regularity also concerns $\widetilde{\Theta}$, and is appropriate only in the case $m > 1$: it is that $d\widetilde{\Theta}$ should be a multisymplectic form, and then we say that L is *multisymplectic regular*. We may also express this in terms of the Hilbert-Carathéodory form Θ: it is easy to see (and we shall shortly show) that the fundamental vector fields Δ^j_i are always characteristic vector fields of $d\Theta$, and the condition for multisymplectic regularity is that the characteristic distribution of $d\Theta$ should be spanned by the Δ^j_i.

It is worth making a remark here about the exceptional case $m = 1$, and comparing this with a similar situation for the De Donder-Weyl theory for Lagrangians defined on jet bundles, where again $m = 1$ is an exceptional case. Let $\pi \colon E \to M$ be a fibration, and let $\lambda = \mathscr{L}\mathrm{d}^m x$ be a Lagrangian m-form (here, we take the split coordinates (x^i, y^α) to be genuine fibred coordinates on E). The De Donder-Weyl theory considers the Cartan form

$$\Theta_C = \mathscr{L}\mathrm{d}^m x + \frac{\partial \mathscr{L}}{\partial y^\alpha_i}(\mathrm{d}y^\alpha - y^\alpha_j \mathrm{d}x^j) \wedge \mathrm{d}^{m-1}x_i;$$

the appropriate notion of regularity for this form is that the Hessian $\partial^2\mathscr{L}/\partial y_i^\alpha \partial y_j^\beta$ should be non-degenerate. If the Cartan form is regular, it is equivalent to the Lagrangian in the sense of Dedecker: the extremals of Θ_C are prolongations of the extremals of λ. In the regular case, $\mathrm{d}\Theta_C$ is multisymplectic provided that $m > 1$, as may be seen easily in coordinates by taking the contraction with an arbitrary vector field on $J^1\pi$. But if $m = 1$ then $\mathrm{d}\Theta_C$ is certainly not symplectic, because it is a 2-form on an odd-dimensional manifold.

A similar situation arises in the homogeneous case, where we consider $\mathrm{d}\Theta$ on $\mathscr{F}^{(m)}E$. The fundamental vector fields Δ_i^j are always annihilated by $\mathrm{d}\Theta$; and so is any second order frame field Γ satisfying the Euler-Lagrange equations. But in the case $m = 1$ a second-order frame field is just a vector field, and so would be a section of the characteristic distribution linearly independent of the (single) fundamental vector field.

4. The structural tensor

Our approach to proving the equivalence, for a non-vanishing homogeneous Lagrangian L, of the three definitions of regularity given above will involve the use of a certain section Q of the bundle $\mathscr{V}^* \odot \mathscr{V}^* \to E$. This section is in general distinct from the Rund metric g, although the coordinate formula for Q, which we shall give below, does appear in slightly different form in Rund's work [9], [10]. It may be checked that this coordinate formula does indeed define a tensorial object, and we call it the *structural tensor* of the Lagrangian. To determine properties of Q we shall use a particular class of coordinate systems on $\mathscr{F}^{(m)}E$, and we first define these in the context of certain local bases of vector fields and 1-forms.

We start with the total derivatives X_i which, as described above, are m linearly independent globally-defined vector fields along the projection $\mathscr{F}^{(m)}E \to E$; we also have m linearly independent globally-defined semi-basic 1-forms

$$\vartheta^i = \frac{1}{L}\frac{\partial L}{\partial u_i^A}\mathrm{d}u^A$$

(this is slightly different from the definition given in [3]) where $\langle X_i, \vartheta^j\rangle = \delta_i^j$. We now extend $\{X_i\}$ to a local basis $\{X_i, X_\alpha\}$, where $m+1 \leqslant \alpha \leqslant m+n$, such that $\langle X_\alpha, \vartheta^i\rangle = 0$. Let

$$X_\alpha = X_\alpha^A \frac{\partial}{\partial u^A},$$

say: the latter contraction condition is then

$$\frac{\partial L}{\partial u_i^A} X_\alpha^A = 0.$$

Finally, let ϑ^α be the semi-basic 1-forms which make $\{\vartheta^i, \vartheta^\alpha\}$ a basis of semi-basic 1-forms dual to the basis $\{X_i, X_\alpha\}$. Since $\langle X_i, \vartheta^\alpha\rangle = 0$, the ϑ^α are necessarily contact 1-forms, and we have

