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DENSELY CONTINUOUS FORMS, POINTWISE TOPOLOGY AND CARDINAL FUNCTIONS

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Abstract. We consider the space $D(X,Y)$ of densely continuous forms introduced by Hammer and McCoy [5] and investigated also by Holá [6]. We show some additional properties of $D(X,Y)$ and investigate the subspace $D^*(X)$ of locally bounded real-valued densely continuous forms equipped with the topology of pointwise convergence τ_p. The largest part of the paper is devoted to the study of various cardinal functions for $(D^*(X),\tau_p)$, in particular: character, pseudocharacter, weight, density, cellularity, diagonal degree, π-weight, π-character, netweight etc.

Keywords: locally bounded densely continuous form, topology of pointwise convergence, cardinal function, weight, density, netweight, cellularity

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1. Introduction

For Hausdorff spaces X, Y, Z we denote by $F(X,Y)$ the space of all functions from X to Y; by 2^Z, $\mathfrak{K}(Z)$ and $\mathfrak{F}(Z)$ we mean the family of all closed, nonempty compact and finite subsets of Z, respectively. Thus $F(X,2^Y)$ denotes the space of all closed-valued multifunctions (set-valued maps) from X to Y.

We also introduce other subspaces of $F(X,2^Y)$ which, however, will for the sake of simplicity be denoted by $G(X,Y)$ and $D(X,Y)$. The former, $G(X,Y)$, denotes the subspace of all multifunctions with closed graphs. To define the latter consider the set $DC(X,Y)$ of the so-called densely continuous functions, i.e. functions from X to Y such that the set $C(f)$ of points of continuity is dense in X. Now, for

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$f \in DC(X,Y)$ let φ_f denote the closure of the graph of $f \upharpoonright C(f)$ in $X \times Y$. The space $D(X,Y)$ of densely continuous forms then consists of the elements φ_f for all $f \in DC(X,Y)$. The elements φ_f of $D(X,Y)$ can also be regarded as closed valued multifunctions if we consider $\varphi_f(x) = \{y \in Y;\ (x,y) \in \varphi_f\}$ for each $x \in X$. Then one can observe that $D(X,Y) \subseteq G(X,Y) \subseteq F(X, 2^Y)$.

In the case $Y = \mathbb{R}$ (with the usual metric) the range space is omitted from the notation, so $D(X)$ means $D(X,\mathbb{R})$. Most of the time we shall be interested in the subspace $D^*(X)$ which consists of locally bounded forms from $D(X)$, i.e. elements $\varphi_f \in D(X)$ which are bounded on some neighborhood of every point $x \in X$. It is interesting to note that for a Baire space X the set $D^*(X)$ coincides with the set of all minimal USCO maps from X to $\mathbb{R}$. These minimal USCO maps are known to have many interesting applications (see Holá [6] or Christensen [3]).

To define the topology τ_p of pointwise convergence on $F(X, 2^Y)$ consider (Y,d) to be a metric space and consider the Hausdorff (extended-valued) metric H on 2^Y defined for nonempty A, B in 2^Y by

$$H(A,B) = \inf\{\varepsilon > 0;\ A \subseteq S_\varepsilon[B],\ B \subseteq S_\varepsilon[A]\},$$

where $S_\varepsilon[A] = \{y \in Y;\ d(y,A) < \varepsilon\}$. We also define $H(A,\emptyset) = H(\emptyset, A) = \infty$ for $A \neq \emptyset$ in 2^Y. Then $(2^Y, H)$ is a metric space. The topology τ_p of pointwise convergence on $F(X,2^Y)$ is induced by the uniformity $\mathfrak{U}_p$ of pointwise convergence which has a base consisting of sets of the form

$$W(A,\varepsilon) = \{(\varphi,\psi);\ \forall x \in A \ \ H(\varphi(x),\psi(x)) < \varepsilon\}$$

for all $A \in \mathfrak{F}(X)$ and $\varepsilon > 0$. The general τ_p-basic neighborhood of $\varphi \in F(X,2^Y)$ will be denoted by $W(\varphi, A, \varepsilon)$, i.e. $W(\varphi, A, \varepsilon) = W(A,\varepsilon)[\varphi]$. If $A = \{a\}$, we may write $W(\varphi, a, \varepsilon)$ instead of $W(\varphi, \{a\}, \varepsilon)$. The space $D^*(X)$ with the induced topology τ_p will be denoted by $D^*_p(X)$ for short.

In the paper's second section we establish some properties of the general space $D(X,Y)$ of densely continuous forms and of the particular subspace $D^*(X)$ of locally bounded real-valued densely continuous forms, including the metrizability and first countability of these spaces.

In the paper's third section we investigate the cardinal characteristics of the space $D^*(X)$ equipped with the topology of pointwise convergence, in particular: character of the space, pseudocharacter, weight, density, diagonal degree, netweight, π-weight, π-character etc.

Throughout the paper we assume all spaces to be Hausdorff and we use a somewhat standard notation of the closure, interior, boundary and the complement of a set A, namely $\bar{A}$, $\operatorname{int} A$, $\operatorname{bd} A$, A^c, respectively. The topology of a space Z is denoted by $\tau(Z)$ and an open local base at $x \in Z$ is denoted by $\mathfrak{B}(x)$. The cardinality of a set A is denoted by $|A|$.

2. Properties of densely continuous forms

The following statement is similar to Proposition 2.2 from the paper of Holá [6] with the local compactness of Y being replaced by Čech-completeness. A multifunction $\varphi\colon X \to Y$ is called upper semicontinuous at a point $x \in X$ (according to Beer [1]), if for each open V containing $\varphi(x)$, there is $O \in \mathfrak{B}(x)$ with $\varphi[O] \subseteq V$. An everywhere upper semicontinuous multifunction with nonempty compact values is called an USCO map (see Christensen [3]).

