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ON THE EULER FUNCTION OF REPDIGITS

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Dedicated to William D. Banks on his $\sqrt{\varphi(2005)}^{\text{th}}$ birthday.

Abstract. For a positive integer n we write $\varphi(n)$ for the Euler function of n. In this note, we show that if $b > 1$ is a fixed positive integer, then the equation

$$\varphi\Big(x\frac{b^n-1}{b-1}\Big) = y\frac{b^m-1}{b-1}, \qquad \text{where } x, y \in \{1, \ldots, b-1\},$$

has only finitely many positive integer solutions (x, y, m, n).

Keywords: Euler function, prime, divisor

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1. Introduction

For a positive integer n we write $\varphi(n)$ for the Euler function of n. In this paper, we prove the following result.

Theorem 1.1. *If $b > 1$ is given, then the equation*

$$\varphi\Big(x\frac{b^n-1}{b-1}\Big) = y\frac{b^m-1}{b-1}, \tag{1}$$

with $x, y \in \{1, \ldots, b-1\}$ has only finitely many positive integer solutions (x, y, m, n).

Some equations of a similar flavor have been treated in [3], [4], [5] and [6].

We use the Vinogradov symbols $\ll$ and $\gg$, and the Landau symbol O with their regular meanings. The constants implied by them may depend on our parameter b. We use p, q and P with or without subscripts to denote prime numbers. For a positive real number x we use $\log x$ for the maximum between 2 and the natural logarithm of x. Note that with this convention, the function log is sub-multiplicative; i.e., $\log(xy) \leqslant \log x \log y$ holds for all positive real numbers x and y. For a positive integer n, we write $P(n)$, $p(n)$, $\omega(n)$, $\Omega(n)$ and $\tau(n)$ for the largest prime factor of n, smallest prime factor of n, the number of distinct prime factors of n, the number of prime power divisors (> 1) of n, and the total number of divisors of n, respectively. We put $u_n = (b^n - 1)/(b - 1)$. Finally, we use $c_0, c_1, \ldots$ for positive constants depending on b which are labeled increasingly throughout the paper.

2. The proof

Since b is fixed, and (x, y) can take only $(b-1)^2$ values, we may assume that both x and y are fixed. Let $N = x(b^n - 1)/(b - 1)$. If $m > n$, then

$$\varphi(N) = y\frac{b^m - 1}{b - 1} \geqslant \frac{b^{n+1} - 1}{b - 1} > b^n - 1 \geqslant N,$$

which is a contradiction. If $m = n$, then $\varphi(N)/N = y/x$. Since $P(N)$ divides the denominator of the rational number $\varphi(N)/N$ in reduced form, it follows that $P(N) \leqslant b - 1$. In particular, $P(u_n) \leqslant b - 1$. Since for $n > 6$, u_n always has a *primitive divisor*, which, in particular, is a prime congruent to 1 modulo n, we get that $n \leqslant \max\{6, b - 2\}$ (see [1] and [2] for the existence and properties of primitive divisors).

From now on, we assume that $n > m$. We will first show that $n - m$ is bounded. Let $k = \gcd(m, n)$. Then k divides $\lambda = n - m$, therefore

$$b^k \leqslant b^\lambda \ll \frac{N}{\varphi(N)} = \prod_{P \mid N} \Big(1 + \frac{1}{P - 1}\Big) \ll \prod_{\substack{P \mid N \\ P > b}} \Big(1 + \frac{1}{P - 1}\Big). \tag{2}$$

Let $P \mid N$ such that $P > b$. Then P does not divide x and there exists a divisor l_P of n minimal with the property that $P \mid u_{l_P}$. The number l_P is called *the order of apparition of P in the sequence* $(u_n)_{n \geqslant 1}$ and P is certainly primitive for u_{l_P}. Furthermore, $P \equiv 1 \pmod{l_P}$. We now fix $d \mid n$ and consider

$$\mathcal{S}_d = \sum_{l_P = d} \frac{1}{P} \qquad \text{and} \qquad \omega_d = \#\{P \colon\ l_P = d\}. \tag{3}$$