$$\mathrm{d}u^A = u_i^A \vartheta^i + X_\alpha^A \vartheta^\alpha.$$

The special coordinate systems mentioned above, which we now define, will be of use in setting up such local bases. At each point $\xi \in \mathscr{F}^{(m)}E$ the matrix (u_i^A) has rank m, so by re-ordering the superscripts A we may define coordinate functions (which we still call u_i^A) where the $m \times m$ matrix (u_i^j) is non-singular at ξ and hence in some neighbourhood of ξ. We shall therefore restrict attention to coordinate systems having this property. (To call these 'special' coordinate systems is perhaps somewhat extravagant: given the rank condition, in reality there is little more to them than notational convenience.) We shall denote the components of the inverse of (u_i^j) by $\bar{u}_i^j$, and set $\bar{u}_i^j u_j^\alpha = v_i^\alpha$.

In such a coordinate system, the homogeneity condition for the Lagrangian is

$$u_j^k \frac{\partial L}{\partial u_i^k} + u_j^\alpha \frac{\partial L}{\partial u_i^\alpha} = \delta_j^i L,$$

so that

$$\frac{\partial L}{\partial u_i^j} = \bar{u}_j^i L - v_j^\alpha \frac{\partial L}{\partial u_i^\alpha},$$

and therefore

$$\vartheta^i = \bar{u}_j^i \mathrm{d}u^j + \frac{1}{L}\frac{\partial L}{\partial u_i^\alpha}(\mathrm{d}u^\alpha - v_j^\alpha du^j).$$

A natural choice for ϑ^α is then

$$\vartheta^\alpha = \mathrm{d}u^\alpha - v_j^\alpha \mathrm{d}u^j,$$

and with this choice we find that the coefficients X_α^A are given by

$$X_\alpha^i = -u_j^i \frac{1}{L}\frac{\partial L}{\partial u_j^\alpha}, \quad X_\alpha^\beta = \delta_\alpha^\beta - u_j^\beta \frac{1}{L}\frac{\partial L}{\partial u_j^\alpha}.$$

We now introduce the section Q of the bundle $\mathscr{V}^* \odot \mathscr{V}^* \to E$, the structural tensor, by its formula in a general coordinate system,

$$Q_{AB}^{ij} = \frac{\partial^2 L}{\partial u_i^A \partial u_j^B} - \frac{1}{L}\Big(\frac{\partial L}{\partial u_i^A}\frac{\partial L}{\partial u_j^B} - \frac{\partial L}{\partial u_j^A}\frac{\partial L}{\partial u_i^B}\Big);$$

at each point ξ of $\mathscr{F}^{(m)}E$ we may regard the Q_{AB}^{ij} as the components of a symmetric bilinear form on $\mathscr{V}_\xi$, as before. Now by differentiating the homogeneity condition we obtain

$$\frac{\partial^2 L}{\partial u_i^A \partial u_j^B} u_k^B = \delta_k^j \frac{\partial L}{\partial u_i^A} - \delta_k^i \frac{\partial L}{\partial u_j^A},$$

so that

$$Q^{ij}_{AB}u^B_k = \Big(\delta^j_k\frac{\partial L}{\partial u^A_i} - \delta^i_k\frac{\partial L}{\partial u^A_j}\Big) - \frac{1}{L}\Big(\frac{\partial L}{\partial u^A_i}(L\delta^j_k) - \frac{\partial L}{\partial u^A_j}(L\delta^i_k)\Big) = 0.$$

Thus

$$Q\big(\cdot, \Delta^j_k\big) = 0,$$

so that Q annihilates the fundamental vector fields. We shall denote by $\mathscr{D}$ the vertical distribution on $\mathscr{F}^{(m)}E$ spanned by the Δ^j_i, so that $\mathscr{D}$ is a vector subbundle of $\mathscr{V} \subset T\mathscr{F}^{(m)}E$; then Q defines a symmetric bilinear form $\widetilde{Q}$ on the quotient bundle $\mathscr{V}/\mathscr{D}$, in other words a section of the bundle $\mathscr{D}^\perp \odot \mathscr{D}^\perp \to \mathscr{F}^{(m)}E$ where $\mathscr{D}^\perp$ is the subbundle of $\mathscr{V}^*$ consisting of the annihilators of $\mathscr{D}$. Using our special coordinates we have $Q^{ij}_{Al}u^l_k + Q^{ij}_{A\beta}u^\beta_k = 0$, or $Q^{ij}_{Al} = -Q^{ij}_{A\beta}v^\beta_l$, so that

$$Q^{ij}_{\alpha l} = -Q^{ij}_{\alpha\beta}v^\beta_l, \quad Q^{ij}_{kl} = Q^{ij}_{\alpha\beta}v^\alpha_k v^\beta_l.$$

