Czechoslovak Mathematical Journal, 58 (133) (2008), 183–202

DIRECT SUMMANDS AND RETRACT MAPPINGS OF GENERALIZED MV-ALGEBRAS

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(Received January 31, 2006)

Abstract. In the present paper we deal with generalized MV-algebras (GMV-algebras, in short) in the sense of Galatos and Tsinakis. According to a result of the mentioned authors, GMV-algebras can be obtained by a truncation construction from lattice ordered groups. We investigate direct summands and retract mappings of GMV-algebras. The relations between GMV-algebras and lattice ordered groups are essential for this investigation.

Keywords: residuated lattice, lattice ordered group, generalized MV-algebra, direct summand

MSC 2000: 06D35, 06F15

1. Introduction

In [5], the notion of generalized MV-algebra (GMV-algebra, in short) has been introduced; it has been studied in the context of residuated lattices.

The fundamental result of [5] is Theorem (A). From this it follows that each GMV-algebra can be represented by using lattice ordered groups. For a detailed formulation of this result, cf. Section 2 below.

In the present paper we apply the mentioned representation for investigating direct summands and retract mappings of GMV-algebras.

Let $\mathbf{M}$ be a GMV-algebra and let $\ell(\mathbf{M})$ be the underlying lattice of $\mathbf{M}$. Further, let $\mathbf{A}$ be a subalgebra of $\mathbf{M}$. We prove that $\mathbf{A}$ is a direct summand of $\mathbf{M}$ iff the underlying lattice $\ell(\mathbf{A})$ of A is an internal direct factor of the lattice $\ell(\mathbf{A})$.

Supported by VEGA Agency grant 1/2002/05.

This work has been partially supported by the Slovak Academy of Sciences via the project Center of Excellence - Physics of Information, grant I/2/2005.

The main result concerning retract mappings of GMV-algebras is Theorem (C) presented in Section 7 below.

We recall that the investigation of direct summands of some types of algebraic structures is frequent in the literature. E.g., a rather large series of papers has dealt with direct summands of abelian groups; cf. the references given in [4].

The related notion of direct product decomposition of MV-algebras was dealt with in [9]; for the case of pseudo MV-algebras cf. [10] and [20] (under a different terminology).

Retract mappings and retracts of lattice ordered groups were investigated in [13], [14], [15], [16]. Retract mappings of MV-algebras were studied in [17].

An important tool in the investigation of the relation between GMV-algebras and lattice ordered groups that is applied in [5] is the negative cone of a lattice ordered group. In the introduction of [5], the authors mention the papers of Chang [1], Mundici [18] and Dvurečenskij [3] on MV-algebras and pseudo MV-algebras; here the authors write: 'It should be noted that all the three authors have expressed their results in terms of the positive cone rather than the negative cone.' Hence in this respect, the method of [5] differs from that of [1], [3], [18].

We also remark that the term 'generalized MV-algebra' was applied in a different sense in [17]; in the sense of [17], this term is equivalent to the notion of pseudo MV-algebra (cf. [3], [6], [7], and also [10], [11], [12], [19] and [20]).

In what follows, the term 'GMV-algebra' will be used in the sense of [5].

2. Preliminaries

For the sake of completeness, we recall some basic definitions. We also quote some results of [5].

A *residuated lattice* is an algebra $\mathbf{L} = (L; \wedge, \vee, \cdot, \backslash, /, e)$ of type $(2,2,2,2,2,0)$ such that $(L; \wedge, \vee)$ is a lattice, $(L; \cdot, e)$ is a monoid and for each $x, y, z \in L$,

$$x \cdot y \leqslant \Leftrightarrow x \leqslant z/y \Leftrightarrow y \leqslant x \setminus z.$$

A residuated lattice is *commutative* if $xy = yx$ for each $x, y \in L$; it is *integral* if $x \wedge e = x$ for each $x \in L$.