Clearly,

$$b^d \gg u_d \geqslant \prod_{l_P = d} P \geqslant d^{\omega_d},$$

giving

$$\omega_d \ll \frac{d}{\log d}. \tag{4}$$

Using estimate (4), we can estimate the sum $\mathcal{S}_d$ defined in (3) as follows

$$\mathcal{S}_d \leqslant \sum_{\substack{l_P=d\\ P<d^2}} \frac{1}{P} + \sum_{\substack{l_P=d\\ P\geqslant d^2}} \frac{1}{P} \ll \sum_{\substack{P\equiv 1 \pmod d\\ P\leqslant d^2}} \frac{1}{P} + \frac{\omega_d}{d^2} \ll \frac{\log\log d}{\varphi(d)}, \tag{5}$$

where in the above inequalities (5) we used the estimate (4), together with the Brun-Titchmarsch Theorem which asserts that the estimate

$$\sum_{\substack{p\equiv a \pmod b\\ p<t}} \frac{1}{p} \ll \frac{\log\log t}{p}$$

holds for all coprime integers $1 \leqslant a \leqslant b$ and all positive real numbers t (see, for example, Lemma 6.3 in [7] or Theorem 1 in [8]). Let c_0 be an upper bound for the constant implied by the Vinogradov symbol appearing in (5), and assume that $c_0 > 1$.

Taking logarithms in the inequality (2) and using the inequality $1+t<e^t$ which is valid for all positive real numbers t, we get

$$\begin{aligned} k \leqslant \lambda &\leqslant O(1) + \sum_{\substack{P\mid u_n\\ P>b}} \frac{1}{P-1} \leqslant \sum_{\substack{d\mid n\\ d>1}} \mathcal{S}_d + O\Bigg(1 + \sum_{P\geqslant 2} \frac{1}{P^2}\Bigg) \\ &\leqslant c_0 \sum_{\substack{d\mid n\\ d>1}} \frac{\log\log d}{\varphi(d)} + O(1). \end{aligned} \tag{6}$$

Since the function $\log\log(\cdot)$ is sub-multiplicative, it follows that the function $c_0 \log\log n/\varphi(n)$ satisfies

$$\frac{c_0\log\log(ab)}{\varphi(ab)} \leqslant \frac{c_0\log\log a}{\varphi(a)} \cdot \frac{c_0\log\log b}{\varphi(b)}, \qquad \text{whenever } \gcd(a,b)=1.$$

Hence, writing $n = p_1^{\nu_1}\dots p_s^{\nu_s}$, with $p(n) = p_1 < \dots < p_s = P(n)$, we have

$$\sum_{\substack{d\mid n\\ d>1}} \frac{c_0\log\log d}{\varphi(d)} \leqslant \prod_{i=1}^{s}\left(1 + \sum_{\nu=1}^{\nu_i} \frac{c_0\log\log(p_i^{\nu})}{p_i^{\nu-1}(p_i-1)}\right) - 1.$$

Since obviously

$$\sum_{\nu \geqslant 1} \frac{\log\log(p^\nu)}{p^{\nu-1}(p-1)} \ll \frac{\log\log p}{p},$$

we get that there exists a positive constant c_1 such that

$$\sum_{\substack{d \mid n \\ d>1}} \frac{c_0 \log\log d}{\varphi(d)} \leqslant \prod_{p \mid n} \left(1 + \frac{c_1 \log\log p}{p}\right) - 1. \tag{7}$$

Combining (7) with the estimate (6), we get

$$k \leqslant \lambda \ll \prod_{p \mid n} \Big(1 + \frac{c_1 \log\log p}{p}\Big),$$

and therefore

$$k \leqslant \lambda \ll \prod_{p \mid n} \Big(1 + \frac{c_1 \log\log p}{p}\Big) \ll \exp\left(c_1 \sum_{p \mid n} \frac{\log\log p}{p}\right),$$

which, after taking logarithms, gives

$$\log k \leqslant \log \lambda \ll 1 + \sum_{p \mid n} \frac{\log\log p}{p}. \tag{8}$$

We now bound the sum

$$\mathcal{T} = \sum_{p \mid n} \frac{\log\log p}{p}.$$

Assume first that $p \mid k$. Clearly, $\omega(k) = O(\log k / \log\log k)$, therefore, by the Prime Number Theorem, there exists an absolute constant c_2, such that

$$\begin{aligned} \mathcal{T}_1 = \sum_{p \mid k} \frac{\log\log p}{p} &\ll \sum_{q \leqslant c_2 \log k} \frac{\log\log q}{q} \\ &\ll \log\log(c_2 \log k) \sum_{q \leqslant c_2 \log k} \frac{1}{q} \ll (\log\log\log k)^2. \end{aligned} \tag{9}$$