Equivalently,

$$Q = Q^{ij}_{\alpha\beta}[\mathrm{d}u^\alpha_i - v^\alpha_k\mathrm{d}u^k_i] \odot [\mathrm{d}u^\beta_j - v^\beta_l\mathrm{d}u^l_j]$$

where the bracketed 1-forms are equivalence classes modulo semi-basic forms. We note that $[\mathrm{d}u^\alpha_i - v^\alpha_k\mathrm{d}u^k_i]$ is just the equivalence class of the total derivative $X_i(\mathrm{d}u^\alpha - v^\alpha_k\mathrm{d}u^k)$ and that these equivalence classes span the annihilators of $\mathscr{D}$. The expression above shows that the $Q^{ij}_{\alpha\beta}$ can be regarded as the coefficients of $\widetilde{Q}$, which we call the *reduced structural tensor*.

5. A condition for multisymplectic regularity

We now use the structural tensor to obtain a condition for the multisymplectic regularity of a non-vanishing homogeneous Lagrangian for the case $m > 1$. We start by obtaining a formula for $Ld\Theta$ in terms of a basis $\{\vartheta^i, \vartheta^\alpha\}$ of semi-basic 1-forms, as described above.

Lemma 1. *We have the following formula for $Ld\Theta$:*

$$L\mathrm{d}\Theta = X^A_\alpha\Big(\frac{\partial L}{\partial u^A} - u^B_i\frac{\partial^2 L}{\partial u^B \partial u^A_i}\Big)\vartheta^\alpha \wedge \Theta + X^A_\alpha X^B_\beta\frac{\partial^2 L}{\partial u^A \partial u^B_i}\vartheta^\alpha \wedge \vartheta^\beta \wedge \Theta_i$$
$$+ X^A_\alpha\frac{\partial^2 L}{\partial u^A_i \partial u^B_j}\mathrm{d}u^B_j \wedge \vartheta^\alpha \wedge \Theta_i,$$

where $\Theta_i = (-1)^i L\vartheta^1 \wedge \vartheta^2 \wedge \ldots \hat{\vartheta}^i \ldots \wedge \vartheta^m = X_i \lrcorner\, \Theta$, and summation over i (from 1 to m) is intended in the second and third terms.

P r o o f. We have $\Theta = L\vartheta^1 \wedge \vartheta^2 \wedge \ldots \wedge \vartheta^m$, whence

$$\mathrm{d}\Theta = \frac{1}{L}\,\mathrm{d}L \wedge \Theta + \mathrm{d}\vartheta^i \wedge \Theta_i.$$

Now

$$\mathrm{d}\vartheta^i = \frac{1}{L}\Big(- \mathrm{d}L \wedge \vartheta^i + \mathrm{d}\Big(\frac{\partial L}{\partial u_i^A}\Big) \wedge \mathrm{d}u^A\Big),$$

and $\mathrm{d}u^A = u_i^A \vartheta^i + X_\alpha^A \vartheta^\alpha$, so that

$$\begin{aligned}\mathrm{d}\vartheta^i &= \frac{1}{L}\Big(\Big(- \delta_j^i \mathrm{d}L + u_j^A \mathrm{d}\Big(\frac{\partial L}{\partial u_i^A}\Big)\Big) \wedge \vartheta^j + \mathrm{d}\Big(\frac{\partial L}{\partial u_i^A}\Big) \wedge X_\alpha^A \vartheta^\alpha\Big)\\ &= \frac{1}{L}\Big(- \frac{\partial L}{\partial u_j^A}\,\mathrm{d}u_i^A \wedge \vartheta^j + \mathrm{d}\Big(\frac{\partial L}{\partial u_i^A}\Big) \wedge X_\alpha^A \vartheta^\alpha\Big).\end{aligned}$$