The *negative cone* of a residuated lattice $\mathbf{L}$ is an algebra $\mathbf{L}^- = (L^-; \wedge, \vee, \cdot, \backslash_{L^-}, /_{L^-}, e)$ where

$$L^- = \{x \in L \colon\ x \leqslant e\},$$
$$x \backslash_{L^-} y = (x \setminus y) \wedge e, \quad x/_{L^-} y = (x/y) \wedge e.$$

Then $\mathbf{L}^-$ is a residuated lattice as well.

A *generalized MV-algebra* (*GMV-algebra*, in short) is a residuated lattice satisfying the identities

$$x/((x \vee y) \setminus x) = x \vee y = (x/(x \vee y)) \setminus x.$$

If $\mathbf{L}$ is a GMV-algebra, then its negative cone $\mathbf{L}^-$ is a GMV-algebra as well.

Let P be a partially ordered set. A mapping $\gamma\colon P \to P$ is a *closure operator* on P if $\gamma(x) \leqslant \gamma(y)$ whenever $x \leqslant y$, $x \leqslant \gamma(x)$ and $\gamma(\gamma(x)) = x$. Put $\gamma(P) = P_\gamma$. Then

$$\gamma(x) = \min\{t \in P_\gamma \colon x \leqslant t\}$$

for each $x \in P$; hence the mapping γ is uniquely determined by the set P_γ.

Let $\mathbf{L}$ be a residuated lattice. A closure operator γ on L satisfying $\gamma(a)\gamma(b) \leqslant \gamma(a, b)$ for each $a, b \in L$ is a *nucleus* on L. If L_γ is the image of a nucleus γ on L, then the set L_γ is endowed with a residuated lattice structure in the following way:

$$L_\gamma = (L_\gamma; \wedge, \vee_\gamma, \circ_\gamma, \setminus, /, \gamma(e)),$$

where

$$\gamma(a) \vee_\gamma \gamma(b) = \gamma(a \vee b), \quad \gamma(a) \circ_\gamma \gamma(b) = \gamma(ab).$$

A residuated lattice $\mathbf{A}$ is a *direct sum* of its subalgebras $\mathbf{B}$ and $\mathbf{C}$, in symbols $\mathbf{A} = \mathbf{B} \oplus \mathbf{C}$, if the map $B \times C \to A$ defined by $f(x, y) = xy$ is an isomorphism. In such case $\mathbf{B}$ and $\mathbf{C}$ are *direct summands* of $\mathbf{A}$. Under the above notation, put $z = xy$; we denote $x = z(\mathbf{B})$ and $y = z(\mathbf{C})$. We say that x and y is the *component* of z in $\mathbf{B}$ or in $\mathbf{C}$, respectively.

For lattice ordered groups we use the terminology and the notation as in [8].

Let $\mathbf{G} = (G; \wedge, \vee, \cdot, ^{-1}, e)$ be a lattice ordered group. The algebra

$$\mathbf{G}^* = (G; \wedge, \vee, \cdot, \setminus, /, e)$$

where $x \setminus y = x^{-1}y$ and $y/x = yx^{-1}$, is a GMV-algebra.

The following theorem is one of the main results of [5]; we use a slightly modified notation.

Theorem 2.1 (cf. [5], Theorem (A)). *A residuated lattice* $\mathbf{M}$ *is a* GMV*-algebra if and only if there are lattice ordered groups* $\mathbf{G}$ *and* $\mathbf{G}_1$ *and a nucleus* γ *on* $(\mathbf{G}_1^*)^-$ *such that*

$$\mathbf{M} = \mathbf{G}^* \oplus \mathbf{L}_\gamma,$$

where $\mathbf{L} = (\mathbf{G}_1^*)^-$.