Proposition 2.1. *If X is a topological space, Y is a Čech-complete space and $\varphi \in D(X,Y)$, then there is a dense G_δ set $U \subseteq X$ such that φ is upper semicontinuous and has nonempty compact values at each $x \in U$.*

Proof. Let $f \in DC(X,Y)$ be such that $\varphi = \overline{f \restriction C(f)}$. By the characterization of Čech-complete spaces from Engelking (see [4], Theorem 3.9.2), there is a countable family $\{\mathfrak{A}_n\}_{n=1}^{\infty}$ of open covers of Y with the property that given a family $\mathfrak{C}$ of closed sets which has the finite-intersection property and for every $n \in \mathbb{N}$ contains sets of diameter less than the cover $\mathfrak{A}_n$, then $\mathfrak{C}$ has nonempty intersection. Define a set $U \subseteq X$ by

$$U = \bigcap_{n\in\mathbb{N}} \Big\{x \in X;\ \ \exists O_n \in \mathfrak{B}(x)\ \ \exists A_n \in \mathfrak{A}_n\colon\ \overline{f\,[O_n \cap C(f)]} \subseteq A_n\Big\}.$$

Clearly U is G_δ, because the intersecting sets are open. If $x \in C(f)$ and $n \in \mathbb{N}$, then $\mathfrak{A}_n$ covers the point $f(x)$, i.e. there is an open set $A_n \in \mathfrak{A}_n$ such that $f(x) \in A_n$. Since f is continuous at x and Y is Tychonoff, there is $O_n \in \mathfrak{B}(x)$ such that $\overline{f\,[O_n \cap C(f)]} \subseteq \overline{f\,[O_n]} \subseteq A_n$. Therefore $x \in U$, that is, U contains $C(f)$ and hence is dense.

Next we show that $\varphi(x) \neq \emptyset$ for each $x \in U$ (every $\varphi \in D(X,Y)$ is closed-valued). So let $x \in U$ and consider the family $\mathfrak{C}(x)$ of closed sets of the form $\mathfrak{C}(x) = \{\overline{f\,[O \cap C(f)]};\ O \in \mathfrak{B}(x)\}$. Then $\mathfrak{C}(x)$ has the finite-intersection property (FIP). Moreover, since $x \in U$, for each $n \in \mathbb{N}$ there are $O_n \in \mathfrak{B}(x)$ and $A_n \in \mathfrak{A}_n$ with $\overline{f\,[O_n \cap C(f)]} \subseteq A_n$ and $\overline{f\,[O_n \cap C(f)]} \in \mathfrak{C}(x)$. Therefore $\mathfrak{C}(x)$ contains a set of diameter less than the cover $\mathfrak{A}_n$ for each $n \in \mathbb{N}$, so by the characterization of Čech-completeness the intersection of $\mathfrak{C}(x)$ is nonvoid, i.e. $\emptyset \neq \bigcap \mathfrak{C}(x) = \varphi(x)$.

To show that φ has compact values we use a variation of this approach and an equivalent definition of compactness. Choose $x \in U$ arbitrarily. Since $\varphi(x)$ is closed in Y, any subset closed in $\varphi(x)$ is also closed in Y. So let $\mathfrak{H} = \{C_\alpha;\ \alpha \in I\}$ be a family of closed subsets of $\varphi(x)$ with FIP. Since $x \in U$, for each $n \in \mathbb{N}$ there are $O_n \in \mathfrak{B}(x)$ and $A_n \in \mathfrak{A}_n$ such that $A_n \supseteq \overline{f\,[O_n \cap C(f)]} \supseteq \varphi(x) \supseteq C_\alpha$ for each $\alpha \in I$. Therefore $\mathfrak{H}$ contains sets of diameter less than the cover $\mathfrak{A}_n$ for each $n \in \mathbb{N}$ and hence by the characterization of Čech-completeness $\mathfrak{H}$ has nonempty intersection in Y which must belong to $\varphi(x)$. So $\varphi(x)$ is compact.

Finally, we show that φ is upper semicontinuous at every $x \in U$. Let $x \in U$ and let $V \subseteq Y$ be open and such that $\varphi(x) \subseteq V$. Considering the family $\mathfrak{C}(x)$ with all its properties from the previous paragraphs we see that the extended family $\mathfrak{C}(x) \cup \{V^c\}$ cannot have FIP, because otherwise its intersection would be nonempty, i.e. $\emptyset \neq \bigcap \mathfrak{C}(x) \cap V^c = \varphi(x) \cap V^c$, which is a contradiction. On the other hand, we know that $\mathfrak{C}(x)$ has FIP, hence there is a finite subfamily $\{O_1, O_2, \dots O_k\} \subseteq \mathfrak{B}(x)$ such that

$$\emptyset = \bigcap_{i=1}^{k} \overline{f\,[O_i \cap C(f)]} \cap V^c \supseteq \overline{f\Big[\Big(\bigcap_i O_i\Big) \cap C(f)\Big]} \cap V^c.$$

By putting $O = \bigcap\limits_{i=1}^{k} O_i$ we see that $O \in \mathfrak{B}(x)$ and for each $y \in O$ we have $\varphi(y) \subseteq \overline{f\,[O \cap C(f)]} \subseteq V$. Consequently, φ is upper semicontinuous at $x \in U$, which proves the proposition. $\square$

Note 2.2. It is easy to show that if Y is locally compact and $\varphi \in G(X, Y)$, then φ is USCO if and only if φ is locally bounded (i.e. for each $x \in X$ there is $O \in \mathfrak{B}(x)$ with $\overline{\varphi[O]}$ compact). We shall use this statement frequently in the later course of our paper mainly as a characterization of elements of $D^*(X)$. A similar condition of Berge (see [2] p. 112 or, additionaly, Holá [6], the remark after Theorem 3.8) saying that any multifunction $\varphi\colon X \to Y$ with closed graph and a compact range $\varphi[X]$ is upper semicontinuous, may also be used later on.