Assume now that $p \nmid k$. Then $p \nmid m$. Thus,

$$\mathrm{ord}_p(bu_m) = \mathrm{ord}_p(b) + \mathrm{ord}_p(u_m) \ll 1 + \mathrm{ord}_p(l_p) \ll 1 + \frac{p}{\log p}. \tag{10}$$

Here, for a positive integer n and a prime p we use $\operatorname{ord}_p(n)$ for the exact order at which p divides n, together with the well-known facts that $p \mid u_m$ if and only if $l_p \mid m$, that $l_p \mid p-1$, and that if $p \nmid m$, then

$$\operatorname{ord}_p(u_m) = \operatorname{ord}_p(u_{l_p}) \leqslant \operatorname{ord}_p(b^{p-1}-1) \leqslant \frac{\log(b^{p-1})}{\log p} \ll \frac{p}{\log p}.$$

Now let t be any positive integer and let us count the contribution to the sum $\mathcal{T}$ from primes in $\mathcal{I}_t = [2^t, 2^{t+1}]$. Let p be a prime in $\mathcal{I}_t$ and let n_t be the number of prime factors of n in $\mathcal{I}_t$ which do not divide k. Then n has at least 2^{n_t-1} distinct divisors which are multiples of p. For each one of these divisors d except $O(1)$ of them (actually, for each one of these divisors except, possibly, the values less than or equal to 6), u_d has a primitive divisor; i.e., a prime $q \mid u_d$ such that $q \nmid u_{d'}$ for any $d' < d$, and $q \equiv 1 \pmod{d}$. This argument shows that u_n has at least $2^{n_t-1} - 6$ distinct divisors congruent to 1 modulo p, giving $\operatorname{ord}_p(\varphi(N)) \geqslant 2^{n_t-1} - 6$. Combining this argument with the estimate (10), we get

$$2^{n_t-1} \ll 1 + \frac{p}{\log p} \ll 1 + \frac{2^t}{t},$$

giving $n_t \ll t$. Thus,

$$\mathcal{T}_2 = \sum_{\substack{p \mid n \\ p \nmid k}} \frac{\log\log p}{p} \ll \sum_{t \geqslant 1} \frac{n_t \log\log(2^{t+1})}{2^t} \ll \sum_{t \geqslant 1} \frac{t \log t}{2^t} \ll 1. \tag{11}$$

Inserting the estimates (9) and (11) into the estimate (8), we get

$$\log k \leqslant \log \lambda \ll 1 + (\log\log\log k)^3,$$

leading to the conclusion that k (hence, also λ) is bounded. We may therefore assume that both k and λ are fixed. Furthermore, by replacing now b by b^k, x by $x(b^k-1)/(b-1)$, and y by $y(b^k-1)/(b-1)$, we may assume that m and n are coprime; i.e., that $k = 1$.

To finish, we shall show in what follows first that $p_1 = p(n)$ is bounded, then that $s = \Omega(n)$ is bounded, and finally that n itself is bounded.

Assume that $p(n) = p_1$ can get arbitrarily large. In particular, we may assume that $p_1 > \min\{6, b\}$. Then the smallest prime factor of u_n is congruent to 1 modulo p_i for some $i \geqslant 1$, therefore it is $\geqslant 2p_1 + 1 > b > x$. Hence, the equation (1) can be written as

$$\varphi(u_n) = \frac{y}{\varphi(x)} u_m,$$

therefore

$$\varphi(u_n)/u_n \text{ } \tag{12}$$

$$\frac{\varphi(u_n)}{u_n} = \frac{yu_m}{\varphi(x)u_n} \leqslant \frac{yu_{n-1}}{\varphi(x)u_n} \leqslant \frac{(b-1)(b^{n-1}-1)}{b^n-1}. \tag{12}$$