Theorem 2.2 (cf. [5], Theorem 3.4). *If $\mathbf{L} = (L; \wedge, \vee, \cdot, \backslash, /, e)$ is a GMV-algebra and γ is a nucleus on γ, then*

(i) $\vee_\gamma = \vee$;

(ii) *γ preserves binary joins;*

(iii) $\gamma(e) = e$;

(iv) *$\mathbf{L}_\gamma = (L_\gamma; \wedge, \vee, \circ_\gamma, \backslash, /, e)$ is a GMV-algebra;*

(v) *L_γ is a filter of the lattice $(L; \wedge, \vee)$.*

3. Internal direct factors of partially ordered sets

Assume that P is a partially ordered set and that $(P_i)_{i\in I}$ is and indexed system of partially ordered sets. The *direct product* $\prod\limits_{i\in I} P_i$ is defined in the usual way. The elements of $\prod\limits_{i\in I} P_i$ are written in the form $t = (t_i)_{i\in I}$. If

$$\varphi\colon\ P \to \prod_{i\in I} P_i$$

is an isomorphism, then we say that φ is a *direct product decomposition* of P. In such case, for each $i \in I$ and each $a \in P$ we put

$$P(i,a) = \{x \in P\colon\ \varphi(x)_j = \varphi(a)_j \quad \text{for each } j \in I \setminus \{i\}\}.$$

The set $P(i,a)$ endowed with the partial order induced from P is an *internal direct factor of P with respect to the element a*. Obviously, $P(i,a)$ is isomorphic to P_i.

For each $y \in P$, we denote by $\varphi_i^a(y)$ the element of $P(i,a)$ such that

$$(\varphi(\varphi_i^a(y))_i = (\varphi(y))_i.$$

Then the mapping

$$\varphi^a\colon\ P \to \prod_{i\in I} P(i,a) \tag{1}$$

where $\varphi^a(y) = (\varphi_i^a(y))_{i\in I}$ for each $y \in P$, is an isomorphism. We say that φ^a defines an *internal direct product decomposition of P with respect to the element a.*

For each $x \in P$ we now put

$$x_i = (\varphi^a(x))_i;$$

x_i is the *i-the component* of x with respect to (1). We also say that x_i is the component of x in $P(i, a)$ and we write $x_i = x(P(i, a))$. Then

$$a_i = a \quad \text{for each} \quad i \in I, \tag{2}$$

$$(x_i)_i = x_i \quad \text{and} \quad (x_i)_j = a \text{ if } j \in I,\ j \neq i. \tag{3}$$

Now let I_1 and I_2 be nonempty subsets of I such that $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = I$. Put

$$P(I_1, a) = \{x \in P\colon\ x_i = a_i \quad \text{for each} \quad i \in I_2\},$$
$$P(I_2, a) = \{x \in P\colon\ x_i = a_i \quad \text{for each} \quad i \in I_1\}.$$

Let $x \in P$. The element $y \in P$ such that

$$y_i = \begin{cases} x_i & \text{if } \ i \in I_1, \\ a_i & \text{if } \ i \in I_2 \end{cases}$$

will be denoted by x_{I_1}. Analogously we define x_{I_2}. Then the mapping

$$x \to (x_{I_1}, x_{I_2})$$

defines an internal direct product decomposition

$$P \to P(I_1, a) \times P(I_2, a). \tag{$*_1$}$$

Further, we have internal direct product decompositions

$$P(I_1, a) \to \prod_{i \in I_1} P(i, a), \tag{$*_2$}$$

$$P(I_2, a) \to \prod_{i \in I_2} P(i, a). \tag{$*_3$}$$

All the internal direct product decompositions $(*_1)$, $(*_2)$ and $(*_3)$ are taken with respect to the element a.

Lemma 3.1. *Assume that P is a partially ordered set and that a is the greatest element of P. Let* (1) *be valid. Then*

$$x = \bigwedge_{i \in I} x_i$$

for each $x \in P$.

