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AN EXTENSION THEOREM FOR MODULAR MEASURES ON EFFECT ALGEBRAS

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Abstract. We prove an extension theorem for modular measures on lattice ordered effect algebras. This is used to obtain a representation of these measures by the classical ones. With the aid of this theorem we transfer control theorems, Vitali-Hahn-Saks, Nikodým theorems and range theorems to this setting.

Keywords: effect algebras, modular measures, extension, Vitali-Hahn-Saks theorem, Nikodým theorem, decomposition theorem, control theorems, range, Liapunoff theorem

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1. Introduction

This note contains an extension theorem for vector-valued modular measures defined on lattice ordered effect algebras. In spite of the presence in [2], [8], [9] of some results announced in the abstract, we consider it worth-while to revisit theorems about the range, control theorems, Vitali-Hahn-Saks and Nikodým theorems in the context of modular measures on effect algebras. Indeed, while in [2], [8], [9] the results are obtained by imitating the proof of the Boolean case, we here transfer the results directly from the case of vector-valued measures on Boolean algebras to the case of vector-valued modular measures on lattice ordered effect algebras. To this end we make use of a method elaborated by H. Weber (see [32]). Moreover, operating in this way, we obtain some results completely new such as the generalization of results on the range obtained by Fischer and Schoeler in the Boolean case or the generalization of results obtained by Kluvanek concerning the weak closure of the range in the Boolean case.

Effect algebras were introduced by Foulis and Bennett in 1994 [15] for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arise in quantum physics and in mathematical economics, in particular of orthomodular lattices in noncommutative measure theory and MV-algebras in fuzzy measure theory.

2. Preliminaries

In this section we give some basic definitions and fix some notation.

Definition 2.1. Let $(L, \leqslant)$ be a poset with a smallest element 0 and a greatest element 1 and let $\ominus$ be a partial operation on L such that $b \ominus a$ is defined if and only if $a \leqslant b$ and for all $a, b, c \in L$:

If $a \leqslant b$ then $b \ominus a \leqslant b$ and $b \ominus (b \ominus a) = a$.

If $a \leqslant b \leqslant c$ then $c \ominus b \leqslant c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Then $(L, \leqslant, \ominus)$ is called a *difference poset (D-poset* for short) or a *difference lattice (D-lattice* for short) if L is a lattice.

For the rest of the paper, let L be a D-lattice and let (E, τ) be a Hausdorff complete locally convex linear space.

One defines in L a partial operation $\oplus$ as follows:

$a \oplus b$ is defined and $a \oplus b = c$ if and only if $c \ominus b$ is defined and $c \ominus b = a$.

The operation $\oplus$ is well-defined by the cancellation law [17, page 13] ($a \leqslant b, c$ and $b \ominus a = c \ominus a$ implies $b = c$), and $(L, \oplus, 0, 1)$ is an effect algebra (see [17, Theorem 1.3.4]), that is, the following conditions are satisfied for all $a, b, c \in L$:

If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$;

if $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;

there exists a unique $a^{\perp} \in E$ such that $a \oplus a^{\perp}$ is defined and $a \oplus a^{\perp} = 1$;

if $a \oplus 1$ is defined, then $a = 0$.

We say that a and b are *orthogonal* if $a \leqslant b^{\perp}$ and we write $a \perp b$. Therefore $a \oplus b$ is defined if and only if $a \perp b$, and in this case $a \oplus b = (a^{\perp} \ominus b)^{\perp}$ by [17, Lemma 1.2.5]. If $a_1, \ldots, a_n \in L$ we inductively define $a_1 \oplus \ldots \oplus a_n = (a_1 \oplus \ldots \oplus a_{n-1}) \oplus a_n$ if the right-hand side exists. The sum is independent of any permutation of the elements. We say that a finite family $(a_i)_{i=1}^n$ of (not necessarily different) elements of L is *orthogonal* if $a_1 \oplus \ldots \oplus a_n$ exists.

We say that a sequence (a_n) of elements of L is orthogonal if the set $\{a_1, \dots a_n\}$ is orthogonal for every $n \in \mathbb{N}$.

Following [20], an element p of L is *central* if for every $a \in L$

$$(a \wedge p) \vee (a \wedge p^{\perp}) = a. \tag{1}$$

The *centre* $C(L)$ of L is the set of all central elements. We stress that it is a Boolean subalgebra of the centre in the lattice theoretical sense. In particular, $C(L)$ is a sublattice of L.

