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THE UPPER TRACEABLE NUMBER OF A GRAPH

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Abstract. For a nontrivial connected graph G of order n and a linear ordering $s\colon v_1, v_2, \ldots, v_n$ of vertices of G, define $d(s) = \sum\limits_{i=1}^{n-1} d(v_i, v_{i+1})$. The traceable number $t(G)$ of a graph G is $t(G) = \min\{d(s)\}$ and the upper traceable number $t^+(G)$ of G is $t^+(G) = \max\{d(s)\}$, where the minimum and maximum are taken over all linear orderings s of vertices of G. We study upper traceable numbers of several classes of graphs and the relationship between the traceable number and upper traceable number of a graph. All connected graphs G for which $t^+(G) - t(G) = 1$ are characterized and a formula for the upper traceable number of a tree is established.

Keywords: traceable number, upper traceable number, Hamiltonian number

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1. Introduction and some known results

We refer to the book [6] for graph-theoretical notation and terminology not described in this paper. For a connected graph G of order $n \geqslant 3$ and a cyclic ordering $s\colon v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ of vertices of G, the number $d(s)$ is defined as

$$d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1}),$$

where $d(v_i, v_{i+1})$ is the distance between v_i and v_{i+1}. Therefore, $d(s) \geqslant n$ for each cyclic ordering s of vertices of G. The *Hamiltonian number* $h(G)$ of G is defined in [5] by

$$h(G) = \min\{d(s)\},$$

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where the minimum is taken over all cyclic orderings s of the vertices of G. Therefore, $h(G) = n$ if and only if G is Hamiltonian. In [7], [8] Goodman and Hedetniemi introduced the concept of a *Hamiltonian walk* in a connected graph G, defined as a closed spanning walk of minimum length in G. During the 10-year period 1973–1983, this concept received considerable attention. For example, Hamiltonian walks were also studied by Asano, Nishizeki and Watanabe [1], [2], Bermond [3], Nebeský [9], and Vacek [12]. It was shown in [5] that the Hamiltonian number of a connected graph G is, in fact, the length of a Hamiltonian walk in G. This concept was studied further in [4], [10], [11].

A concept related to the Hamiltonian number of a graph was introduced in [10]. A graph has been called *traceable* if it contains a Hamiltonian path. Therefore, every Hamiltonian graph is traceable. The converse is not true of course. For a connected graph G of order $n \geqslant 2$ and an ordering (also called a *linear ordering*) $s\colon v_1, v_2, \ldots, v_n$ of vertices of G, the number $d(s)$ is defined as

$$d(s) = \sum_{i=1}^{n-1} d(v_i, v_{i+1}).$$

The *traceable number* $t(G)$ of G is defined in [10] by

$$t(G) = \min\{d(s)\},$$

where the minimum is taken over all linear orderings s of vertices of G. Thus if G is a connected graph of order $n \geqslant 2$, then $t(G) \geqslant n-1$. Furthermore, $t(G) = n-1$ if and only if G is traceable. As with Hamiltonian numbers of graphs, there is an alternative way to define the traceable number of a connected graph. It was shown in [10] that the traceable number of a connected graph G is the minimum length of a spanning walk in G. All of the results stated in this section appear in [10].

Theorem 1.1. *For every nontrivial connected graph G,*

$$1 \leqslant h(G) - t(G) \leqslant \operatorname{diam}(G).$$

Furthermore, $h(G) - t(G) = 1$ if and only if G is Hamiltonian.

Theorem 1.2. *Let G be a nontrivial connected graph of order n such that l is the length of a longest path in G and p is the maximum size of a spanning linear forest in G. Then*

$$2n - 2 - p \leqslant t(G) \leqslant 2n - 2 - l.$$

For a vertex v in a connected graph G, the *eccentricity* $e(v)$ of v is the largest distance between v and a vertex of G. The *diameter* $\operatorname{diam}(G)$ of a connected graph G is the largest eccentricity among all vertices of G.

Theorem 1.3. *If T is a nontrivial tree of order n, then*

$$t(T) = 2n - 2 - \operatorname{diam}(T).$$

If G is a connected graph and H is a connected spanning subgraph of G, then $d_G(u,v) \leqslant d_H(u,v)$ for every two vertices u and v of G and so $t(G) \leqslant t(H)$. In particular, if G is a connected graph and T is a spanning tree of G, then $t(G) \leqslant t(T)$.

Theorem 1.4. *If G is a connected graph of order $n \geqslant 3$, then*

$$n - 1 \leqslant t(G) \leqslant 2n - 4.$$

Furthermore,

(a) $t(G) = 2n - 4$ *if and only if* $G = K_3$ *or* $G = K_{1,n-1}$;

(b) $t(G) = 2n - 5$ *if and only if* (1) $n = 4$ *and* $G \neq K_{1,3}$, *or* (2) $n \geqslant 5$ *and* $G = K_{1,n-1} + e$ *or* G *is a double star of order* n; *and*

(c) *for each pair* k, n *of integers with* $3 \leqslant n - 1 \leqslant k \leqslant 2n - 4$, *there exists a connected graph of order* n *with traceable number* k.