Now we present the main theorem of this section. The theorem investigates the metrizability of the space $(D(X, Y), \tau_p)$ analogously to similar results for the topology of uniform convergence on compacta from the paper of Holá [6].

Theorem 2.3. *For a topological space X and a metric space (Y, d) the following statements are equivalent:*

(a) *$(F(X, 2^Y), \mathfrak{U}_p)$ is metrizable,*
(b) *$(F(X, 2^Y), \tau_p)$ is first countable,*
(c) *$(G(X, Y), \mathfrak{U}_p)$ is metrizable,*

(d) $(G(X,Y),\tau_p)$ *is first countable,*
(e) $(D(X,Y),\mathfrak{U}_p)$ *is metrizable,*
(f) $(D(X,Y),\tau_p)$ *is first countable,*
(g) X *is countable.*

Proof. Since we have $D(X,Y) \subseteq G(X,Y) \subseteq F(X,2^Y)$, it is clear that $(a) \Rightarrow (c) \Rightarrow (e) \Rightarrow (f)$ and $(a) \Rightarrow (b) \Rightarrow (d) \Rightarrow (f)$. We first show that $(f) \Rightarrow (g)$. Let $y_0 \in Y$ be arbitrary but fixed and let $\varphi_f \in D(X,Y)$ be the densely continuous form generated by the constant function $f \equiv y_0$. Without loss of generality we may suppose that the local base at φ_f is of the form $\{W(\varphi_f, A_n, 1/m);\ n,m \in \mathbb{N}\}$ for some finite sets $\{A_n;\ n \in \mathbb{N}\} \subseteq \mathfrak{F}(X)$. Suppose that X is not countable; hence $X \setminus \bigcup A_n \neq \emptyset$ and there is $x \notin \bigcup A_n$. For each $n \in \mathbb{N}$ the set A_n is compact so there is an open $U_n \subseteq X$ such that $x \in U_n \subseteq \overline{U_n} \subseteq A_n^c$. Choose a point $y_1 \neq y_0$ in Y and define a function $g_n\colon X \to Y$ by

$$g_n(z) = \begin{cases} y_1, & \text{if } z \in \overline{U_n}, \\ y_0, & \text{otherwise.} \end{cases}$$

Then g_n is continuous at the points of the set $(\operatorname{bd}\overline{U_n})^c$, so $g_n \in DC(X,Y)$. The induced form $\varphi_n \in D(X,Y)$ satisfies $\varphi_n(x) = \{y_1\}$ and $\varphi_n(z) = \{y_0\}$ for each $z \in A_n \subseteq \overline{U_n}^c$. Therefore for each $n \in \mathbb{N}$ and $m \in \mathbb{N}$ we have $\varphi_n \notin W(\varphi_f, x, d(y_o, y_1))$ and $\varphi_n \in W(\varphi_f, A_n, 1/m)$. This contradicts the claim of the local base at φ_f, because then $W(\varphi_f, x, d(y_o, y_1))$ cannot contain any basic set $W(\varphi_f, A_n, 1/m)$.

Now it suffices to prove $(g) \Rightarrow (a)$, i.e. to show that the uniformity $\mathfrak{U}_p$ of pointwise convergence has a countable base on $F(X,2^Y)$. If X is countable, then it has countably many finite subsets, so $\{W(A,1/n);\ A \in \mathfrak{F}(X), n \in \mathbb{N}\}$ is a countable base for $\mathfrak{U}_p$. This proves the theorem. □

Corollary 2.4. *For a topological space X the following statements are equivalent:*

(a) $(D^*(X),\mathfrak{U}_p)$ *is metrizable,*
(b) $(D^*(X),\tau_p)$ *is first countable,*
(c) X *is countable.*

Proof. The implication $(c) \Rightarrow (a)$ follows from the definition of $D^*(X)$ and the preceding theorem; $(a) \Rightarrow (b)$ is obvious. For the proof of $(b) \Rightarrow (c)$ observe that in the proof of Theorem 2.3 the induced multifunctions φ_n as well as φ_f in fact belong to $D^*(X)$ (in view of Note 2.2) and hence this proof also works in the space $D^*(X)$. □

Note 2.5. Since the space $C(X,Y)$ of continuous functions from X to Y is contained in $D(X,Y)$, if X is a Tychonoff space, the implication $(f) \Rightarrow (g)$ of Theorem 2.3 follows from a statement of McCoy and Ntantu (see [8], Exercise 1, p. 68): the space $(C(X,Y),\tau_p)$ is metrizable (or first countable) iff X is countable.

3. Cardinal Functions on $D_p^*(X)$

Definition. For a topological space Z we define
the weight of Z: $w(Z) = \aleph_0 + \min\{|\mathfrak{B}|;\ \mathfrak{B} \text{ is a base in } Z\}$,
the density of Z: $d(Z) = \aleph_0 + \min\{|D|;\ D \text{ is dense in } Z\}$,
the cellularity of Z:

$$c(Z) = \aleph_0 + \sup\{|\mathfrak{U}|;\ \mathfrak{U} \subseteq \tau(Z) \text{ is a pairwise disjoint family}\};$$

a *network* in Z is a family $\mathfrak{N} = \{N_s;\ s \in S\}$ of subsets of Z such that if $z \in Z$ and $U \in \mathfrak{B}(z)$, then there is $s \in S$ with $z \in N_s \subseteq U$. The *network weight* of Z is

$$nw(Z) = \aleph_0 + \min\{|\mathfrak{N}|;\ \mathfrak{N} \text{ is a network in } Z\}.$$

Then obviously $c(Z) \leqslant d(Z) \leqslant nw(Z) \leqslant w(Z)$. Throughout this section we assume that the space X *is not finite*, i.e. $|X| \geqslant \aleph_0$.