The limit of the expression appearing on the right hand side of the above inequality (12) when $n \to \infty$ is $1 - 1/b$. Hence, if $n > c_3$, then the right-hand side of the above inequality is $\leqslant c_4 = 1 - 1/(2b)$. Thus,

$$c_4^{-1} \leqslant \frac{u_n}{\varphi(u_n)} = \prod_{P|n}\Big(1 + \frac{1}{P-1}\Big) \leqslant \exp\Bigg(\sum_{\substack{d|n \\ d>1}} \mathcal{S}_d + O\Big(\sum_{p \geqslant p_1} \frac{1}{p^2}\Big)\Bigg),$$

giving

$$c_5 \leqslant \sum_{d|n} \mathcal{S}_d + O\Big(\frac{1}{p_1}\Big),$$

where $c_5 = \log(c_4^{-1}) > 0$. Thus, if c_6 is the constant implied by the above Landau symbol, and if $p_1 > c_7 = 2c_6c_5^{-1}$, then we get

$$1 \ll \sum_{\substack{d|n \\ d>1}} \mathcal{S}_d,$$

where the constant implied in the above Vinogradov symbol is $c_8 = 2c_5^{-1}$. Using the estimates (5) and (7), we get that

$$1 \ll \sum_{\substack{d|n \\ d>1}} \mathcal{S}_d \leqslant \sum_{\substack{d|n \\ d>1}} \frac{c_0 \log\log d}{\varphi(d)} \leqslant \prod_{i=1}^{s}\Big(1 + \frac{c_1 \log\log p_i}{p_i}\Big) - 1. \tag{13}$$

The same argument employed to bound the number of prime factors of n in the interval $\mathcal{I}_t$ which do not divide k, shows that n has at least $2^{s-1} - 6$ prime factors which are congruent to 1 modulo p_1. Hence, $\text{ord}_{p_1}(\varphi(N)) \geqslant 2^{s-1} - 6$, while by the inequality (10), the number $\text{ord}_{p_1}(\varphi(N))$ cannot exceed $\text{ord}_{p_1}(b^{p_1-1} - 1) = O(p_1/\log p_1)$. This shows that $s = \omega(n) \leqslant c_9 \log p_1$. Hence,

$$\begin{aligned} \prod_{i=1}^{s}\Big(1 + \frac{c_1 \log\log p_i}{p_i}\Big) - 1 &\leqslant \Big(1 + \frac{c_1 \log\log p_1}{p_1}\Big)^{c_9 \log p_1} - 1 \\ &\leqslant \exp\Big(c_{10} \frac{\log p_1 \log\log p_1}{p_1}\Big) - 1. \end{aligned} \tag{14}$$

Here, $c_{10} = c_1 c_9$. Since the function $(\log p_1 \log\log p_1)/p_1$ is bounded, we conclude that there exists a constant c_{11} such that

$$\exp\Big(c_{10}\frac{\log p_1 \log\log p_1}{p_1}\Big) - 1 \leqslant c_{11}\frac{\log p_1 \log\log p_1}{p_1}. \tag{15}$$

The combination of the inequalities (13), (14) and (15) leads to the conclusion that

$$p_1 \ll \log p_1 \log\log p_1,$$

which shows that p_1 is bounded. Now n has at least $\tau(n/p_1) - 6$ divisors which are multiples of p_1 and which are > 6. For each such divisor, u_n has a primitive divisor which is congruent to 1 modulo p_1, which shows that $\mathrm{ord}_{p_1}(\varphi(N)) \geqslant \tau(n/p_1) - 6$. Since by the estimate (10) this p_1-adic order is $\ll 1 + p_1/\log p_1 \ll 1$, we get that $\tau(n/p_1) \ll 1$, therefore $\tau(n) \ll 1$. In particular, $\Omega(n)$ is bounded.

To finish the proof, it suffices to show that for each $i \leqslant s$, p_i is bounded. We proceed by induction on i, the case $i = 1$ being obvious. Fix s, $1 \leqslant i \leqslant s-1$, and assume inductively that p_i is bounded. Since the numbers ν_j for $j = 1, \ldots, s$ are also bounded, we may assume that the first i distinct primes as well as their multiplicities are all fixed. Write $n_1 = \prod_{j=1}^{i} p_j^{\nu_j}$. Then $p_{i+1} = p(n/n_1)$. Assume that p_{i+1} can get arbitrarily large. Suppose, in particular, that it is larger than $\min\{6, b^{n_1} - 1\}$. Then writing $u_n = (b^{n_1} - 1)/(b-1) \cdot (b^n - 1)/(b^{n_1} - 1)$, and observing that every prime factor of $(b^n - 1)/(b^{n_1} - 1)$ is congruent to 1 modulo p_j for some $j \geqslant i+1$; hence, larger that $b^{n_1} - 1$, we get that

$$yu_m = \varphi(N) = \varphi(xu_{n_1})\varphi\Big(\frac{b^n - 1}{b^{n_1} - 1}\Big),$$

so writing $N_1 = (b^n - 1)/(b^{n_1} - 1)$, we get

$$\frac{\varphi(N_1)}{N_1} = \frac{yu_m}{\varphi(xu_{n_1})N_1} = \frac{y(b^m - 1)(b^{n_1} - 1)}{(b-1)\varphi(xu_{n_1})(b^n - 1)}.$$