For a vertex v of a nontrivial connected graph G, the *traceable number* $t(v)$ of v is defined by

$$t(v) = \min\{d(s)\},$$

where the minimum is taken over all linear orderings s of vertices of G whose first term is v. Thus $t(v) \geqslant n - 1$ for every vertex v of G. Furthermore, $t(v) = n - 1$ if and only if G contains a Hamiltonian path with initial vertex v. Observe that

$$t(G) = \min\{t(v)\colon\ v \in V(G)\}.$$

Moreover, the traceable number of a vertex v in a connected graph G is the minimum length of a spanning walk in G whose initial vertex is v.

Theorem 1.5. *Let G be a connected graph and let u and v be adjacent vertices of G. Then*

$$|t(u) - t(v)| \leqslant 1.$$

Therefore, if k is an integer such that

$$\min\{t(v)\colon\ v \in V(G)\} \leqslant k \leqslant \max\{t(v)\colon\ v \in V(G)\},$$

then there exists a vertex w of G such that $t(w) = k$.

Theorem 1.6. *If T is a nontrivial tree of order n and v is a vertex of T, then*

$$t(v) = 2(n-1) - e(v).$$

It was observed in [10] that Theorem 1.6 is not true in general for a nontrivial connected graph that is not a tree.

2. Basic definitions and preliminary results

For a connected graph G, the *upper Hamiltonian number* $h^+(G)$ is defined in [5] by

$$h^+(G) = \max\{d(s)\},$$

where the maximum is taken over all cyclic orderings s of vertices of G. Obviously, $h(G) \leqslant h^+(G)$ for every connected graph G. The upper Hamiltonian number of a graph has been studied in [4], [5]. As expected, for a connected graph G, the *upper traceable number* $t^+(G)$ is defined by

$$t^+(G) = \max\{d(s)\},$$

where the maximum is taken over all linear orderings s of vertices of G. Consequently, $t(G) \leqslant t^+(G)$ for every connected graph G. For each integer $n \geqslant 3$, it was shown in [5] that K_n and $K_{1,n-1}$ are the only connected graphs G of order n for which $h(G) = h^+(G)$. In fact, there is only one nontrivial connected graph G of order n for which $t(G) = t^+(G)$. Observe that $t(K_n) = t^+(K_n) = n-1$ for $n \geqslant 2$. On the other hand, if $G \neq K_n$ is a connected graph of order $n \geqslant 3$, then G contains two nonadjacent vertices x and y such that $d(x,y) = 2$. Let x, z, y be an $x-y$ path in G. Let $s\colon x, z, y, w_1, w_2, \ldots, w_{n-3}$ and $s'\colon z, x, y, w_1, w_2, \ldots, w_{n-3}$ be two linear orderings of vertices of G. Then $d(s') = d(s) + 1$ and so $t(G) \neq t^+(G)$. We state this observation as follows.

Observation 2.1. *Let G be a nontrivial connected graph of order n. Then*

$$t(G) = t^+(G) \text{ if and only if } G = K_n.$$

As an illustration, we now establish the upper traceable numbers of complete multipartite graphs and the hypercubes.

Proposition 2.2. *If $G = K_{n_1,n_2,\ldots,n_k}$, where $n = n_1 + n_2 + \ldots + n_k$ and $k \geqslant 2$, then*

$$t^+(G) = 2n - k - 1.$$

P r o o f. For each integer i with $1 \leqslant i \leqslant k$, let $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}\}$ be a partite set of G. Then

$$s_0\colon\ v_{1,1}, v_{1,2}, \ldots, v_{1,n_1}, v_{2,1}, v_{2,2}, \ldots, v_{2,n_2}, \ldots, v_{k,1}, v_{k,2}, \ldots, v_{k,n_k}$$

is a linear ordering of vertices of G. Since

$$d(s_0) = (k-1) + \sum_{i=1}^{k} 2(n_i - 1) = 2n - k - 1,$$

it follows that $t^+(G) \geqslant 2n - k - 1$. On the other hand, let $s\colon\ x_1, x_2, \ldots, x_n$ be an arbitrary linear ordering of vertices of G. Since $\operatorname{diam}(G) = 2$, it follows that $d(x_j, x_{j+1}) = 1$ or $d(x_j, x_{j+1}) = 2$ for $1 \leqslant j \leqslant n-1$. Furthermore, there are at most $\sum\limits_{i=1}^{k}(n_i - 1) = n - k$ pairs x_j, x_{j+1} $(1 \leqslant j \leqslant n-1)$ for which $d(x_j, x_{j+1}) = 2$. Thus

$$d(s) \leqslant 2(n-k) + 1 \cdot [(n-1) - (n-k)] = 2n - k - 1$$

and so $t^+(G) \leqslant 2n - k - 1$. Therefore, $t^+(G) = 2n - k - 1$. $\square$

Proposition 2.3. *For each integer $n \geqslant 2$,*

$$t^+(Q_n) = 2^{n-1}(2n-1) - n + 1.$$

P r o o f. First, we show that $t^+(Q_n) \leqslant 2^{n-1}(2n-1) - n + 1$. Let s be an arbitrary linear ordering of $V(Q_n)$ with $d(s) = t^+(Q_n)$. Since $\operatorname{diam}(Q_n) = n$ and for each vertex v in Q_n there is exactly one vertex in Q_n whose distance from v is n, it follows that there are at most 2^{n-1} terms in $d(s)$ equal to n. Consequently, each of the remaining $2^{n-1} - 1$ terms in $d(s)$ is at most $n-1$. Thus

$$d(s) \leqslant 2^{n-1}n + (2^{n-1} - 1)(n-1) = 2^{n-1}(2n-1) - n + 1$$

and so $t^+(Q_n) \leqslant 2^{n-1}(2n-1) - n + 1$.