Corollary 3.1. *If X is countable, then $c(D_p^*(X)) = d(D_p^*(X)) = nw(D_p^*(X)) = w(D_p^*(X))$.*

P r o o f. Since the desired equality is true for metrizable spaces, the statement is clear in view of Corollary 2.4. □

Definition. For a topological space Z we define:
the pseudocharacter of Z is $\Psi(Z) = \sup\{\Psi(Z,z);\ z \in Z\}$, where

$$\Psi(Z,z) = \aleph_0 + \min\{|\mathfrak{G}|;\ \mathfrak{G} \subseteq \tau(Z)\colon \bigcap\mathfrak{G} = \{z\}\},$$

the diagonal degree of Z is

$$\Delta(Z) = \aleph_0 + \min\{|\mathfrak{G}|;\ \mathfrak{G} \subseteq \tau(Z \times Z)\colon \bigcap\mathfrak{G} = \Delta_Z\}.$$

Obviously for any space Z we have $\Psi(Z) \leqslant \Delta(Z)$.

Theorem 3.2. *If X is regular, then $\Psi(D_p^*(X)) = \Delta(D_p^*(X)) = d(X)$.*

Proof. First we want to show that $d(X) \leqslant \Psi(D_p^*(X))$. So let φ_f be the zero form induced by the zero function $f \equiv 0$ on X and let $\mathfrak{G} = \{G_m;\ m \in M\}$ be a family of open sets in $D_p^*(X)$ such that $\bigcap \mathfrak{G} = \{\varphi_f\}$ and $|\mathfrak{G}| = |M| = \Psi(D_p^*(X), \varphi_f)$. Since the pointwise topology τ_p has a particular local base at φ_f, we may, without loss of generality, suppose that each open G_m is of the form $W(\varphi_f, A_m, \varepsilon_m)$ for some $A_m \in \mathfrak{F}(X)$ and $\varepsilon_m > 0$. Consider the set $A = \bigcup\{A_m;\ m \in M\}$ and suppose that A is not dense in X. Thus there is $x \notin \bar{A}$ and since X is regular, there is $V \in \mathfrak{B}(x)$ such that $\overline{V}$ is disjoint from $\bar{A}$. If we define $g\colon X \to \mathbb{R}$ by

$$g(y) = \begin{cases} 1, & y \in \overline{V}, \\ 0, & \text{otherwise} \end{cases}$$

then $C(g) = (\operatorname{bd} V)^c$. Since the boundary of V is nowhere dense, $C(g)$ is dense in X, so $g \in DC(X)$. In view of Note 2.2 the induced form φ_g belongs to $D^*(X)$ and satisfies the following condition: if $y \in \bigcup A_m \subseteq \overline{V}^c$, then $\varphi_g(y) = 0 = \varphi_f(y)$ and hence $\varphi_g \in \bigcap\limits_m W(\varphi_f, A_m, \varepsilon_m) = \{\varphi_f\}$, but $\varphi_g \neq \varphi_f$ because $\varphi_g(x) = 1$ and $\varphi_f(x) = 0$; a contradiction. Consequently, the set A is dense in X and has cardinality $|A| = |M|$, since each A_m is finite. Therefore $d(X) \leqslant |M| \leqslant \Psi(D_p^*(X))$.

Now it suffices to show $\Delta(D_p^*(X)) \leqslant d(X)$. To prove that consider $D \subseteq X$ which is dense in X and such that $|D| = d(X)$. For each $a \in D$ and $n \in \mathbb{N}$ consider the set $G(a,n) = \bigcup\{W(\varphi, a, 1/n) \times W(\varphi, a, 1/n);\ \varphi \in D^*(X)\}$ which is open in the product topology of $D_p^*(X) \times D_p^*(X)$ and contains the diagonal $\Delta_{D^*(X)}$. Moreover, the family $\mathfrak{G} = \{G(a,n);\ a \in D,\ n \in \mathbb{N}\}$ has cardinality $|\mathfrak{G}| = |D|$, so if we show that $\bigcap \mathfrak{G} = \Delta_{D^*(X)}$, then by the definition of the diagonal degree $\Delta(D_p^*(X)) \leqslant |\mathfrak{G}| = |D| = d(X)$. So suppose the converse, i.e. let $(\varphi_f, \varphi_g) \in \bigcap \mathfrak{G}$ and $\varphi_f \neq \varphi_g$. Without loss of generality we may suppose that there is $y \in \varphi_g(z) \setminus \varphi_f(z)$ for some $z \in X$, which means that there is $\varepsilon > 0$ such that the distance of y and $\varphi_f(z)$ exceeds 2ε or, in other words, $S_\varepsilon(y) \cap S_\varepsilon[\varphi_f(z)] = \emptyset$. Moreover, since the elements of $D^*(X)$ are upper semicontinuous (Note 2.2), there is $O \in \mathfrak{B}(z)$ such that $\varphi_f[O] \subseteq S_{\varepsilon/2}[\varphi_f(z)]$.