A function μ on a D-lattice with values in E or in $[0, +\infty]$ is called a *measure* if for every $a, b \in L$, with $a \perp b$,

$$\mu(a \oplus b) = \mu(a) + \mu(b).$$

A *modular measure* is a measure which also satisfies the modular law, that is, for all $a, b \in L$

$$\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b).$$

A measure μ is called *exhaustive* if $\mu(x_n) \to 0$ for every orthogonal sequence (x_n).

A sequence of measures (μ_n) is called *uniformly exhaustive* if for every orthogonal sequence (x_n), $\mu_m(x_n) \to 0$ uniformly in $m \in \mathbb{N}$. A measure μ is called *σ-order continuous* if the order convergence of sequences implies the convergence with respect to μ, (i.e. $a_n \uparrow a$ or $a_n \downarrow a$ implies that $\mu(a_n)$ converges to $\mu(a)$); μ is called *order continuous* if the same condition holds for nets.

A *uniform D-lattice* is a D-lattice endowed with a uniformity which makes the operations $\vee, \wedge$ and $\ominus$ uniformly continuous. We call this uniformity a *D-uniformity.*

If μ is an E-valued modular measure, it is known that the sets

$$\{(a, b) \in L \times L\colon\ \forall c, d \in [a \wedge b, a \vee b] \quad \mu(c) - \mu(d) \in W\},$$

where W is a neighborhood of 0 in E, form a base for the *μ-uniformity* $\mathscr{U}(\mu)$, which is the weakest D-uniformity that makes μ uniformly continuous (see [1, Theor. 3.2]; compare also [19] and [30, §3.1]). The topology induced by the μ-uniformity is called the μ-topology. If Λ is a set of modular functions on L, then the supremum of the λ-uniformities, $\lambda \in \Lambda$, is called the Λ-uniformity or the uniformity generated by Λ.

The proof of the following proposition is given in 4.3 of [6].

Proposition 2.2. *Let $\mathscr{U}$ be a D-uniformity. Then $N(\mathscr{U}) = \bigcap_{U\in\mathscr{U}} U$ is a D-congruence and the quotient $(\hat{L}, \hat{\mathscr{U}}) = (L, \mathscr{U})/N(\mathscr{U})$ is a D-lattice endowed with a Hausdorff D-uniformity.*

We recall that, if $\|\cdot\|$ is a seminorm on E, for a modular measure $\mu\colon L \to E$, the *total variation* is the function

$$|\mu|(a) = \sup\left\{\sum_{i=0}^{n-1} \|\mu(a_{i+1}) - \mu(a_i)\|\colon a_0 \leqslant a_1 \leqslant \ldots \leqslant a_n = a\right\}, \quad a \in L.$$

By 1.3.10 of [31], $|\mu|$ is a modular function and by 3.11 of [13] it is a measure.

For real-valued modular measures on D-lattices the following equivalence holds ([4, Lemma 2.6]):

Proposition 2.3. *Let $\mu\colon L \to \mathbb{R}$ be a modular measure. Then μ is bounded if and only if μ is exhaustive if and only if μ has bounded variation.*

3. The representation theorem

The next theorem repeats "verbatim" [33, Theorem 2.2]. Repeating the proof, we have only to observe that $\mu_D(p) = \sum_{d\in D} (\tilde{\lambda}(d\wedge p)/\tilde{\lambda}(d))\tilde{\mu}(d)$ (where D is a suitable decomposition) are measures and this follows from [4, 2.2 and 2.4]. We notice that the above net is involved in the definition of the extension $\bar{\mu}$ that is its weak limit and so it is a measure, too.

Theorem 3.1. *Let A be a Boolean sublattice of $C(L)$, $\Lambda \subset \{\lambda\colon L \to \mathbb{R}^+ : \lambda$ is a modular measure$\}$, let $\mathscr{U}$ be the Λ-uniformity and $\mu\colon (A, \mathscr{U}|_A) \to (E, \tau)$ a continuous modular measure. Then μ has an extension to an exhaustive continuous modular measure $\bar{\mu}\colon (L, \mathscr{U}) \to (E, \tau)$ with $\bar{\mu}(L) \subset \overline{\mathrm{co}}\,\mu(A)$.*

It turns out that Theorem 3.1 remains true for measures with values in a linear topological space which is not locally convex, i.e. ℓ_p $0 < p < 1$; we are going to show this fact.