It follows that

$$L\mathrm{d}\Theta = \frac{\partial L}{\partial u^A}\,\mathrm{d}u^A \wedge \Theta + \mathrm{d}\Big(\frac{\partial L}{\partial u_i^A}\Big) \wedge X_\alpha^A \vartheta^\alpha \wedge \Theta_i.$$

Now $\vartheta^i \wedge \Theta = 0$, so we can write

$$\begin{aligned}L\mathrm{d}\Theta &= \frac{\partial L}{\partial u^A} X_\alpha^A \vartheta^\alpha \wedge \Theta + \mathrm{d}\Big(\frac{\partial L}{\partial u_i^A}\Big) \wedge X_\alpha^A \vartheta^\alpha \wedge \Theta_i\\ &= \frac{\partial L}{\partial u^A} X_\alpha^A \vartheta^\alpha \wedge \Theta + \Big(\frac{\partial^2 L}{\partial u^B \partial u_i^A}\mathrm{d}u^B + \frac{\partial^2 L}{\partial u_i^A \partial u_j^B}\mathrm{d}u_j^B\Big) \wedge X_\alpha^A \vartheta^\alpha \wedge \Theta_i\\ &= \frac{\partial L}{\partial u^A} X_\alpha^A \vartheta^\alpha \wedge \Theta + \Big(\frac{\partial^2 L}{\partial u^B \partial u_i^A}(u_j^B \vartheta^j + X_\beta^B \vartheta^\beta) + \frac{\partial^2 L}{\partial u_i^A \partial u_j^B}\mathrm{d}u_j^B\Big) \wedge X_\alpha^A \vartheta^\alpha \wedge \Theta_i\\ &= X_\alpha^A \Big(\frac{\partial L}{\partial u^A} - u_i^B \frac{\partial^2 L}{\partial u^B \partial u_i^A}\Big)\vartheta^\alpha \wedge \Theta + X_\alpha^A X_\beta^B \frac{\partial^2 L}{\partial u^A \partial u_i^B}\vartheta^\alpha \wedge \vartheta^\beta \wedge \Theta_i\\ &\quad + X_\alpha^A \frac{\partial^2 L}{\partial u_i^A \partial u_j^B}\mathrm{d}u_j^B \wedge \vartheta^\alpha \wedge \Theta_i.\end{aligned}$$

□

In the expression above for $Ld\Theta$ the first two terms are semi-basic. Note that

$$\begin{aligned}L(\Delta_i^j \lrcorner\, \mathrm{d}\Theta) &= X_\alpha^A u_i^B \frac{\partial^2 L}{\partial u_k^A \partial u_j^B}\vartheta^\alpha \wedge \Theta_k\\ &= X_\alpha^A \Big(\delta_i^j \frac{\partial L}{\partial u_k^A} - \delta_i^k \frac{\partial L}{\partial u_j^A}\Big)\vartheta^\alpha \wedge \Theta_k = 0,\end{aligned}$$

in view of the properties of X_α^A, so the Δ_i^j are characteristic, as we mentioned before.

The point at issue is whether the exterior derivative of the corresponding Lagrangian form $\widetilde{\Theta}$ defines a multisymplectic structure on the sphere bundle. The question is, therefore, if χ is a characteristic vector of $\mathrm{d}\Theta$ (so that $\chi \lrcorner \mathrm{d}\Theta = 0$), is it the case that χ must be a linear combination of the Δ^i_j?

The answer hinges on the properties of the final term in the expression for $L\mathrm{d}\Theta$, the one involving $\mathrm{d}u^B_j$. We shall therefore consider the 1-form

$$X^A_\alpha \frac{\partial^2 L}{\partial u^A_i \partial u^B_j}\, \mathrm{d}u^B_j.$$

Lemma 2. *When expressed in special coordinates,*

$$X^A_\alpha \frac{\partial^2 L}{\partial u^A_i \partial u^B_j}\, \mathrm{d}u^B_j = Q^{ij}_{\alpha\beta}(\mathrm{d}u^\beta_j - v^\beta_k\, \mathrm{d}u^k_j).$$