On the other hand, since $(z, y) \in \varphi_g = \overline{g \restriction C(g)}$, there is $t \in C(g) \cap O$ with $g(t) \in S_{\varepsilon/2}(y)$. By the continuity of g at t there is $U \in \mathfrak{B}(t)$, $U \subseteq O$, such that $g[U] \subseteq S_{\varepsilon/2}(y)$. This (in view of Observation 2.1 from Holá [6]) implies $\varphi_g[U] \subseteq \overline{S_{\varepsilon/2}(y)}$. Finally, since D is dense, there is $a \in D$ with $a \in U \subseteq O$. Choose $n \in \mathbb{N}$ such that $1/n < \varepsilon/2$. Since $(\varphi_f, \varphi_g) \in G(a,n)$, there is $\varphi \in D^*(X)$ with $(\varphi_f, \varphi_g) \in W(\varphi, a, 1/n) \times W(\varphi, a, 1/n)$, which by the definition of the Hausdorff metric H implies $\varphi(a) \subseteq S_{\varepsilon/2}[\varphi_g(a)] \subset S_\varepsilon(y)$ and $\varphi(a) \subset S_{\varepsilon/2}[\varphi_f(a)] \subset S_\varepsilon[\varphi_f(z)]$, a contradiction with our assumption of 2ε-disjointness. Therefore $\Delta(D_p^*(X)) \leqslant d(X)$ and the proof of the theorem is complete. $\square$

Corollary 3.3. *Let X be regular. The following statements are equivalent:*

(a) *Each element of $D_p^*(X)$ is a G_δ-set,*
(b) *Each compact subset of $D_p^*(X)$ is a G_δ-set,*
(c) *$D_p^*(X)$ has a G_δ-diagonal,*
(d) *X is separable.*

Note 3.4. If X is a Tychonoff space, then by a result of McCoy and Ntantu (see [8], Theorem 4.3.1) we have $\Psi(C_p(X)) = \Delta(C_p(X)) = d(X)$, which extends our Theorem 3.2.

Definition. In a topological space Z a family $\mathfrak{P}(z)$ of nonempty open sets is called a *local π-base* at $z \in Z$, if for each $U \in \mathfrak{B}(z)$ there is $P \in \mathfrak{P}(z)$ with $P \subseteq U$. We define the *π-character* of Z by $\pi_\chi(Z) = \aleph_0 + \sup\{\pi_\chi(Z, z);\ z \in Z\}$, where

$$\pi_\chi(Z, z) = \min\{|\mathfrak{P}(z)|;\ \mathfrak{P}(z) \text{ is a local } \pi\text{-base at } z\}.$$

Define the *character* of Z by $\chi(Z) = \aleph_0 + \sup\{\chi(Z, z);\ z \in Z\}$, where

$$\chi(Z, z) = \min\{|\mathfrak{B}(z)|;\ \mathfrak{B}(z) \text{ is a local base at } z\}.$$

For every space Z we have $\pi_\chi(Z) \leqslant \chi(Z)$.

Theorem 3.5. *For every X we have $\pi_\chi(D_p^*(X)) = \chi(D_p^*(X)) = |X|$.*

Proof. We shall proceed similarly as in the proof of Theorem 3.2. First we would like to prove $|X| \leqslant \pi_\chi(D_p^*(X))$. Once again, let φ_f be the zero form induced by the zero function $f \equiv 0$ and let $\mathfrak{P} = \{P_m;\ m \in M\}$ be a local π-base at φ_f such that $|\mathfrak{P}| = |M| = \pi_\chi(D_p^*(X), \varphi_f)$. Since each P_m is nonempty and open in $D_p^*(X)$, for each $m \in M$ there are $\varphi_{f_m} \in D^*(X)$, $A_m \in \mathfrak{F}(X)$ and $\varepsilon_m > 0$ such that $W(\varphi_{f_m}, A_m, \varepsilon_m) \subseteq P_m$, that is, $\{W(\varphi_{f_m}, A_m, \varepsilon_m);\ m \in M\}$ is also a local π-base at φ_f. We would like to show that $X = \bigcup\{A_m;\ m \in M\}$.

So let $x \in X$ and choose $\varepsilon > 0$ arbitrarily. Thus there is $m \in M$ with the property $W(\varphi_{f_m}, A_m, \varepsilon_m) \subseteq W(\varphi_f, x, \varepsilon)$. Suppose now that $x \notin A_m$. Since A_m is finite (compact) and X is Hausdorff, there is $U \in \mathfrak{B}(x)$ with $\overline{U} \subseteq A_m^c$. Define a function $g\colon X \to \mathbb{R}$ by

$$g(y) = \begin{cases} \varepsilon, & y \in \overline{U}, \\ f_m(y), & \text{otherwise.} \end{cases}$$

Again, $C(g)$ is dense and it induces a form $\varphi_g \in D^*(X)$. Since $A_m \subseteq \overline{U}^c$ we see that $\varphi_g = \varphi_{f_m}$ on A_m and hence $\varphi_g \in W(\varphi_{f_m}, A_m, \varepsilon_m)$. On the other hand, $\varphi_g(x) = \varepsilon$ and $\varphi_f(x) = 0$ shows that $\varphi_g \notin W(\varphi_f, x, \varepsilon)$, which contradicts the inclusion between the basic sets. Therefore $x \in A_m$, i.e. $X = \bigcup\{A_m;\ m \in M\}$, which implies $|X| \leqslant |M| \leqslant \pi_\chi(D^*_p(X))$.

Now, since we know that the family of all finite subsets of X has again cardinality $|X|$, it is easy to see that for any $\varphi_f \in D^*_p(X)$ the family $\{W(\varphi_f, A, 1/n);\ A \in \mathfrak{F}(X),\ n \in \mathbb{N}\}$ is a local base at φ_f of cardinality at most $|X|$ and hence $\chi(D^*_p(X)) \leqslant |X|$. This concludes the proof of the theorem. $\square$

Corollary 3.6. *The following statements are equivalent:*

(a) $D^*_p(X)$ *is first countable,*
(b) $\pi_\chi(D^*_p(X))$ *is countable,*
(c) X *is countable.*

Note 3.7. Analogously as before we get an extension of our Theorem 3.5, since by Theorem 4.4.1 of McCoy and Ntantu [8], if X is Tychonoff, then $\pi_\chi(C_p(X)) = \chi(C_p(X)) = |X|$.