Theorem 3.2. *Let A be a Boolean sublattice of $C(L)$, $\Lambda \subset \{\lambda\colon L \to \mathbb{R}^+ : \lambda$ is a modular measure$\}$, let $\mathscr{U}$ be the Λ-uniformity and $\mu\colon (A, \mathscr{U}|_A) \to \ell_p$ $(0 < p < 1)$ a continuous modular measure. Then μ has an extension to an exhaustive continuous modular measure $\bar{\mu}\colon (L, \mathscr{U}) \to \ell_p$ $(0 < p < 1)$ with $\bar{\mu}(L) \subset \overline{\mathrm{co}}^p\,\mu(A)$, where the closure is understood in $(\ell_p, \|\ \|_p)$.*

Proof. The proof proceeds by inspection from 3.1. Since ℓ_p is continuously embedded in ℓ_1 and our extension theorem holds true for measures with values in ℓ_1, we have only to ensure that starting with $\mu_D(L) \subseteq \ell_p$, its limit $\bar{\mu}$ lies in ℓ_p $(0 < p < 1)$.

We claim that the values of μ_D lie in a bounded set of ℓ_p: Indeed, by Labuda [18, Proposition on p. 58] the convex hull of the range of a bounded measure with values in ℓ_p is bounded and (e.g. see [30, 2.3]) this implies that the range of a group-valued measure is bounded (we point out that in ℓ_p $(0 < p < 1)$ a subset is norm bounded if and only if it is bounded).

For every $a \in L$, the *bounded* net $\mu_D(a)$ converges to $\bar{\mu}(a)$ weakly, hence coordinatewise. Therefore the values of $\bar{\mu}$ lie in ℓ_p, as desired.

The last inclusion derives from $\bar{\mu}(L) \subset \overline{\mathrm{co}}^1\, \mu(A)$ proved in Theorem 3.1 and from the equality $\overline{\mathrm{co}}^1\, \mu(A) = \overline{\mathrm{co}}^p\, \mu(A)$. For, $\overline{\mathrm{co}}^1\, \mu(A)$ is a closed, bounded, convex set in ℓ_p. It follows from [24, Theorem 1] that $\overline{\mathrm{co}}^1\, \mu(A)$ is compact in ℓ_p. As the inclusion of ℓ_p into ℓ_1 is continuous we have $\overline{\mathrm{co}}^1\, \mu(A) = \overline{\mathrm{co}}^p\, \mu(A)$.

Finally, the exhaustivity derives from the compactness and [27, 3.2.5]. The extension is $\mathscr{U}$ continuous by [30, 6.1] since μ is exhaustive. □

Notation 3.3. *Let $\mathscr{V}, \mathscr{U}$ be D-uniformities and μ, ν modular measures on L, then $\mu \ll \mathscr{U}$ means that μ-uniformity is weaker than $\mathscr{U}$ and $\mu \perp \mathscr{U}$ means that the infimum between the μ-uniformity and $\mathscr{U}$ is the trivial uniformity. If $\mathscr{U} = \nu$-uniformity, we write $\mu \ll \nu$ or $\mu \perp \nu$ instead of $\mu \ll \mathscr{U}$ or $\mu \perp \mathscr{U}$.*

Repeating line by line [33, Theorem 3.1.7] and applying 3.1 when Weber applies his Theorem 2.2 and [4, 2.4] when Weber applies his Theorem 3.1.5, one obtains

Theorem 3.4. *Let $\Lambda \subset \{\lambda\colon L \to \mathbb{R}^+ : \lambda$ is a modular measure$\}$ and let $\mathscr{U}$ be the Λ-uniformity and $(\widetilde{L}, \widetilde{\mathscr{U}})$ the uniform completion of the quotient $(\hat{L}, \hat{\mathscr{U}}) := (L, \mathscr{U})/N(\mathscr{U})$. Then:*

(a) *For any continuous modular measure $\mu\colon (L, \mathscr{U}) \to (E, \tau)$, the function $\hat{\mu}\colon \hat{L} \to E$ defined by $\hat{\mu}(\hat{x}) = \mu(x)$ whenever $x \in \hat{x} \in \hat{L}$, has a unique continuous extension $\tilde{\mu}\colon (\widetilde{L}, \widetilde{\mathscr{U}}) \to E$. Denote by $\bar{\mu}\colon C(\widetilde{L}) \to E$ the restriction of $\tilde{\mu}$ to $C(\widetilde{L})$. Then $\overline{\mu(L)} = \overline{\tilde{\mu}(\widetilde{L})}$ and $\overline{\mathrm{co}}\, \mu(L) = \overline{\mathrm{co}}\, \bar{\mu}(C(\widetilde{L}))$.*

(b) *If $\mathscr{U}$ is generated by the set of all real-valued bounded modular measures on L, then the map $\mu \mapsto \bar{\mu}$ defines an isomorphism from the linear space of all exhaustive modular measures $\mu\colon L \to E$ onto the linear space of all order continuous measures $\bar{\mu}\colon C(\widetilde{L}) \to E$.*