Proof. This is a consequence of the relations (2) and (3). $\square$

If (1) holds and $i \in I$, then we put

$$P'(i, a) = \{x \in P \colon\ x_i = a\}.$$

Then in view of $(*_2)$ we have an internal product decomposition

$$P'(i, a) \to \prod_{j \in I \setminus \{i\}} P(j, a).$$

Moreover, according to $(*_1)$ we obtain a two-factor internal direct product decomposition

$$P \to P(i, a) \times P'(i, a). \tag{4}$$

For $x \in P$ we put $x(P'(i, a)) = x'_i$. $\square$

Lemma 3.2. *Let P be as in Lemma* 3.1. *Further, let x and y be elements of P. Then*

$$x_i \wedge x'_i = x, \quad x_i \vee y'_i = a.$$

Proof. The validity of the first relation is a consequence of Lemma 3.1 and of (4). In view of (2) and (3), the second relation holds. $\square$

4. The negative cone

Let $\mathbf{G}$ be a lattice ordered group. The algebra

$$\mathbf{G}^- = \{G^-; \wedge, \vee, \cdot, e\},$$

where $\mathbf{G}^- = \{g \in G\colon\ g \leqslant e\}$ is the *negative cone* of G. For $x, y \in G^-$ we put $x \setminus y = (x^{-1}y) \wedge e$ and $y/x = (yx^{-1}) \wedge e$. An elementary calculation shows that the algebra

$$(\mathbf{G}^-)^* = (G^-; \wedge, \vee, \cdot, \setminus, /, e)$$

is a GMV-algebra; moreover, under the notation as in Section 2 we have $(\mathbf{G}^-)^* = (\mathbf{G}^*)^-$.

We denote by $\ell(\mathbf{G})$ and $\ell(\mathbf{G}^-)$ the underlying lattice of $\mathbf{G}$ or of $\mathbf{G}^-$, respectively.

A filter C of $\ell(G^-)$ will be called regular if for each $x \in G^-$, the set $\{c \in C\colon\ x \leqslant c\}$ has a minimal element; in such case, this minimal element will be denoted by $\gamma_C(x)$. Clearly,

$$\gamma_C(G^-) = C.$$

Lemma 4.1. *Let C be a regular filter of the lattice $\ell(\mathbf{G}^-)$. Then γ_C is a nucleus on G^- (with respect to the GMV-algebra $(\mathbf{G}^-)^*$).*

P r o o f. It is obvious that γ_C is a closure operator on the lattice $\ell(\mathbf{G}^-)$. Let $a, b \in G^-$. In view of the definition of the operation $/_{L^-}$ we have

$$\gamma(a)/_{L^-}b = (\gamma(a)b^{-1}) \wedge e.$$

From $b \in G^-$ we obtain $b^{-1} \geqslant e$, thus $\gamma(a)b^{-1} \geqslant \gamma(a)$, whence

$$\gamma(a) \leqslant (\gamma(a)b^{-1}) \wedge e \leqslant e$$

and thus $\gamma(a)/_{L^-}b \in \gamma_C(G^-)$. Analogously, $b \setminus_{L^-} \gamma(a) \in \gamma_C(G^-)$. Hence in view of [5], Lemma 1.3, γ_C is a nucleus. $\square$

Let C be as in Lemma 4.1. Denote

$$\gamma_C = \gamma, \quad \mathbf{L} = (\mathbf{G}^-)^*, \quad P = \ell(\mathbf{L}_\gamma), \quad a = e. \tag{5}$$

Lemma 4.2. *Assume that* (1) *is valid. Let $x, y \in P$ and $i \in I$. Then*

$$x \circ_\gamma y = (x_i \circ_\gamma y_i) \circ_\gamma (x_i' \circ_\gamma y_i').$$

P r o o f. We use the relation (4). In view of Lemma 3.2,

$$x = x_i \wedge x_i', \quad y = y_i \wedge y_i'.$$

Also, $x_i \vee x_i' = e$. From this and from Lemma 2.10 in [5] we obtain $x = x_i \circ_\gamma x_i'$. Similarly, $y = y_i \circ_\gamma y_i'$. Thus

$$x \circ_\gamma y = (x_i \circ_\gamma x_i') \circ_\gamma (y_i \circ_\gamma y_i') = x_i \circ_\gamma (x_i' \circ_\gamma y_i) \circ_\gamma y_i'.$$