Consider the set $\mathfrak{K}(\mathbb{R})$ of nonempty compact subsets of the reals and equip the set with the already defined Hausdorff metric H. Since $\mathbb{R}$ is a complete metric space, so is $(\mathfrak{K}(\mathbb{R}), H)$ (see Beer [1], Theorem 3.2.4 and Exercise 3.2.4(b)). We also know that the second countability of $\mathbb{R}$ is inherited by $(\mathfrak{K}(\mathbb{R}), H)$ (see Engelking [4], exercises 4.5.22(a) and 3.12.26(b)). So $(\mathfrak{K}(\mathbb{R}), H)$ is a second countable complete metric space.

Theorem 3.8. *For each space X we have $w(D^*_p(X)) = |X|$.*

P r o o f. We first show $w(D^*_p(X)) \leqslant |X|$. Let $\mathfrak{B}$ be a countable base for the topology for $(\mathfrak{K}(\mathbb{R}), H)$. For $x \in X$ and $V \in \mathfrak{B}$ define $W[x, V] = \{\varphi \in D^*(X);\ \varphi(x) \in V\}$. In view of Note 2.2 each form in $D^*(X)$ has only nonempty compact values. It is easy to see (since $\mathfrak{K}(\mathbb{R})$ has the metric topology induced by H) that the family $\mathfrak{A} = \{W[x, V];\ x \in X,\ V \in \mathfrak{B}\}$ is just another subbase for the topology of pointwise convergence on $D^*(X)$, i.e. it generates the same topology as our usual subbase $\{W(\varphi, x, \varepsilon);\ \varphi \in D^*(X),\ x \in X,\ \varepsilon > 0\}$. Since $\mathfrak{B}$ is countable, the subbase $\mathfrak{A}$ has cardinality $|\mathfrak{A}| = |X| \cdot |\mathfrak{B}| = |X|$. We know that the family of finite intersections of sets in $\mathfrak{A}$, which is then a base for $D^*_p(X)$, has again cardinality $|X|$, which implies $w(D^*_p(X)) \leqslant |X|$.

On the other hand, since the character of a space is less than or equal to the weight, the desired statement follows from Theorem 3.5. $\square$

Definition. A family $\mathfrak{P}$ of nonempty open subsets of a space Z is called a *π-base* for Z, if every nonempty open subset of Z contains an element of the family $\mathfrak{P}$. The *π-weight* of Z is then defined to be $\pi w(Z) = \aleph_0 + \min\{|\mathfrak{P}|;\ \mathfrak{P} \text{ is a } \pi\text{-base for } Z\}$. Obviously $d(Z) \leqslant \pi w(Z) \leqslant w(Z)$.

Theorem 3.9. *For any topological space X we have*

$$\begin{aligned}|X| &= w(D_p^*(X)) = \pi w(D_p^*(X)) \\ &= |X| \cdot c(D_p^*(X)) = |X| \cdot d(D_p^*(X)) = |X| \cdot nw(D_p^*(X)).\end{aligned}$$

Proof. From Theorems 3.5, 3.8 and from our previous remarks we deduce the inequalities

$$\begin{gathered}|X| = \pi_\chi(D_p^*(X)) \leqslant \pi w(D_p^*(X)), \\ w(D_p^*(X)) = |X| = |X| \cdot \aleph_0, \\ \aleph_0 \leqslant c(D_p^*(X)) \leqslant d(D_p^*(X)) \leqslant nw(D_p^*(X)) \leqslant w(D_p^*(X)), \\ d(D_p^*(X)) \leqslant \pi w(D_p^*(X)) \leqslant w(D_p^*(X)),\end{gathered}$$

which imply the desired result. □

The next corollary follows immediately from the previous results and extends Corollaries 2.4 and 3.6.

Corollary 3.10. *$D_p^*(X)$ is second countable if and only if X is countable.*

The next statement is (together with its proof) analogous to Theorem 3.6 from the paper of Holá and McCoy [7] where a similar topic is investigated for the topology of uniform convergence on compacta.

Theorem 3.11. *If X is regular, then $nw(X) \leqslant nw(D_p^*(X))$.*

Proof. Let $\mathfrak{N} = \{N_m;\ m \in M\}$ be a network in $D_p^*(X)$ such that $|\mathfrak{N}| = |M| = nw(D_p^*(X))$. For each $m \in M$ define a set

$$N_m^* = \{x \in X;\ \forall \varphi \in N_m \quad \varphi(x) \cap (0, \infty) \neq \emptyset\}.$$

We would like to show that the family $\mathfrak{N}^* = \{N_m^*;\ m \in M\}$ is a network in X. So let $x \in X$ and let $U \subseteq X$ be an open neighborhood of x. Since X is regular, there is an open V with $x \in V \subseteq \overline{V} \subseteq U$. Define a function $f\colon X \to \mathbb{R}$ by

$$f(y) = \begin{cases} 1, & y \in V, \\ 0, & y \in V^c. \end{cases}$$

Then $f \in DC(X)$ and $\varphi_f \in D^*(X)$. Since $\mathfrak{N}$ is a network in $D_p^*(X)$, there is $m \in M$ with $\varphi_f \in N_m \subseteq W(\varphi_f, x, 1)$. Is it true that $x \in N_m^* \subseteq U$? Well, if $y \in U^c$, then $\varphi_f(y) = \{0\}$, which together with $\varphi_f \in N_m$ implies $y \notin N_m^*$. Therefore $N_m^* \subseteq U$. Furthermore, we have $\varphi_f(x) = \{1\}$ and since $N_m \subseteq W(\varphi_f, x, 1)$, for each $\varphi \in N_m$ we get $\varphi(x) \subseteq (0, 2)$, which implies $x \in N_m^*$. This shows that $\mathfrak{N}^*$ is a network in X of cardinality at most $|M|$ and hence $nw(X) \leqslant nw(D_p^*(X))$. This completes the proof of the theorem. $\square$