(c) *Let F be a complete locally convex Hausdorff linear space and let $\nu\colon (L, \mathscr{U}) \to F$, $\mu\colon (L, \mathscr{U}) \to E$ be modular measures. Then $\mu \ll \nu$ if and only if $\bar{\mu} \ll \overline{\nu}$ and*

$\mu \perp \nu$ if and only if $\bar{\mu} \perp \overline{\nu}$. Moreover, if $T\colon E \to F$ is a continuous additive map and $\nu = T \circ \mu$, then $\hat{\nu} = T \circ \hat{\mu}, \tilde{\nu} = T \circ \tilde{\mu}$ and $\overline{\nu} = T \circ \bar{\mu}$.

(d) *Let E be a Banach space and $\mu\colon (L, \mathscr{U}) \to E$ be a continuous modular measure. Then μ has bounded variation iff $\tilde{\mu}$ has bounded variation. Moreover, if $\nu := |\mu|$ is bounded, then $\tilde{\nu} = |\tilde{\mu}|$ and $\overline{\nu} = |\bar{\mu}|$.*

Remark 3.5. Applying Theorem 3.2 instead of 3.1, one can observe that Theorem 3.4(a) remains true when one substitutes ℓ_p $(0 < p < 1)$ for a complete locally convex Hausdorff linear space.

4. Applications

We illustrate the method of transferring results from the Boolean case to this setting:

4.1. The control. A measure ν is a control for a set of measures Λ if

$$\nu - \text{uniformity} = \Lambda - \text{uniformity}.$$

Theorem 4.1. *Let E be a Banach space and $\mu\colon L \to E$ an exhaustive modular measure. Then there exists an $x' \in E'$ such that $x' \circ \mu$ is a control of μ.*

Proof. By Ribakov's theorem there exists $x' \in E'$ such that $x' \circ \bar{\mu}$ is a control for $\bar{\mu}$, by 3.4 $x' \circ \mu$ is a control of μ. □

In order to prove Theorem 4.4, we need some preparatory stuff.

The following proposition is contained in [30, 6.2].

Proposition 4.2. *Let $\mathscr{U}$ be a D-uniformity and $\mu_n\colon (L, \mathscr{U}) \to (E, \tau)$, $n \in \mathbb{N}$, a pointwise bounded, uniformly exhaustive sequence of continuous modular measures. Then $\mu_n\colon (L, \mathscr{U}) \to (E, \tau)$, $n \in \mathbb{N}$, is equicontinuous.*

Proposition 4.3. *Let $\mathscr{U}$ be the uniformity generated by $\{\mu_n\colon n \in \mathbb{N}\}$ where (μ_n) is a pointwise bounded sequence of exhaustive modular measures on L with values in E. Assume the same notation as in 3.4. Then $(\mu_n)_{n\in\mathbb{N}}$ is uniformly exhaustive if and only if $(\bar{\mu}_n)_{n\in\mathbb{N}}$ is uniformly exhaustive.*

Proof. $\Rightarrow$: We suppose that (μ_n) is uniformly exhaustive. Then $\nu := (\mu_n)_{n\in\mathbb{N}}\colon L \to (l_\infty(E), \tau_\infty)$ is exhaustive, besides, from 4.2 (μ_n) is equicontinuous with respect to $\mathscr{U}$. Hence $\hat{\nu} = (\hat{\mu}_n)_{n\in\mathbb{N}}\colon (\hat{L}, \hat{\mathscr{U}}) \to (l_\infty(E), \tau_\infty)$ is continuous. Let $\tilde{\nu}\colon (\widetilde{L}, \widetilde{\mathscr{U}}) \to$

$(l_\infty(E), \tau_\infty)$ be the continuous extension of $\hat{\nu}$. Then $\tilde{\nu} = (\tilde{\mu}_n)_{n\in\mathbb{N}}$. Since $\tilde{\nu}$ is exhaustive, $(\tilde{\mu}_n)$ is uniformly exhaustive and hence $(\bar{\mu}_n)$ is so.