Proof. We have

$$X^i_\alpha = -u^i_j \frac{1}{L}\frac{\partial L}{\partial u^\alpha_j}, \qquad X^\beta_\alpha = \delta^\beta_\alpha - u^\beta_j \frac{1}{L}\frac{\partial L}{\partial u^\alpha_j},$$

from which it follows that

$$\begin{aligned} X^A_\alpha \frac{\partial^2 L}{\partial u^A_i \partial u^B_j} &= \frac{\partial^2 L}{\partial u^\alpha_i \partial u^B_j} - \frac{1}{L}\frac{\partial L}{\partial u^\alpha_k} u^C_k \frac{\partial^2 L}{\partial u^C_i \partial u^B_j} \\ &= \frac{\partial^2 L}{\partial u^\alpha_i \partial u^B_j} - \frac{1}{L}\frac{\partial L}{\partial u^\alpha_k}\Big(\delta^i_k \frac{\partial L}{\partial u^B_j} - \delta^j_k \frac{\partial L}{\partial u^B_i}\Big) \\ &= Q^{ij}_{\alpha B}. \end{aligned}$$

The result then follows using the expression for $Q^{ij}_{\alpha l}$ in terms of $Q^{ij}_{\alpha\beta}$. $\square$

Theorem 1. *For $m > 1$ the form $d\widetilde{\Theta}$ defines a multisymplectic structure on $\mathscr{S}^{(m)}E$ if and only if the reduced structural tensor $\widetilde{Q}$ is non-degenerate, in other words if and only if the $mn \times mn$ matrix $(Q^{ij}_{\alpha\beta})$ is non-singular.*

Proof. We need to consider the characteristic vectors of $\mathrm{d}\Theta$.

Suppose that $d\widetilde{\Theta}$ is multisymplectic, so that the only characteristic vectors of $d\Theta$ are the linear combinations of the Δ^i_j. Suppose that $Q^{ij}_{\alpha\beta}\chi^\beta_j = 0$: then if

$$\chi = \chi^\beta_j \frac{\partial}{\partial u^\beta_j},$$

χ is a characteristic vector of $\mathrm{d}\Theta$. But if χ is non-zero it cannot be expressed as a linear combination of the Δ_i^j. It follows that $(Q^{ij}_{\alpha\beta})$ is non-singular.

Suppose conversely that $(Q^{ij}_{\alpha\beta})$ is non-singular, and that $\chi\lrcorner\,\mathrm{d}\Theta = 0$ with

$$\chi = \chi^i X_i + \chi^\alpha X_\alpha + \chi_i^A \frac{\partial}{\partial u_i^A}.$$

In the expression for $\chi\lrcorner\,\mathrm{d}\Theta$ there will be just one term in $\mathrm{d}u_j^B \wedge \Theta_i$, which comes from contracting $\chi^\alpha X_\alpha$ with the last term in $L\mathrm{d}\Theta$. Thus $Q^{ij}_{\alpha\beta}\chi^\beta = 0$. But then $Q^{ij}_{\alpha\beta}\chi^\beta\zeta_j = 0$ for any ζ_j, whence $\chi^\beta\zeta_j = 0$ for any ζ_j, and $\chi^\beta = 0$.

We may therefore assume that $\chi^\alpha = 0$: then there remains only one term in $\chi\lrcorner\,\mathrm{d}\Theta$ involving $\mathrm{d}u_j^B$, which is obtained by contracting $\chi^i X_i$ with the last term in $L\mathrm{d}\Theta$, and is

$$\chi^i Q^{jk}_{\alpha\beta}(\mathrm{d}u_k^\beta - v_l^\beta\,\mathrm{d}u_k^l) \wedge \vartheta^\alpha \wedge \Theta_{ij}$$

where $\Theta_{ij} = X_i\lrcorner\,\Theta_j = -\Theta_{ji}$. It follows that χ^i must satisfy

$$\chi^i Q^{jk}_{\alpha\beta} = \chi^j Q^{ik}_{\alpha\beta}.$$

Let $\bar{Q}^{\alpha\beta}_{ij}$ be the coefficients of the inverse matrix, so that $Q^{jk}_{\alpha\beta}\bar{Q}^{\gamma\beta}_{lk} = \delta^\gamma_\alpha\delta^j_l$. Multiply through by the inverse:

$$\chi^i\delta^\gamma_\alpha\delta^j_l = \chi^j\delta^\gamma_\alpha\delta^i_l.$$

Sum over α and γ, and again over j and l, to obtain $m\chi^i = \chi^i$, whence $\chi^i = 0$ (for $m \neq 1$).