Next we show that $t^+(Q_n) \geqslant 2^{n-1}(2n-1) - n + 1$. Since the result is true for Q_2, we may assume that $n \geqslant 3$. Let $G = Q_n$. Then G consists of two disjoint copies G_1 and G_2 of Q_{n-1}, where the corresponding vertices of G_1 and G_2 are adjacent.

For each vertex v of G, there is a unique vertex $\overline{v}$ of G such that $d(v, \overline{v}) = n = \operatorname{diam}(Q_n)$. Necessarily, exactly one of v and $\overline{v}$ belongs to G_1 for each vertex v of G. It is well-known that Q_n is Hamiltonian for $n \geqslant 2$ and so Q_n is traceable. Let $P\colon v_1, v_2, \ldots, v_{2^{n-1}}$ be a Hamiltonian path in G_1. Now define a linear ordering s of $V(G)$ by

$$s\colon v_1, \overline{v}_1, v_2, \overline{v}_2, \ldots, v_{2^{n-1}}, \overline{v}_{2^{n-1}}.$$

Since $d(v_i, \overline{v}_i) = n$ and $d(v_i, v_{i+1}) = 1$ for $1 \leqslant i \leqslant 2^{n-1} - 1$, it follows by the triangle inequality that

$$n = d(v_i, \overline{v}_i) \leqslant d(v_i, v_{i+1}) + d(v_{i+1}, \overline{v}_i) = 1 + d(v_{i+1}, \overline{v}_i).$$

Thus $d(v_{i+1}, \overline{v}_i) \geqslant n - 1$, which implies that $d(v_{i+1}, \overline{v}_i) = n - 1$. Hence

$$t^+(Q_n) \geqslant d(s) = 2^{n-1}n + (2^{n-1} - 1)(n - 1) = 2^{n-1}(2n - 1) - n + 1,$$

as desired. $\square$

If $s\colon v_1, v_2, \ldots, v_n$ is an arbitrary linear ordering of vertices of a connected graph, then for each vertex v_i, both $d(v_{i-1}, v_i) \leqslant e(v_i)$ $(2 \leqslant i \leqslant n)$ and $d(v_i, v_{i+1}) \leqslant e(v_i)$ $(1 \leqslant i \leqslant n - 1)$. Thus, If G is a connected graph of order $n \geqslant 2$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$, then

$$t^+(G) \leqslant \sum_{i=1}^{n-1} e(v_i).$$

Since the eccentricity of a vertex in G is at most the diameter of G, we have the following observation, which provides an upper bound for the upper traceable number of a graph in terms of its order and diameter.

Observation 2.4. *If G is a nontrivial connected graph of order n, then*

$$t^+(G) \leqslant (n - 1) \operatorname{diam}(G).$$

The upper bound for the upper traceable number of a graph described in Observation 2.4 is sharp. For example, $t^+(C_n) = (n - 1) \operatorname{diam}(C_n)$ for each odd integer $n \geqslant 3$, as we show next.

Proposition 2.5. *For each integer $n \geqslant 3$,*

$$t^+(C_n) = \left\lceil \frac{(n-1)^2}{2} \right\rceil.$$

Proof. Let $C_n\colon\ v_1, v_2, \ldots, v_n, v_1$ and let $d = \operatorname{diam}(C_n) = \lfloor n/2 \rfloor$ be the diameter of C_n. We consider two cases according to whether n is odd or n is even.

Case 1. *n is odd.* Then $n = 2k+1$ for some positive integer k and so $d = k = (n-1)/2$. By Observation 2.4, $t^+(C_n) \leqslant (n-1)d$. Let

$$s_0\colon\ v_1, v_{k+1}, v_{2k+1}, v_{3k+1}, \ldots, v_{(2k+1)k+1}$$

be a linear ordering of elements of $V(C_n)$, where each subscript is expressed modulo $2k+1$ as one of the integers $1, 2, \ldots, 2k+1$. Since $d(s_0) = (2k)k = (n-1)d$, it follows that $t^+(C_n) \geqslant (n-1)d$. Thus

$$t^+(C_n) = (n-1)d = \frac{(n-1)^2}{2} = \left\lceil \frac{(n-1)^2}{2} \right\rceil$$

if n is odd.

Case 2. *n is even.* Then $n = 2k$ for some integer $k \geqslant 2$ and so $d = k = n/2$. Let s be a linear ordering of vertices of C_n with $d(s) = t^+(C_n)$. Since $\operatorname{diam}(C_n) = k$ and for each $v \in V(C_n)$ there is exactly one vertex in C_n whose distance from v is k, it follows that at most k terms in $d(s)$ equal k. Consequently, at least $k-1$ terms in $d(s)$ are $k-1$ or less. Thus

$$d(s) \leqslant k^2 + (k-1)^2 = 2k^2 - 2k + 1 = \frac{(n-1)^2 + 1}{2}$$

and so $t^+(C_n) \leqslant \frac{1}{2}\big((n-1)^2 + 1\big)$. On the other hand, let

$$s_1\colon\ v_1, v_{k+1}, v_2, v_{k+2}, v_3, v_{k+3}, \ldots, v_{k-1}, v_{(k-1)+k}, v_k, v_{2k}$$

be a linear ordering of the vertices of C_n. Since $d(s_1) = k^2 + (k-1)^2 = \frac{1}{2}\big((n-1)^2+1\big)$, it follows that $t^+(C_n) \geqslant d(s_1) = \frac{1}{2}\big((n-1)^2 + 1\big)$. Therefore,

$$t^+(C_n) = \frac{(n-1)^2 + 1}{2} = \left\lceil \frac{(n-1)^2}{2} \right\rceil$$

if n is even. $\square$

3. A Characterization of graphs whose traceable and upper traceable numbers differ by 1

By Observation 2.1, the complete graph K_n of order $n \geqslant 2$ is the only nontrivial connected graph G of order n for which $t(G) = t^+(G)$. In this section we first present a characterization of those connected graphs G for which $t^+(G) - t(G) = 1$.