The left-hand side of the above-equality is < 1, while the right hand side tends to (assuming that $n \to \infty$)

$$L = \frac{y(b^{n_1} - 1)b^{-\lambda}}{(b-1)\varphi(xu_{n_1})}.$$

Note that the above number is < 1, for if it were equal to 1, we would then get the equation

$$(b-1)\varphi(xu_{n_1}) = \frac{y(b^{n_1} - 1)}{b^{\lambda}},$$

which is impossible since its left-hand side is an integer and its right-hand side is not. Hence, $L < 1$. Thus, choosing c_{12} to be some constant in the interval $(L, 1)$, we get that

$$c_{12}^{-1} \leqslant \frac{N_1}{\varphi(N_1)} = \prod_{P \mid N_1} \Big(1 + \frac{1}{P-1}\Big). \tag{16}$$

It is clear that $P \mid N_1$ if and only if $P \mid u_n$, $n_1 \mid l_P$ and $l_P > n_1$. Hence, using again the fact that $1 + t < \mathrm{e}^t$ for all $t > 0$, and the estimate (5), we get

$$\prod_{P \mid N_1} \Big(1 + \frac{1}{P-1}\Big) \leqslant \exp\Bigg(c_0 \sum_{\substack{n_1 \mid d \\ d > n_1}} \frac{\log\log d}{\varphi(d)} + O\Big(\sum_{P \geqslant p_{i+1}} \frac{1}{P^2}\Big)\Bigg),$$

which together with the estimates (16) and (7) leads to

$$\begin{aligned} c_{13} &\leqslant c_0 \sum_{\substack{n_1 \mid d \\ d > n_1}} \frac{\log\log d}{\varphi(d)} + O\Big(\frac{1}{p_{i+1}}\Big) \\ &\leqslant c_0 \mathcal{S}_{n_1} \Big(\prod_{j=i+1}^{s} \Big(1 + \frac{c_1 \log\log p_j}{p_j}\Big) - 1\Big) + O\Big(\frac{1}{p_{i+1}}\Big), \end{aligned}$$

where $c_{13} = \log(c_{12}^{-1}) > 0$. Writing c_{14} for an upper bound for $c_0 \mathcal{S}_{n_1}$, and c_{15} for the constant implied by the above Landau symbol, we get that if $p_{i+1} > 2c_{15}c_{13}^{-1}$, then

$$1 \ll \prod_{j=i+1}^{s} \Big(1 + \frac{c_1 \log\log p_j}{p_j}\Big) - 1 \leqslant \Big(1 + \frac{c_1 \log\log p_{i+1}}{p_{i+1}}\Big)^s - 1,$$

where the constant implied in the above Vinogradov symbol is $c_{16} = 2c_{14}c_{13}^{-1}$. The above inequality certainly implies that

$$1 \ll \frac{s \log\log p_{i+1}}{p_{i+1}},$$

which leads to $p_{i+1} \ll 1$, thus completing the induction and finishing the proof of the theorem.

3. Comments and remarks

If one replaces the condition that x and y belong to $\{1, \ldots, b-1\}$ with the weaker condition that x and y are fixed (or bounded), then it is perhaps not true that the equation (1) has only finitely many such solutions (m, n). For example, taking $b = 2$, $x = 1$, $y = 2$, we note that the equation (1) is always satisfied when $m = n - 1$ and $2^n - 1$ is prime. Of course, we do not know that there are infinitely many *Mersenne primes*; i.e., primes of the form $2^n - 1$, but the general belief is that this is indeed so. Note further that when $m = n = 1$, then the equation (1) is trivially satisfied with $y = \varphi(x)$. It would be interesting to study the nontrivial solutions of the equation (1) in all five variables (x, y, b, m, n); i.e., where the base b is also variable. We conjecture that there exists an absolute constant n_0 such that all such solutions have $n \leqslant n_0$. We leave this conjecture as an open problem for the reader.

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