$\Leftarrow$: Suppose that $(\bar{\mu}_n)_{n\in\mathbb{N}}$ is uniformly exhaustive. Then $\overline{\nu} := (\bar{\mu}_n)_{n\in\mathbb{N}}\colon C(\widetilde{L}) \to (l_\infty(E), \tau_\infty)$ is exhaustive. Moreover, by 4.2 here applied to the Boolean case, $(\bar{\mu}_n)$ is equicontinuous with respect to $\widetilde{\mathscr{U}}|_{C(\widetilde{L})}$, i.e. $\overline{\nu}\colon (C(\widetilde{L}), \widetilde{\mathscr{U}}|_{C(\widetilde{L})}) \to l_\infty(E)$ is continuous. Let $\nu\colon (L, \mathscr{U}) \to l_\infty(E)$ be the measure which corresponds to $\overline{\nu}$ according to 3.4. Applying 3.4(c) to the projections $(x_n)_{n\in\mathbb{N}} \mapsto x_n$ which carry from $l_\infty(E)$ to E, one obtains that $\nu = (\mu_n)_{n\in\mathbb{N}}$. As $\nu\colon (L, \mathscr{U}) \to l_\infty(E)$ is continuous and hence exhaustive, (μ_n) is uniformly exhaustive. $\square$

Theorem 4.4. *Let $\mu_n\colon L \to E$, $n \in \mathbb{N}$, be a pointwise bounded, uniformly exhaustive sequence of modular measures. Then there is a modular measure $\nu\colon L \to E$ which controls (μ_n).*

Proof. By 4.3 $\{\bar{\mu}_n\colon n \in \mathbb{N}\}$ is uniformly exhaustive and by [11, Theorem 2] it has a control $\overline{\nu}\colon C(\widetilde{L}) \to E$. By 3.4 the corresponding measure $\nu\colon L \to E$ is a control for $\{\mu_n\colon n \in \mathbb{N}\}$. $\square$

4.2. The Vitali-Hahn-Saks theorem.

Theorem 4.5. *Let L be σ-complete and let $\mu_n\colon L \to E$, $n \in \mathbb{N}$, be a sequence of σ-order continuous modular measures which converges pointwise to $\mu\colon L \to E$.*

(a) *Then the sequence (μ_n) is uniformly exhaustive and μ is a σ-order continuous modular measure.*
(b) *If for every $n \in \mathbb{N}$, μ_n is continuous with respect to a D-uniformity $\mathscr{V}$ on L, then $\{\mu_n\colon n \in \mathbb{N}\} \cup \{\mu\}$ is equicontinuous with respect to $\mathscr{V}$.*

Proof. Let $\mathscr{U}$ be the uniformity generated by $\{\mu_n\colon n \in \mathbb{N}\}$. As E is a subspace of a product of Banach spaces, we may assume that E is a Banach space. Then $(L, \mathscr{U})$ is complete by [31, 1.1.4]. Passing to the quotient $(\hat{L}, \hat{\mathscr{U}}) := (L, \mathscr{U})/N(\mathscr{U})$ we can suppose that $\mathscr{U}$ is Hausdorff. So $(L, \mathscr{U}) = (\hat{L}, \hat{\mathscr{U}}) = (\widetilde{L}, \widetilde{\mathscr{U}})$. By the classical Vitali-Hahn-Saks-Nikodým theorem the restrictions $\bar{\mu}_n := \mu_n|_{C(L)}$, $n \in \mathbb{N}$, are uniformly exhaustive. Hence μ_n, $n \in \mathbb{N}$, are uniformly exhaustive by 4.3 and therefore $\{\mu_n\colon n \in \mathbb{N}\} \cup \{\mu\}$ is uniformly exhaustive. Hence by 4.2, $\{\mu_n\colon n \in \mathbb{N}\} \cup \{\mu\}$ is equicontinuous with respect to $\mathscr{U}$ if $\mu_n \ll \mathscr{U}$ for any $n \in \mathbb{N}$. $\square$

4.3. The Nikodým theorem. The classical Nikodým boundedness theorem asserts that for families of exhaustive measures defined on a σ-algebra, pointwise boundedness and uniform boundedness are equivalent. It is known that an analogous result is not true for measures on effect algebras, but it is true for modular functions on orthomodular lattices.

Theorem 4.6. *Let L be σ-complete and let M be a pointwise bounded set of σ-order continuous modular measures on L. Then M is uniformly bounded.*