Theorem 3.1. *Let G be a connected graph of order $n \geqslant 3$. Then*

$$t^+(G) - t(G) = 1 \text{ if and only if } G = K_n - e \text{ or } G = K_{1,n-1}.$$

P r o o f. First observe that for $n \geqslant 3$, $t^+(K_n - e) = n$ and $t(K_n - e) = n-1$, while $t^+(K_{1,n-1}) = 2n-3$ and $t(K_{1,n-1}) = 2n-4$. Hence, if $G = K_n - e$ or $G = K_{1,n-1}$, then $t^+(G) - t(G) = 1$. It remains therefore to verify the converse.

Let G be a connected graph of order $n \geqslant 3$ such that $t^+(G) - t(G) = 1$. We claim that $\operatorname{diam}(G) = 2$. Assume, to the contrary, that $\operatorname{diam}(G) \neq 2$. If $\operatorname{diam}(G) = 1$, then $G = K_n$. However, $t^+(K_n) = t(K_n) = n-1$. If $\operatorname{diam}(G) \geqslant 3$, then G contains two vertices u and v such that $d(u,v) = 3$. Let u, x, y, v be a $u - v$ path in G and let $v_1, v_2, \ldots, v_{n-4}$ be the remaining vertices of G. Also, let $v_0 = v$ and

$$\sum_{i=0}^{n-5} d(v_i, v_{i+1}) = a.$$

For the linear orderings

$$s_1\colon\ u, x, y, v, v_1, v_2, \ldots, v_{n-4}$$

and

$$s_2\colon\ u, y, x, v, v_1, v_2, \ldots, v_{n-4}, \quad d(s_1) = a+3 \quad \text{and} \quad d(s_2) = a+5.$$

Since $t(G) \leqslant d(s_1)$ and $t^+(G) \geqslant d(s_2)$, it follows that $t^+(G) - t(G) \geqslant 2$, a contradiction. Thus, $\operatorname{diam}(G) = 2$, as claimed.

We now consider two cases, depending on whether G is traceable.

Case 1. *G is traceable.* Then $t(G) = n-1$. Since $G \neq K_n$, the graph G contains at least one pair of nonadjacent vertices. Suppose that G contains two pairs u, v and x, y of nonadjacent vertices. If the vertices $\{u,v\} \cap \{x,y\} = \emptyset$, then every linear ordering s' beginning with u, v, x, y has $d(s') \geqslant n+1$, which is a contradiction. If $\{u,v\} \cap \{x,y\} \neq \emptyset$, say $v = x$, then every linear ordering s'' beginning with u, v, y has $d(s'') \geqslant n+1$, a contradiction. Hence G contains exactly one pair of nonadjacent vertices and so $G = K_n - e$.

Case 2. *G is not traceable.* Then $t(G) = n + k - 2$ for some integer $k \geqslant 2$. Thus G contains k pairwise vertex-disjoint paths $G_1, G_2, \ldots, G_k$ such that $\{V(G_1), V(G_2), \ldots, V(G_k)\}$ is a partition of $V(G)$. However, G does not contain fewer than k pairwise vertex-disjoint paths with these properties. Suppose that G_i is an $x_i - y_i$ path for $1 \leqslant i \leqslant k$. Furthermore, let $x_i, \ldots, y_i$ denote the $x_i - y_i$ path G_i for $1 \leqslant i \leqslant k$. Then the linear ordering

$$s\colon\ x_1, \ldots, y_1, x_2, \ldots, y_2, \ldots, y_{k-1}, x_k, \ldots, y_k$$

of the vertices of G has the property that $d(s) = t(G) = n + k - 2$. Furthermore, $d(s)$ contains exactly $k - 1$ terms, namely $d(y_i, x_{i+1})$ for $1 \leqslant i \leqslant k - 1$, that equal 2, with all other terms equal to 1.

Observe that $x_i x_j, x_i y_j, y_i y_j \notin E(G)$ for all i and j with $1 \leqslant i, j \leqslant k$ and $i \neq j$, for otherwise G contains fewer than k vertex-disjoint paths whose vertex sets form a partition of $V(G)$.