We are left with

$$\chi = \chi_i^A \frac{\partial}{\partial u_i^A},$$

so that

$$L(\chi\lrcorner\,\mathrm{d}\Theta) = Q^{ij}_{\alpha\beta}(\chi_j^\beta - v_k^\beta\chi_j^k) \wedge \vartheta^\alpha \wedge \Theta_i = 0.$$

Thus χ must satisfy $\chi_j^\beta = v_k^\beta\chi_j^k$, so that

$$\chi_i^A\frac{\partial}{\partial u_i^A} = \chi_i^j\Big(\frac{\partial}{\partial u_i^j} + v_j^\beta\frac{\partial}{\partial u_i^\beta}\Big) = \chi_i^k\bar{u}_k^j\Delta_j^i$$

as required. $\square$

We now relate the other two concepts of regularity to the structural tensor Q.

In order to consider Rund regularity it will be convenient to examine a quite general class of sections P of $\mathscr{V}^* \odot \mathscr{V}^* \to \mathscr{F}^{(m)}E$, with coefficients P^{ij}_{AB}, namely those for which

$$P^{ij}_{AB} = Q^{ij}_{AB} + K^{ij}_{kl}\frac{1}{L}\frac{\partial L}{\partial u^A_k}\frac{\partial L}{\partial u^B_l}$$

for some coefficients K^{ij}_{kl}, symmetric under the interchange of k, i with l, j. In terms of our special coordinates, P involves terms of the form

$$\frac{\partial L}{\partial u^A_k}[\mathrm{d}u^A_i] = L\bar{u}^k_l[\mathrm{d}u^l_i] + \frac{\partial L}{\partial u^\alpha_k}[\mathrm{d}u^\alpha_i - v^\alpha_l \mathrm{d}u^l_i],$$

where as before the brackets indicate equivalence classes modulo semi-basic 1-forms. Set

$$\omega^\alpha_i = [\mathrm{d}u^\alpha_i - v^\alpha_l \mathrm{d}u^l_i], \quad \omega^k_i = \bar{u}^k_l[\mathrm{d}u^l_i] + \frac{1}{L}\frac{\partial L}{\partial u^\alpha_k}\omega^\alpha_i;$$

these 1-form classes are clearly linearly independent and have the same span as the $[\mathrm{d}u^A_i]$. In terms of this basis

$$P = Q^{ij}_{\alpha\beta}\omega^\alpha_i \odot \omega^\beta_j + LK^{ij}_{kl}\omega^k_i \odot \omega^l_j,$$

so that with respect to the ω^A_i the matrix of coefficients of P is in block-diagonal form. Thus P is non-degenerate if and only if both diagonal components are non-degenerate.

We pointed out earlier that $\omega^\alpha_i = X_i(\vartheta^\alpha)$ modulo semi-basic 1-forms. It is interesting to note in passing that a similar result holds for the relationship between ω^k_i and ϑ^k.

In the case of interest, where $P = L^{1-2/m}g$,

$$K^{ij}_{kl} = \frac{2}{m}\delta^i_k\delta^j_l - \delta^j_k\delta^i_l.$$

In this case, if V^l_j satisfies $K^{ij}_{kl}V^l_j = (2/m)\delta^i_k \operatorname{tr} V - V^i_k = 0$ then V must be diagonal; we find by taking the trace that $\operatorname{tr} V = 0$, and therefore $V^l_j = 0$. Thus g will be non-degenerate, and L will be Rund-regular, if and only if $\widetilde{Q}$ is non-degenerate.

One virtue of dealing with P of general form is that it allows us to draw further conclusions. In particular, consider the case with $K^{ij}_{kl} = \delta^i_k\delta^j_l - \delta^j_k\delta^i_l$: now P is just the Hessian

$$\frac{\partial^2 L}{\partial u^A_i \partial u^B_j}.$$

If V^l_j satisfies $K^{ij}_{kl}V^l_j = 0$ in this case, then $\delta^i_k \operatorname{tr} V - V^i_k = 0$; on taking the trace this time we find that $(m-1)\operatorname{tr} V = 0$, and therefore $V^l_j = 0$ provided that $m \neq 1$. Thus for $m > 1$, we may equally well say that L is Rund-regular if and only if the (full) Hessian of L is non-degenerate. This provides yet another insight into Rund's theory, of which he was apparently unaware in general terms; it is interesting to note, however, that for $m = 2$, g coincides with the Hessian.