Proof. Since a subset A of E is bounded if and only if all countable subsets of A are bounded, we may assume that M is countable, i.e. $M := \{\mu_n \colon n \in \mathbb{N}\}$. As in the proof of 4.5 we may assume that E is a Banach space. Let $\mathscr{U}$ be the uniformity generated by $\{\mu_n \colon n \in \mathbb{N}\}$. Then $(L, \mathscr{U})$ is complete by [31, 1.1.4]. Passing to the quotient $(\hat{L}, \hat{\mathscr{U}}) := (L, \mathscr{U})/N(\mathscr{U})$ we may assume that $\mathscr{U}$ is Hausdorff. So $(L, \mathscr{U}) = (\hat{L}, \hat{\mathscr{U}}) = (\widetilde{L}, \widetilde{\mathscr{U}})$. By the classical Nikodým theorem, $\{\bar{\mu}_n \colon n \in \mathbb{N}\}$ is uniformly bounded, i.e., for some positive number r we have that $\bar{\mu}_n(C(L)) \subset \{x \in E \colon \|x\| \leqslant r\}$ for every $n \in \mathbb{N}$. By 3.4 $\mu_n(L) \subset \overline{\mathrm{co}}\, \bar{\mu}_n(C(L)) \subset \{x \in E \colon \|x\| \leqslant r\}$. Hence $\{\mu_n \colon n \in \mathbb{N}\}$ is uniformly bounded. $\square$

4.4. The range. First, we present the following theorem which generalizes Kluvánek's result (see [23]). We recall that Kluvánek proved that if m is an E-valued measure on a σ-algebra and $R(m)$ denotes the range of m, then the weak closure of $R(m)$ coincides with the closed convex hull of $R(m)$.

We will make use of the concept of "chained". A uniform space $(X, \mathscr{U})$ is called *chained* if for every $x, y \in X$ and every $U \in \mathscr{U}$ there is a finite sequence $x_0, \ldots, x_n \in X$ with $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in U$ for $i = 1, \ldots, n$. If $\mu \colon L \to E$ is a modular measure, we say that L is *μ-chained* if L is chained with respect to the μ-uniformity. In [13, 2.6] it is proved that L is μ-chained if and only if for every 0-neighborhood U in E there is an orthogonal family $(a_i)_{i=1}^n$ in L such that $\bigoplus\limits_{i=1}^n a_i = 1$ and $\mu([0, a_i]) \subseteq U$ for $i = 1, \ldots, n$. The latter condition for a real-valued measure μ on a Boolean algebra means that μ is "strongly continuous" according to [14].

Theorem 4.7. *Let $\mu \colon L \to E$ be a modular measure such that L is μ-chained. Then for every $a \in L$, the weak closure of $\mu([0, a])$ is convex.*

Proof. For $A \subseteq E$ we denote by $\bar{A}^w$ the weak closure of A. Moreover, let E' be the topological dual of E. Let $a \in L$. We prove that $\mathrm{co}\, \mu([0, a]) \subseteq \overline{\mu([0, a])}^w$. Let $y \in \mathrm{co}\, \mu([0, a])$ and $f_1, \ldots, f_n \in E'$. Set $\nu = (f_1 \circ \mu, \ldots, f_n \circ \mu)$. Then $\nu \colon L \to \mathbb{R}^n$ is a modular measure such that L is ν-chained, because ν-uniformity $\leqslant$ μ-uniformity. By [13] $\nu([0, a])$ is convex. Then $(f_1(y), \ldots, f_n(y)) \in \mathrm{co}\, \nu([0, a]) = \nu([0, a])$. Therefore there exists $x \leqslant a$ such that $f_i(y) = f_i(\mu(x))$ for every $i \leqslant n$. Hence $y \in \overline{\mu([0, a])}^w$. $\square$

Proposition 4.8. *Let L be irreducible and complete. Suppose that there exists a nontrivial order continuous modular measure $\nu\colon L \to E$.*

(a) *Then there is a unique order continuous modular measure $\lambda\colon L \to \mathbb{R}$ with $\lambda(1) = 1$, λ is strictly increasing and every order continuous modular measure has the form $\mu = \lambda\mu(1)$.*

(b) *If L is atomless, then the range of λ is the closed real unit interval.*

Proof. (a) Let $\mu\colon L \to E$ be a nontrivial order continuous modular measure. Then for some $x' \in E'$, $x' \circ \mu$ is not trivial and has bounded variation by 2.3. Therefore $\lambda := |x' \circ \mu|/|x' \circ \mu|(1)$ is an increasing modular measure with $\lambda(1) = 1$. Since $\sup\{x \in L\colon \lambda(x) = 0\} \in C(L) = \{0, 1\}$ by [10, 5.3], λ is strictly positive and therefore strictly increasing. Moreover, λ is order continuous since λ-uniformity= $x' \circ \mu$-uniformity by [31, 1.3.11].

We will now apply 3.4 to $\mathscr{U}$ being the λ-uniformity. Observe that $(L, \mathscr{U})$ is complete by [31, 1.1.4] and therefore $L = \hat{L} = \widetilde{L}$. Since the modular measures satisfy $\mu(x) = \lambda(x)\mu(1)$ for $x \in C(L)$, they are equal by 3.4. Hence $\mu = \lambda\mu(1)$.