Next we claim that at most one of the paths G_i $(1 \leqslant i \leqslant k)$ has order 2 or more. Suppose to the contrary that there are two such paths, say G_1 and G_2. Let s_0 be a linear ordering of the vertices of G beginning with x_1, x_2, y_1, y_2 and containing the pairs y_i, x_{i+1} $(2 \leqslant i \leqslant k - 1)$ as consecutive terms. Then $d(s_0)$ contains at least $3 + (k - 2) = k + 1$ terms equal to 2. Thus

$$d(s_0) \geqslant 2(k + 1) + [(n - 1) - (k + 1)] = n + k,$$

which is a contradiction. Thus, as claimed, at most one of the paths G_i $(1 \leqslant i \leqslant k)$ has order 2 or more, say G_1. Since G is connected and none of $x_i x_j, x_i y_j, y_i y_j$ are edges of G for i and j with $1 \leqslant i, j \leqslant k$ and $i \neq j$, the path G_1 has order 3 or more. If G_1 has order 3, say G_1 is the path x_1, v, y_1, then $v x_i \in E(G)$ for $2 \leqslant i \leqslant k$ and $x_1 y_1 \notin E(G)$ and so $G = K_{1,n-1}$.

Suppose then that G_1 has order 4 or more. Each of the vertices x_i $(2 \leqslant i \leqslant k)$ must be adjacent to an interior vertex of G_1. Thus $x_1 y_1 \notin E(G)$, for otherwise, G contains fewer than k vertex-disjoint paths whose vertex sets form a partition of $V(G)$, which is a contradiction. Indeed, we claim that each vertex x_i $(2 \leqslant i \leqslant k)$ must be adjacent to every interior vertex of G_1; assume, to the contrary, that some vertex x_i, say x_2, is not adjacent to the interior vertex v of G_1. Let s^* be a linear ordering of vertices of G beginning with $v, x_2, y_1, x_1, x_3, x_4, \ldots, x_k$. Then $d(s^*)$ contains at least $k + 1$ terms equal to 2. Thus

$$d(s^*) \geqslant 2(k + 1) + [(n - 1) - (k + 1)] = n + k,$$

which is a contradiction. Since x_2 is adjacent to all interior vertices of G_1, there is a path in G with the vertex set $V(G_1) \cup \{x_2\}$. However then G contains fewer than k vertex-disjoint paths whose vertex sets form a partition of $V(G)$, which is a contradiction. $\square$

4. The upper traceable number of a tree

In this section we establish a formula for the upper traceable number of a tree. In order to do this, we first study the relationship between the upper traceable number and upper Hamiltonian number of a graph.

Proposition 4.1. *For every connected graph G of order $n \geqslant 3$,*

$$1 \leqslant h^+(G) - t^+(G) \leqslant \operatorname{diam}(G).$$

Proof. Let $s_c\colon v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ be a cyclic ordering of vertices of G with $d(s_c) = h^+(G)$. Then $s_l\colon v_1, v_2, \ldots, v_n$ is a linear ordering of vertices of G. Since

$$t^+(G) \geqslant d(s_l) = d(s_c) - d(v_1, v_n) \geqslant h^+(G) - \operatorname{diam}(G),$$

it follows that $h^+(G) - t^+(G) \leqslant \operatorname{diam}(G)$. On the other hand, let $s'_l\colon v'_1, v'_2, \ldots, v'_n$ be a linear ordering of vertices of G with $d(s'_l) = t^+(G)$. Then $s'_c\colon v'_1, v'_2, \ldots, v'_n, v'_{n+1} = v'_1$ is a cyclic ordering of vertices of G. Since

$$h^+(G) \geqslant d(s'_c) = d(s'_l) + d(v_1, v_n) \geqslant t^+(G) + 1,$$

it follows that $h^+(G) - t^+(G) \geqslant 1$. $\square$

Proposition 4.2. *For every nontrivial connected graph G of order n,*

$$h^+(G) - t^+(G) = \operatorname{diam}(G) \text{ if and only if } h^+(G) = n \operatorname{diam}(G).$$

Proof. Let $s_c\colon v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ be a cyclic ordering of the vertices of G with $d(s_c) = h^+(G)$. Then $s_l\colon v_1, v_2, \ldots, v_n$ is a linear ordering of the vertices of G. First assume that $h^+(G) - t^+(G) = \operatorname{diam}(G)$. We will show that $d(v_i, v_{i+1}) = \operatorname{diam}(G)$ for $1 \leqslant i \leqslant n$. For each i with $1 \leqslant i \leqslant n$, let

$$s_i\colon v_{i+1}, v_{i+2}, \ldots, v_n, v_{n+1} = v_1, v_2, \ldots, v_i.$$

Then

$$t^+(G) \geqslant d(s_i) = d(s_c) - d(v_i, v_{i+1}) = h^+(G) - d(v_i, v_{i+1}).$$

Thus $d(v_i, v_{i+1}) \geqslant h^+(G) - t^+(G) = \operatorname{diam}(G)$, implying that $d(v_i, v_{i+1}) = \operatorname{diam}(G)$ for each i with $1 \leqslant i \leqslant n$. Therefore, $h^+(G) = d(s) = n \operatorname{diam}(G)$.

For the converse, assume that $h^+(G) = n \operatorname{diam}(G)$. Since

$$t^+(G) \geqslant d(s_l) = d(s_c) - d(v_1, v_n) = n \operatorname{diam}(G) - d(v_1, v_n) \geqslant (n-1) \operatorname{diam}(G),$$

it follows by Observation 2.4 that $t^+(G) = (n-1) \operatorname{diam}(G)$. Therefore, $h^+(G) - t^+(G) = \operatorname{diam}(G)$. □

It was shown in [5] that

$$h^+(P_n) = \left\lfloor \frac{n^2}{2} \right\rfloor \tag{1}$$

for $n \geqslant 2$. We now determine the upper traceable number of the path P_n for $n \geqslant 2$.