We now consider Dedecker regularity. Given the form of the coordinate expression of the criterion for Dedecker regularity derived earlier, it would be easy to conclude that nothing much needs to be done here to establish equivalence. However, it must be borne in mind that that criterion is expressed in terms of $\mathscr{L}$ rather than L, and is supposed to hold in an affine coordinate neighbourhood in $\mathscr{S}^{(m)}E$, rather than on $\mathscr{F}^{(m)}E$. We now address these differences.

It will be convenient to think of the affine coordinates in terms of a local imbedding ι of $J^i\pi$ into $\mathscr{F}^{(m)}E$, given in relation to coordinates $(u^i, u^\alpha, u^j_i, u^\alpha_i)$ on $\mathscr{F}^{(m)}E$, split appropriately, by $\iota(x^i, y^\alpha, y^\alpha_i) = (x^i, y^\alpha, \delta^j_i, y^\alpha_i)$. The image of ι lies within the domain of the special coordinates we have been using. Then if L is any homogeneous Lagrangian on $\mathscr{F}^{(m)}E$, $\mathscr{L} = L \circ \iota$. Conversely, given $\mathscr{L}$, we can find at least locally a homogeneous Lagrangian L such that $\mathscr{L} = L \circ \iota$, by using the homogeneity formula in the form $L(u^j_i, u^\alpha_i) = \det(u^j_i) L(\delta^j_i, v^\alpha_i)$ (suppressing the base coordinates).

Let us denote by $\mathscr{Q}$ the tensor occuring in the criterion for Dedecker regularity. We seek to relate $\mathscr{Q}$ and Q.

Now in a neighbourhood in $\mathscr{F}^{(m)}E$ in which (u^j_i) is non-singular, we can use v^α_i as a coordinate instead of u^α_i. Let us in fact change fibre coordinates to $v^j_i = u^j_i$, $v^\alpha_i = \bar{u}^j_i u^\alpha_j$. Then $\mathrm{d}u^j_i = \mathrm{d}v^j_i$ and $\mathrm{d}u^\alpha_i = u^j_i\, \mathrm{d}v^\alpha_j + v^\alpha_j\, \mathrm{d}u^j_i$, so that

$$\frac{\partial}{\partial v^\alpha_i} = u^i_j \frac{\partial}{\partial u^\alpha_j}.$$

Moreover, $\partial/\partial v^\alpha_i$ is tangent to the image of ι, and equals $\iota_*(\partial/\partial y^\alpha_i)$ there. Now set $\check{L}(v^\alpha_i) = L(\delta^j_i, v^\alpha_i) = \det(u^j_i)^{-1} L(u^j_i, u^\alpha_i)$. Then

$$\frac{\partial \check{L}}{\partial v^\alpha_i} = \det(u^j_i)^{-1} u^i_k \frac{\partial L}{\partial u^\alpha_k}, \quad \frac{\partial^2 \check{L}}{\partial v^\alpha_i \partial v^\beta_j} = \det(u^j_i)^{-1} u^i_k u^j_l \frac{\partial^2 L}{\partial u^\alpha_k \partial u^\beta_l},$$

so that

$$\frac{\partial^2 \check{L}}{\partial v^\alpha_i \partial v^\beta_j} - \frac{1}{\check{L}}\left(\frac{\partial \check{L}}{\partial v^\alpha_i}\frac{\partial \check{L}}{\partial v^\beta_j} - \frac{\partial \check{L}}{\partial v^\alpha_j}\frac{\partial \check{L}}{\partial v^\beta_i}\right) = \det(u^j_i)^{-1} u^i_k u^j_l Q^{kl}_{\alpha\beta}.$$

We may conclude first that $\det(u^j_i)^{-1} u^i_k u^j_l Q^{kl}_{\alpha\beta}$ is a function just of v^α_i (so far as dependence on fibre coordinates goes); secondly, that $Q^{ij}_{\alpha\beta}$ is non-degenerate everywhere if it is non-degenerate at $u^j_i = \delta^j_i$; and thirdly, that $\mathscr{Q}^{ij}_{\alpha\beta} = Q^{ij}_{\alpha\beta} \circ \iota$. In fact