(b) If L is atomless, then L is connected by [5, 4.3]. Therefore the continuous image $\lambda(L)$ is an interval. Since λ is increasing, $\lambda(1) = 1$, we get $\lambda(L) = [0, 1]$. □

Remark 4.9. Proposition 4.8 remains true for modular measures with values in ℓ_p $(p < 0 < 1)$.

We continue by recalling the following theorem which is contained in [5].

Theorem 4.10. *Let G be a complete Hausdorff topological Abelian group and let $\mu\colon L \to G$ be an exhaustive modular measure. Then there are exhaustive modular measures λ and μ_a $(a \in A)$ on L and elements $g_a \in G$ $(a \in A)$ such that*

(a) *$(\mu_a(x))_{a\in A}$ is uniformly summable in $x \in L$.*

(b) *$\mu = \lambda + \sum\limits_{a\in A} \mu_a$.*

(c) *L is λ-chained.*

(d) *For any $a \in A$, the quotient $L_a := L/N(\mu_a)$ is an irreducible modular D-lattice of finite length; $\mu_a(x) = h(\hat{x}) \cdot g_a$ where $x \in L$, $\hat{x}$ is the corresponding element of the quotient L_a and $h(\hat{x})$ is the height of $\hat{x}$ in L_a.*

(e) *$\lambda(L)$ is dense in an arcwise connected subset of G, in particular $\overline{\lambda(L)}$ is connected. The range of $\sum\limits_{a\in A} \mu_a$ is relatively compact.*

We can refine the previous theorem in the following way:

Theorem 4.11. *Let L be complete and let $\mu\colon L \to E$ be an order continuous modular measure such that the μ-uniformity is Hausdorff. Then there is a μ-continuous modular measure $\nu\colon L \to E$ and there are μ-continuous increasing modular measures $\varrho_a\colon L \to \mathbb{R}$ and $\sigma_b\colon L \to \mathbb{R}$ and elements y_a, $z_b \in E$ $(a \in A, b \in B)$ with the following properties:*

(a) *$(\varrho_a(x)y_a)_{a\in A}$ and $(\sigma_b(x)z_b)_{b\in B}$ are summable uniformly in $x \in L$; $\mu = \nu + \varrho + \sigma$ where $\varrho = \sum\limits_{a\in A} \varrho_a \cdot y_a$ and $\sigma = \sum\limits_{b\in B} \sigma_b \cdot z_b$.*

(b) *$\varrho_a(L) = [0,1]$ for $a \in A$; $\varrho(L)$ is convex and compact; $\sigma(L)$ is compact and $\overline{\mathrm{co}}\,\nu(L) = \overline{\mathrm{co}}\,\nu(C(L))$.*

(c) *The restriction $\nu|_{C(L)}$ is an atomless measure.*

(d) *$\sigma = 0$ if and only if L is atomless.*

Proof. Let A be the set of all atoms a of $C(L)$ for which $[0,a]$ is atomless and let B be the set of the other atoms of $C(L)$. For $p \in A \cup B$, the interval $[0,p]$ is an irreducible lattice. Therefore by 4.8 there is an increasing modular measure $\lambda_p\colon [0,p] \to \mathbb{R}$ with $\lambda_p(p) = 1$ and $\mu(x) = \lambda_p(x)\mu(p)$ $(x \in [0,p])$; $\lambda_p([0,p]) = [0,1]$ if $p \in A$. Let t be the unique complement of $\sup(A\cup B)$ in $C(L)$. We put $\nu(x) = \mu(x\wedge t)$, $\varrho_a(x) = \lambda_a(x \wedge a)$, $\sigma_b(x) = \lambda_b(x \wedge b)$, $y_a = \mu(a)$, $z_b = \mu(b)$ for $a \in A$, $b \in B$, $x \in L$. Then the measures just defined have the desired properties. Indeed, $\varrho(L)$ is the image of the compact convex set I^A under the continuous affine map $(t_a)_{a\in A} \mapsto \sum\limits_{a\in A} t_a \cdot y_a$ and therefore it is compact and convex. Analogously, using (d) of 4.10 we obtain that $\sigma(L)$ is compact; the last assertion of item (b) derives from 3.4(a). Items (c) and (d) are obvious. $\square$

Now we furnish a tool for transferring immediately results known for classical measures on Boolean algebras to the context of modular measures defined on D-lattices.

Theorem 4.12.