Proposition 4.3. *For each integer $n \geqslant 2$,*

$$t^+(P_n) = \left\lfloor \frac{n^2}{2} \right\rfloor - 1.$$

Proof. Since $h^+(P_n) = \left\lfloor \frac{1}{2}n^2 \right\rfloor$, it follows by Proposition 4.1 that $t^+(P_n) \leqslant \left\lfloor \frac{1}{2}n^2 \right\rfloor - 1$. To verify that $t^+(P_n) \geqslant \left\lfloor \frac{1}{2}n^2 \right\rfloor - 1$, it suffices to show that there exists a linear ordering s of the vertices of P_n for which $d(s) = \left\lfloor \frac{1}{2}n^2 \right\rfloor - 1$. Let $P_n\colon u_1, u_2, \ldots, u_n$ and let us consider two cases according to whether n is odd or n is even.

Case 1. *n is odd.* Then $n = 2k+1$ for some positive integer k. Let

$$s_0\colon\ u_{k+1}, u_1, u_{2k+1}, u_2, u_{2k}, u_3, u_{2k-1}, \ldots, u_k, u_{k+2}$$

be a linear ordering of vertices of P_n. Since

$$\begin{aligned} d(s_0) &= k + (2k) + (2k-1) + (2k-2) + \ldots + 2 \\ &= k + (1 + 2 + 3 + \ldots + 2k) - 1 = k + \binom{2k+1}{2} - 1 \\ &= k(2k+2) - 1 = \frac{n^2-1}{2} - 1 = \left\lfloor \frac{n^2}{2} \right\rfloor - 1, \end{aligned}$$

it follows that $t^+(P_n) \geqslant \left\lfloor \frac{1}{2}n^2 \right\rfloor - 1$. Thus $t^+(P_n) = \left\lfloor \frac{1}{2}n^2 \right\rfloor - 1$ if n is odd.

Case 2. *n is even.* Then $n = 2k$ for some integer $k \geqslant 2$. Let

$$s_1\colon\ u_{k+1}, u_1, u_{2k}, u_2, u_{2k-1}, u_3, u_{2k-2}, \ldots, u_{k-1}, u_{k+2}, u_k$$

be a linear ordering of vertices of P_n. Since

$$\begin{aligned} d(s_1) &= k + (2k-1) + (2k-2) + \ldots + 2 \\ &= k + [1 + 2 + 3 + \ldots + (2k-1)] - 1 = k + \binom{2k}{2} - 1 \\ &= 2k^2 - 1 = \frac{n^2}{2} - 1 = \left\lfloor \frac{n^2}{2} \right\rfloor - 1, \end{aligned}$$

it follows that $t^+(P_n) \geqslant \lfloor \frac{1}{2}n^2 \rfloor - 1$. Thus $t^+(P_n) = \lfloor \frac{1}{2}n^2 \rfloor - 1$ if n is even. $\square$

We will now consider trees in general. For each edge e of a tree T, the *component number* $\operatorname{cn}(e)$ of e is defined in [5] as the minimum order of a component of $T - e$. For example, the edge e_5 of the tree T of Figure 1(a) has component number 3 since the order of the smaller component of $T - e_5$ is 3. Each edge of this tree is labeled with its component number in Figure 1(b).

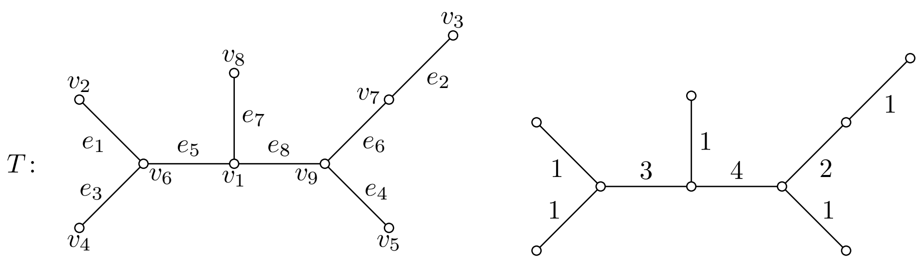

Figure 1. Component numbers of edges

An upper bound for the upper Hamiltonian number of a tree was established in [5] in terms of the component numbers of its edges, which we state as follows.

Theorem 4.4. *If T is a nontrivial tree, then*

$$h^+(T) \leqslant 2 \sum_{e \in E(T)} \operatorname{cn}(e).$$

For the tree T of Figure 1,

$$\sum_{i=1}^{8} \operatorname{cn}(e_i) = 1 + 1 + 3 + 1 + 4 + 1 + 2 + 1 = 14.$$

Thus $h^+(T) \leqslant 28$ by Theorem 4.4. With the aid of Theorem 4.4 and Proposition 4.1, we are able to establish a formula for the upper traceable number of a tree.