$\mathrm{d}u_i^\alpha - v_j^\alpha\,\mathrm{d}u_i^j = u_i^j\,\mathrm{d}v_j^\alpha$, as follows immediately from $u_i^\alpha = u_i^j v_j^\alpha$. Thus

$$\mathscr{Q}^{ij}_{\alpha\beta}[\mathrm{d}y_i^\alpha] \odot [\mathrm{d}y_j^\beta] = \iota^*(Q^{ij}_{\alpha\beta}\omega_i^\alpha \odot \omega_j^\beta).$$

So we see that L is Dedecker-regular exactly when $(Q^{ij}_{\alpha\beta})$ is non-singular. We have therefore established our main result.

Theorem 2. *If L is a non-vanishing homogeneous Lagrangian on $\mathscr{F}^{(m)}E$ where $m > 1$ then the three conditions of Rund regularity, Dedecker regularity and multisymplectic regularity are equivalent.*

7. Conclusions

In this paper we have demonstrated a relationship between three apparently different notions of regularity for a parametric variational problem: Rund regularity, on the one hand, which is a condition on the homogeneous Lagrangian function defined on $\mathscr{F}^{(m)}E$, and multisymplectic and Dedecker regularities, on the other, which are conditions on an $(m+1)$-form defined on $\mathscr{S}^{(m)}E$. We have also seen that, in some respects, this relationship for genuine multiple-integral problems is simpler than that for single-integral problems. It would, however, be too much to expect that a unified notion of regularity should be appropriate for any parametric multiple-integral problem.

A comparison with the case of affine jet bundles, considered in [8], is instructive. In that work, an *arbitrary* Lepage equivalent of a given Lagrangian m-form is considered, and it is shown that Dedecker regularity of different forms may give strictly different results. This is used to advantage by defining a Lagrangian to be *regularizable* if at least one regular Lepage equivalent exists. It is, nevertheless, desirable for the regularity condition to hold for one of the geometrically-constructed Lepage equivalents, such as the Carathéodory form or the truncated Cartan form.

In the present context there are two important geometrically-constructed m-forms associated with a homogeneous Lagrangian: the Hilbert-Carathéodory form considered above, and the fundamental form described in [5]. Both forms have the same extremals as the original Lagrangian, but the latter has the advantage that it is closed precisely when the Lagrangian is null. It would therefore be of interest to compare the regularity properties of the fundamental form with those described above, and we hope to do this in some forthcoming work.

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References

[1] *F. Cantrijn, A. Ibort and M. de Léon*: On the geometry of multisymplectic manifolds. J. Australian Math. Soc. *66* (1999), 303–330.
[2] *J. F. Cariñena, M. Crampin and L. A. Ibort*: On the multisymplectic formalism for first order field theories. Diff. Geom. Appl. *1* (1991), 345–374.
[3] *M. Crampin and D. J. Saunders*: The Hilbert-Carathéodory form for parametric multiple integral problems in the calculus of variations. Acta Applicandae Math. *76* (2003), 37–55.
[4] *M. Crampin and D. J. Saunders*: The Hilbert-Carathéodory and Poincaré-Cartan forms for higher-order multiple-integral variational problems. Houston J. Math. *30* (2004), 657–689.
[5] *M. Crampin and D. J. Saunders*: On null Lagrangians. Diff. Geom. Appl. *22* (2005), 131–146.
[6] *P. M. Dedecker*: On the generalization of symplectic geometry to multiple integrals in the calculus of variations. Lecture Notes in Mathematics, Springer *570* (1977), 395–456.
[7] *M. Giaquinta and S. Hildenbrandt*: Calculus of Variations II. Springer, 1996.
[8] *O. Krupková*: Hamiltonian field theory. J. Geom. Phys. *43* (2002), 93–132.
[9] *H. Rund*: The Hamilton-Jacobi Equation in the Calculus of Variations. Van Nostrand, 1966.
[10] *H. Rund*: A geometrical theory of multiple integral problems in the calculus of variations. Canadian J. Math. *20* (1968), 639–657.

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