To determine the cellularity of $D_p^*(X)$ first recall that if Y is a dense subspace of a topological space Z, then $c(Y) = c(Z)$. Now, since every form $\varphi \in D^*(X)$ assumes only values from $(\mathfrak{K}(\mathbb{R}), H)$, the set $D^*(X)$ is contained in the product space $F_p(X, \mathfrak{K}(\mathbb{R}))$ of all nonempty-compact-valued multifunctions on X. Using the reasoning from Theorem 3.8 we see that the topology on $D_p^*(X)$ is the relative topology from $F_p(X, \mathfrak{K}(\mathbb{R}))$, i.e. $D_p^*(X)$ is a subspace of $F_p(X, \mathfrak{K}(\mathbb{R}))$. In view of the results 2.3.17 and 2.3.18 from Engelking [4] we are able to determine the cellularity of certain product spaces. Hence we would be interested in knowing under what condition $D^*(X)$ is dense in $F_p(X, \mathfrak{K}(\mathbb{R}))$. This analysis, however, depends on whether X has isolated points or not.

Lemma 3.12. *If X is first countable regular without isolated points, then $D^*(X)$ is dense in $F_p(X, \mathfrak{K}(\mathbb{R}))$.*

Proof. Since this proof is rather long and technical, we shall state only the main idea here. The full details will be available in the second author's subsequent dissertation which is due to appear in 2006. So if $W(\psi, A, \varepsilon)$ is an arbitrary basic open set in $F_p(X, \mathfrak{K}(\mathbb{R}))$ and $A = \{x_1, \dots x_k\}$, then for each $i = 1, \dots k$ there is a finite subset $A^i = \{y_1, \dots y_{l_i}\}$ of $\psi(x_i)$ such that $\psi(x_i) \subseteq S_{\varepsilon/2}[A^i]$. We want to construct a form $\varphi \in D^*(X)$ such that $\varphi(x_i) = A^i$ for every $i = 1, \dots k$, because this would imply $\psi(x_i) \subseteq S_{\varepsilon/2}[\varphi(x_i)]$ and $\varphi(x_i) = A^i \subseteq \psi(x_i)$, which gives $H(\psi(x_i), \varphi(x_i)) \leqslant \varepsilon/2 < \varepsilon$ for every such i, or equivalently, $\varphi \in D^*(X) \cap W(\varphi, A, \varepsilon)$, which proves the lemma.

In order to do that we need to define a generating function $f\colon X \to \mathbb{R}$ of the form φ which is continuous at the points of a dense subset $C(f)$ of X. We proceed as follows: for each $i = 1, \dots k$ there is an open neighborhood O_i of the point $x_i \in A$ such that $\overline{O_i} \cap \overline{O_j} = \emptyset$ if $i \neq j$. Moreover, for each fixed index $i \in \{1, \dots k\}$ there is a sequence of open neighborhoods $\{B_n;\ n \in \mathbb{N}\}$ of x_i which are contained in O_i and satisfy $\overline{B_{n+1}} \subseteq B_n$ and $B_n \setminus \overline{B_{n+1}} \neq \emptyset$. On each of the open sets $B_n \setminus \overline{B_{n+1}}$ let f assume the value from $A^i = \{y_1, \dots y_{l_i}\}$ with the index $l = (n \bmod l_i) + 1$, i.e. f assumes y_2 on $B_1 \setminus \overline{B_2}$, y_3 on $B_2 \setminus \overline{B_3}$ and so on with the values from A^i repeated infinitely many times on the sequence of sets $\{B_n \setminus \overline{B_{n+1}};\ n \in \mathbb{N}\}$. Elsewhere in O_i

the function f may assume any constant from the set A^i. Outside $\bigcup O_i$ the function f can be put arbitrarily constant.

Now, it is true that for each $i = 1, \dots k$ the closed set $C_i = \{x_i\} \cup \bigcup_{n=1}^{\infty} \operatorname{bd} B_n$ is nowhere dense in X. Hence the set $C = \bigcup_{i=1}^{k} C_i$ is also closed and nowhere dense. The definition of f implies that it is continuous at the points of the complement of C which is dense in X. So f induces a form $\varphi = \overline{f \restriction C(f)}$ (taking the closure of the graph). Moreover, since f assumes only finitely many values, in view of our Note 2.2, φ belongs to $D^*(X)$. From the construction of f we see that $\varphi(x_i) = \{y_1, \dots y_{l_i}\} = A^i$ as requested. In view of the initial remarks, this completes the proof of the lemma.

$\square$

In the case when X contains isolated points we are not able to directly use the preceding approach, because at an isolated point every generated densely continuous form is just an ordinary (singleton-valued) function. Thus, in order to prove the following general statement, we have to modify slightly the product space considered.

Theorem 3.13. *If X is a first countable regular space, then $c(D_p^*(X)) = \aleph_0$.*

Proof. Let $I(X)$ denote the set of all isolated points in X. Consider the product space $Z = \prod_{x \in X} Y_x$ where $Y_x = \mathbb{R}$ with the usual topology for $x \in I(X)$ and $Y_x = (\mathfrak{K}(\mathbb{R}), H)$ if $x \notin I(X)$. According to Corollary 2.3.18 from Engelking [4] we know that $c(Z) = \aleph_0$. Since $I(X)$ has "the discrete topology", every locally bounded densely continuous form $\varphi \in D^*(X)$ is a single-valued function on $I(X)$, so again $D_p^*(X)$ is a subspace of the product space Z. To show that $D^*(X)$ is dense in Z we would have to find a form $\varphi \in D^*(X)$ which is ε-close to some $\psi \in Z$ at the points of a finite set $A \subseteq X$ for any given ψ, A, $\varepsilon > 0$. This means defining a generating function f for such a form $\varphi \in D^*(X)$. To define f at the points of $X \setminus I(X)$ we can make use of the approach from Lemma 3.12. Since $\psi \restriction I(X)$ is single-valued, we may put $f(x) = \psi(x)$ for $x \in I(X)$. Such function f clearly generates the desired form $\varphi \in D^*(X)$ and hence $D^*(X)$ is dense in Z. In view of the initial remarks we obtain $c(D_p^*(X)) = \aleph_0$. $\square$

Example 4.1. We have $d(D_p^*(\mathbb{R})) > \aleph_0$.