Theorem 4.5. *If T is a nontrivial tree, then*

$$t^+(T) = 2 \sum_{e\in E(T)} \operatorname{cn}(e) - 1.$$

Proof. By Theorem 4.4 and Proposition 4.1,

$$t^+(T) \leqslant h^+(T) - 1 \leqslant 2 \sum_{e\in E(T)} \operatorname{cn}(e) - 1.$$

Thus it remains to show that $t^+(T) \geqslant 2 \sum_{e\in E(T)} \operatorname{cn}(e) - 1$. Since the theorem holds if T has order 2, we may assume that T has order 3 or more. Suppose that $T_1 = T$ has order $n \geqslant 3$. Let v_2 be an end-vertex of T. Furthermore, let Q_2 be a maximal path in T whose initial edge e_1 is incident with v_2 and such that each successive edge in Q_2 is chosen so that it has the maximum component number (among all edges available). Suppose that Q_2 is a $v_2 - v_3$ path. Necessarily, v_3 is an end-vertex of T. Let $T_2 = T - \{v_2\}$ and let Q_3 be a maximal path in T_2 whose initial edge e_2 is incident with v_3 and such that each successive edge in Q_3 is chosen so that it has the maximum component number in T_2 (among all edges available). We continue this process until we arrive at the $v_{n-1} - v_n$ path Q_{n-1}. The final vertex of T is denoted by v_1, which is necessarily adjacent to v_n. Let $e_{n-1} = v_n v_1$. This procedure is illustrated in Figure 2 for the tree T of Figure 1, where each $v_{i+1} - v_{i+2}$ path Q_{i+1} for $1 \leqslant i \leqslant n-2$ is indicated in bold.

For $2 \leqslant i \leqslant n-2$, the edge e_i is the initial edge of the $v_{i+1} - v_{i+2}$ path Q_{i+1} in the tree $T_i = T - \{v_2, v_3, \dots, v_i\}$. Furthermore, let Q_1 be the $v_1 - v_2$ path in $T = T_1$. Consider the linear ordering

$$s\colon\ v_1, v_2, \dots, v_n$$

of vertices of T. We show that

$$d(s) = 2 \sum_{e\in E(T)} \operatorname{cn}(e) - 1. \tag{2}$$

To verify (2), we show that for every integer i with $1 \leqslant i \leqslant n-2$, the edge e_i is traversed $2\operatorname{cn}(e_i)$ times by the paths $Q_1, Q_2, \dots, Q_{n-1}$, while e_{n-1} is traversed $2\operatorname{cn}(e_{n-1}) - 1$ times by the paths $Q_1, Q_2, \dots, Q_{n-1}$. It is certainly the case when an edge is a pendant edge, so suppose that e is an edge of T that is not a pendant edge.

For each tree T_j containing e, let $T_{j,1}$ and $T_{j,2}$ be the components of $T_j - e$ such that $|V(T_{j,1})| \leqslant |V(T_{j,2})| + 1$. We claim that if the initial vertex v_{j+1} of the path Q_{j+1} belongs to $T_{j,1}$, then the terminal vertex v_{j+2} belongs to $T_{j,2}$, that is, the edge e is traversed by Q_{j+1}. Let $c_j = \operatorname{cn}_{T_j}(e)$ and $e = xy$ such that x belongs to $T_{j,1}$.

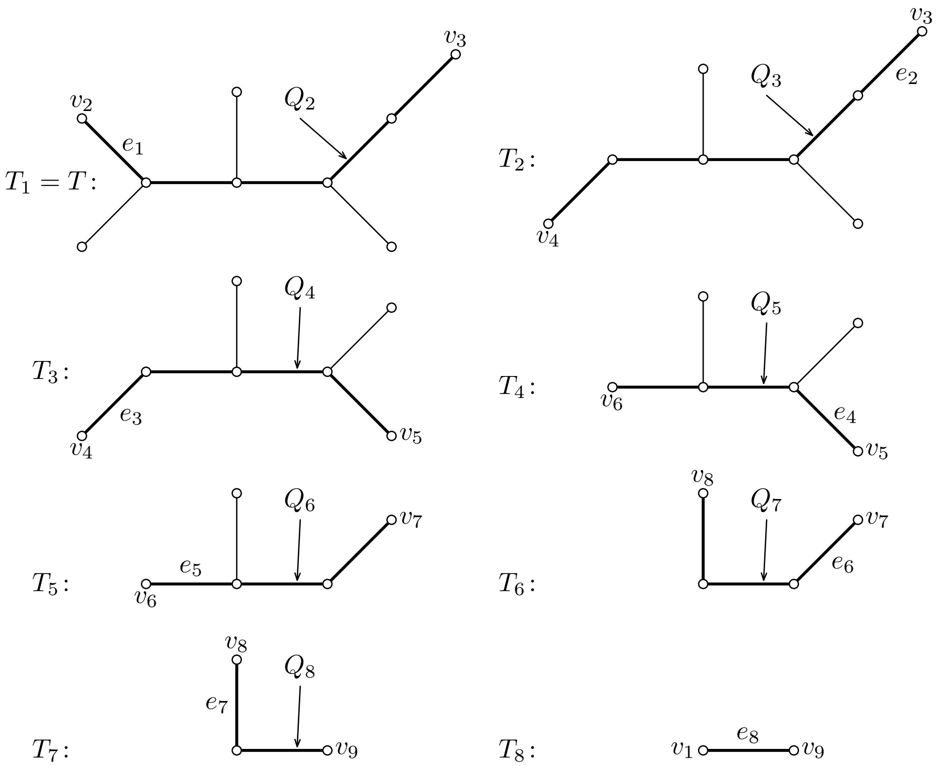

Figure 2. A step in the proof of Theorem 4.5

If $|V(T_{j,1})| \leqslant |V(T_{j,2})|$, then note first that every edge in $T_{j,1}$ has component number at most $c_j - 1$. Assume, to the contrary, that the terminal vertex v_{j+2} of the path Q_{j+1} belongs to $T_{j,1}$. Let $Q_A\colon\ v_{j+1} = u_1, u_2, \ldots, u_k = x$ and $Q_B\colon\ v_{j+2} = w_1, w_2, \ldots, w_l = x$ be the $v_{j+1} - x$ path and $v_{j+2} - x$ path, respectively. Obviously, both Q_A and Q_B are entirely contained in $T_{j,1}$. Furthermore,