(a) *Assume that E has the property that $\overline{\mu(A)}$ is compact for every order continuous atomless measure $\mu\colon A \to E$ defined on a complete Boolean algebra. Then $\overline{\mu(L)}$ is compact (convex, respectively) for every exhaustive modular measure $\mu\colon L \to E$ (with L being μ-chained).*

(b) *Assume that E is a Banach space and $\overline{\mu(A)}$ is compact (convex) for every order continuous atomless measure $\mu\colon A \to E$ of bounded variation defined on a complete Boolean algebra. Then $\overline{\mu(L)}$ is compact (convex, respectively) for every modular measure $\mu\colon L \to E$ of bounded variation (with L being μ-chained).*

P r o o f. We prove only the convexity assertion in (b). The other three assertions can be proved analogously. Let $\mu\colon L \to E$ be a modular measure of bounded variation such that L is μ-chained. We can apply 3.4 to $\mathscr{U} = \mu$-uniformity, observing that by [30, 6.3] μ-uniformity=$\sup x' \circ \mu$-uniformity. Since $\overline{\mu(L)} = \overline{\tilde{\mu}(\widetilde{L})}$, it is enough to show that $\overline{\tilde{\mu}(\widetilde{L})}$ is convex. Replacing μ and L by $\tilde{\mu}$ and $\widetilde{L}$ we may assume that μ is order continuous and L is complete. Since L is atomless, with the notation of 4.11 we have $\overline{\mu(L)} = \overline{\nu(L)} + \varrho(L)$. Observe that $\overline{\nu(L)}$ is convex since $\overline{\nu(C(L))}$ is convex by the assumptions and therefore $\overline{\nu(L)} = \overline{\nu(C(L))}$. Moreover, $\varrho(L)$ is convex. Then $\overline{\mu(L)}$ is convex. $\square$

Observe that in Theorem 4.12 one can replace the order continuous measures on complete Boolean algebras by σ-additive measures on σ-fields of sets. This follows from the representation of σ-complete Boolean algebras.

Theorem 4.12 allows us to transfer e.g. the following theorem of Uhl [25], Kadets [21], Kadets-Shekhtman [22] and Fischer-Schoeler [18] to the setting of modular measures on D-lattices.

Recall that E is said to have the *Radon-Nikodým property* if for every σ-algebra of sets Σ, for every σ-additive nonnegative measure ν on Σ and for every E-valued ν-continuous measure μ of bounded variation on Σ, there exists a ν-integrable function f such that $\mu(A) = \int_A f \, \mathrm{d}\nu$ for all $A \in \Sigma$.

E is said to be *B-convex* if there exists an integer $n \geqslant 2$ and a real number $0 < k < 1$ such that for every $x_1, \ldots, x_n$ in E, $\min\limits_{\alpha_i = \pm 1} \| \sum\limits_{i=1}^{n} \alpha_i x_i \| \leqslant kn \sup\limits_{i \leqslant n} \|x_i\|$.

By virtue of Theorem 4.12 we are able to generalize the following theorem to the setting of modular measures on D-lattices.

Theorem 4.13. *Let $\mathcal{F}$ be a field, E be a Banach space and $\mu\colon \mathscr{F} \to E$ an atomless σ-additive measure.*

If E has the Radon-Nikodým property and μ has bounded variation or if E is B-convex and μ has bounded variation or if $E = c_0$ or $E = \ell_p$ for some $p \in]1, +\infty[\setminus\{2\}$, then $\overline{\mu(\mathscr{F})}$ is convex.

Remark 4.14. Since Theorem 4.12 is still valid for $E = \ell_p$ $(0 < p < 1)$, the same holds true for Theorem 4.13 in ℓ_p $(0 < p < 1)$.

4.5. The decomposition theorem.

Theorem 4.15. *Let F be a locally convex Hausdorff linear space. Let $\mu\colon L \to E$ and $\nu\colon L \to F$ be exhaustive modular measures. Then there are exhaustive modular measures μ_1 and μ_2 such that $\mu = \mu_1 + \mu_2$, $\mu_1 \ll \nu$ and $\mu_2 \perp \nu$.*

P r o o f. We use the notation as in 3.4 where $\mathscr{U}$ is generated by the set of all real-valued bounded modular measures on L. By [26, 5.1] $\bar{\mu}$ has a unique decomposition of the form $\bar{\mu} = \lambda_1 + \lambda_2$ where λ_1 and λ_2 are E-valued $\bar{\mu}$-continuous measures on $C(\widetilde{L})$, $\lambda_1 \ll \overline{\nu}$ and $\lambda_2 \perp \overline{\nu}$. By 3.4 there are exhaustive modular measures $\mu_i \colon L \to E$ with $\bar{\mu}_i = \lambda_i$ decomposing μ. $\square$

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