P r o o f. Suppose by the contrary that the subset $D = \{\psi_n;\ n \in \mathbb{N}\}$ is dense in $D_p^*(\mathbb{R})$. For every $t \in \mathbb{R}$ define a function $f_t\colon\ \mathbb{R} \to \mathbb{R}$ by

$$f_t(x) = \begin{cases} \cos\left(\dfrac{1}{x-t}\right), & x \neq t,\\ 0, & x = t. \end{cases}$$

Since $C(f_t) = \mathbb{R} \setminus \{t\}$, every f_t generates a form $\varphi_t \in D^*(\mathbb{R})$ for which $\varphi_t(t) = [-1,1]$. Now, for each $n \in \mathbb{N}$ define a set $H_n = \{t \in \mathbb{R};\ \psi_n \in W(\varphi_t, t, 1/2)\}$. Since D is dense, for each $t \in \mathbb{R}$ there is $n \in \mathbb{N}$ such that $t \in H_n$ or, in other words, $\mathbb{R} = \bigcup\{H_n;\ n \in \mathbb{N}\}$.

On the other hand, if $\psi_n \in W(\varphi_t, t, 1/2)$, then the Hausdorff metric satisfies $H(\psi_n(t), \varphi_t(t)) < 1/2$, which implies $[-1,1] \subseteq S_{1/2}(\psi_n(t))$ and hence $\psi_n(t)$ cannot be a singleton (if it were, the diameter of $S_{1/2}(\psi_n(t))$ would equal 1, a contradiction). Consequently, if $t \in H_n$, then $\psi_n(t)$ is not a singleton. Now, if $g_n\colon\ \mathbb{R} \to \mathbb{R}$ is the generating function of ψ_n, then we know that $\psi_n \restriction C(g_n) = g_n$ is a singleton-valued function (see Observation 2.1 in Holá [6]). By a classic result of real analysis the set $C(g_n)$ of points of continuity of g_n is a G_δ set in $\mathbb{R}$ (and must be dense by our assumption), i.e. its complement is of first Baire category. Therefore every H_n is of first category, which implies that also $\mathbb{R}$ is of first category, a contradiction. $\square$

Example 4.2. Considering $D(\mathfrak{m})$ as the discrete space of cardinality $\mathfrak{m} \geqslant \aleph_0$ and $\beta D(\mathfrak{m})$ as the Stone-Čech compactification of $D(\mathfrak{m})$ we claim that

$$d(D_p^*(\beta D(\mathfrak{m}))) \leqslant nw(D_p^*(\beta D(\mathfrak{m}))) < w(D_p^*(\beta D(\mathfrak{m}))).$$

P r o o f. First, by our Theorem 3.8 and Theorem 3.6.11 from Engelking [4] we have $w(D_p^*(\beta D(\mathfrak{m}))) = |\beta D(\mathfrak{m})| = 2^{2^{\mathfrak{m}}}$. On the other hand, it is (generally) $d(D_p^*(\beta D(\mathfrak{m}))) \leqslant nw(D_p^*(\beta D(\mathfrak{m}))) \leqslant |D^*(\beta D(\mathfrak{m}))|$. Now, consider the mapping Ψ which assigns to a densely continuous form $\varphi \in D^*(\beta D(\mathfrak{m}))$ its restriction $\varphi \restriction D(\mathfrak{m})$. Since every dense subset of $\beta D(\mathfrak{m})$ contains $D(\mathfrak{m})$ and $D(\mathfrak{m})$ is discrete, each such restriction $\varphi \restriction D(\mathfrak{m})$ is an ordinary continuous function on $D(\mathfrak{m})$. In view of Example 4.3 of the paper of Holá and McCoy [7], our mapping $\Psi\colon\ D^*(\beta D(\mathfrak{m})) \to C(D(\mathfrak{m}), \mathbb{R})$ is injective.

More precisely, since $D(\mathfrak{m})$ is open and dense in $\beta D(\mathfrak{m})$, each continuous function $g \in C(D(\mathfrak{m}), \mathbb{R})$ generates a densely continuous form $\varphi_g = \overline{g \restriction D(\mathfrak{m})}$ for which $\Psi(\varphi_g) = g$, i.e. the mapping Ψ is a bijection from $D(\beta D(\mathfrak{m}))$ onto $C(D(\mathfrak{m}), \mathbb{R})$. This implies $|D(\beta D(\mathfrak{m}))| = |C(D(\mathfrak{m}),\mathbb{R})| = |F(D(\mathfrak{m}),\mathbb{R})| = \mathfrak{c}^{\mathfrak{m}} = 2^{\aleph_0\cdot\mathfrak{m}} = 2^{\mathfrak{m}}$. Combining all our inequalities we obtain

$$d(D_p^*(\beta D(\mathfrak{m}))) \leqslant nw(D_p^*(\beta D(\mathfrak{m}))) \leqslant |D(\beta D(\mathfrak{m}))| = 2^{\mathfrak{m}} < 2^{2^{\mathfrak{m}}} = w(D^*(\beta D(\mathfrak{m}))).$$

This completes the proof. □

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