$$Q_{j+1}\colon\ v_{j+1} = u_1, u_2, \ldots, u_{k'} = w_{l'}, w_{l'-1}, \ldots, w_1 = v_{j+2}$$

for some integers k' and l' with $2 \leqslant k' \leqslant k$ and $2 \leqslant l' \leqslant l$. This implies that

$$\mathrm{cn}_{T_j}(u_{k'}u_{k'+1}) \leqslant \mathrm{cn}_{T_j}(w_{l'}w_{l'-1}).$$

On the other hand, however, observe that

$$\mathrm{cn}_{T_j}(u_{k'}u_{k'+1}) \geqslant \mathrm{cn}_{T_j}(u_{k'-1}u_{k'}) + \mathrm{cn}_{T_j}(w_{l'}w_{l'-1}) > \mathrm{cn}_{T_j}(w_{l'}w_{l'-1}),$$

a contradiction.

If $|V(T_{j,1})| = |V(T_{j,2})| + 1$, then at most one edge in $T_{j,1}$ has component number c_j and each of the remaining edges in $T_{j,1}$ has component number at most $c_j - 1$. Then by a similar argument given for the case where $|V(T_{j,1})| \leqslant |V(T_{j,2})|$, if v_{j+1} belongs to $T_{j,1}$, then v_{j+2} must belong to $T_{j,2}$.

Now let T' and T'' be the components of $T - e$, where the order of T' is $c = \mathrm{cn}(e)$. Suppose that $V(T') = \{v_{n_1}, v_{n_2}, \ldots, v_{n_c}\}$, where $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_c$. Furthermore, let $e = xy$ such that x belongs to T'. Necessarily then, $x = v_{n_c}$. In each tree T_j containing e, let T'_j and T''_j be the components of $T_j - e$ containing x and y, respectively. Then by the claim given above, we have the following:

(1) $|V(T'_j)| \leqslant |V(T''_j)|$.

(2) v_1 belongs to T''.

(3) No two vertices of T' are consecutive in s.

If $x \neq v_n$, then $e \neq e_{n-1}$. Since $v_{n_1+1}, v_{n_2+1}, \ldots, v_{n_c+1}$ belong to T'', it follows that e is traversed $2c$ times by the paths $Q_1, Q_2, \ldots, Q_{n-1}$. On the other hand, if $x = v_n$, then $e = e_{n-1}$. Since $v_{n_1+1}, v_{n_2+1}, \ldots, v_{n_{c-1}+1}$ belong to T'', it follows that e is traversed $2c - 1$ times by the paths $Q_1, Q_2, \ldots, Q_{n-1}$. Thus, as claimed, $d(s) = 2 \sum\limits_{e \in E(T)} \mathrm{cn}(e) - 1$. Therefore,

$$t^+(T) \geqslant d(s) = 2 \sum_{e \in E(T)} \mathrm{cn}(e) - 1,$$

providing the desired result. □

Since $h^+(T) \geqslant t^+(T) + 1$ for every nontrivial tree T by Proposition 4.1, the following corollary is a consequence of Theorems 4.4 and 4.5.

Corollary 4.6. *If T is a nontrivial tree, then*

$$h^+(T) = 2 \sum_{e \in E(T)} \mathrm{cn}(e).$$

We now illustrate Theorem 4.5 and Corollary 4.6. For the tree T of Figure 1, we have seen that $\sum\limits_{i=1}^{8} \mathrm{cn}(e_i) = 14$. Thus by Theorem 4.5 and Corollary 4.6, $t^+(T) = 28 - 1 = 27$ and $h^+(T) = 28$. On the other hand, using the technique described in the proof of Theorem 4.5, we obtain a linear ordering $s\colon\ v_1, v_2, \ldots, v_9$ of vertices of T with $d(s) = t^+(T) = 27$. Observe that for the cyclic ordering $s_c\colon\ v_1, v_2, \ldots, v_9, v_1$ of vertices of T, $d(s_c) = h^+(T) = 28$.

Upper and lower bounds for the upper Hamiltonian number of a tree was established in [5] in terms of its order, as we state now.

Theorem 4.7. *Let T be a tree of order $n \geqslant 3$. Then*

$$2n-2 \leqslant h^+(T) \leqslant \lfloor n^2/2 \rfloor.$$

Moreover,
(a) $h^+(T) = 2n-2$ *if and only if* $T = K_{1,n-1}$,
(b) $h^+(T) = \lfloor n^2/2 \rfloor$ *if and only if* $T = P_n$.

The following corollary is a consequence of Proposition 4.1, Theorems 4.5 and 4.7, and Corollary 4.6.

Corollary 4.8. *Let T be a tree of order $n \geqslant 3$. Then*

$$2n-3 \leqslant t^+(T) \leqslant \lfloor n^2/2 \rfloor - 1.$$

Furthermore,
(a) $t^+(T) = 2n-3$ *if and only if* $T = K_{1,n-1}$,
(b) $t^+(T) = \lfloor n^2/2 \rfloor - 1$ *if and only if* $T = P_n$.

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