Using Lemma 3.2 again we get

$$x_i' \circ_\gamma y_i = x_i' \wedge y_i = y_i \wedge x_i' = y_i \circ_\gamma x_i',$$

whence

$$x \circ_\gamma y = (x_i \circ_\gamma y_i) \circ_\gamma (x_i' \circ_\gamma y_i').$$

□

Lemma 4.3. *Assume that* (1) *is valid. Let $i \in I$ and $x, y \in P_i$. Then $x \circ_\gamma y \in P_i$.*

P r o o f. Put $x \circ_\gamma y = z$. In view of (1),

$$z = z_i \wedge z_i' = z_i \circ_\gamma z_i'.$$

From the relations $x \in P_i$, $z_i' \in P_i'$ we get

$$x \vee z_i' = e.$$

Hence

$$(x \vee z_i) \circ_\gamma y = (x \circ_\gamma y) \vee (z_i' \circ_\gamma y) = e \circ_\gamma y = y.$$

Further, $z_i' \circ_\gamma y = z_i' \wedge y$, thus

$$z \vee (z_i' \wedge y) = y.$$

By the distributivity of ℓ-groups, we have

$$z \vee (z_i' \wedge y) = (z \vee z_i') \wedge (z \vee y) = z_i' \wedge y.$$

Therefore $z_i' \geqslant y$. Since $z_i' \vee y = e$ we get $z_i' = e$. This yields $z = z_i$, whence $z \in P_i$. □

Lemma 4.4. *Let* (1) *be valid. We use the notation as in* (5). *Let* $a \in P$, $i \in I$, $a_1 \in P_i$, $a_2 \in P_i'$ *and* $a = a_1 \wedge a_2$. *Then* $a_1 = a_i$ *and* $a_2 = a_i'$.

Proof. We have

$$a_1 = a_1 \vee a = a_1 \vee (a_i \wedge a_i') = (a_1 \vee a_i) \wedge (a_1 \vee a_i').$$

Since $a_1 \vee a_i' = e$ we get $a_1 = a_1 \vee a_i$, whence $a_1 \geqslant a_i$. By an analogous argument we obtain $a_i \geqslant a_1$, thus $a_i = a_1$. Similarly, $a_2 = a_1'$. □

Lemma 4.5. *Let* (1) *and* (5) *be valid. Let* $i \in I$ *and* $x, y \in P$. *Then*

$$(x \circ_\gamma y)_i = x_i \circ_\gamma y_i, \quad (x \circ_\gamma y)_i' = x_i' \circ_\gamma y_i'.$$

Proof. In view of Lemma 4.3 we have $x_i \circ_\gamma y_i \in P_i$. Analogously, $x_i' \circ_\gamma y_i' \in P_i'$. Now it suffices to apply Lemma 4.2 and Lemma 4.4. □

Again, let us suppose that (1) and (5) are valid.

Let $y, z \in P$. Put $z/y = t$. In view of the definition of residuated lattice we have

$$x \circ_\gamma y \leqslant z \Leftrightarrow x \leqslant t.$$

This yields

$$t = \max P(y, z),$$

where

$$P(y, z) = \{x \in P \colon\ x \circ_\gamma y \leqslant z\}.$$

According to Lemma 4.5 the relation $x \circ_\gamma y \leqslant z$ is equivalent with the validity of the two following conditions:

$$x_i \circ_\gamma y_i \leqslant z_i \tag{6a}$$

$$x_i' \circ_\gamma y_i' \leqslant z_i'. \tag{6b}$$

Lemma 4.6. *Let* (1) *and* (5) *be valid. Let* $i \in I$ *and* $y, z \in P_i$. *Then* $z/y \in P_i$.

Proof. Let us apply the notation as above. From $y, z \in P_i$ we obtain $y_i = y$ and $z_i = z$; thus in view of (6a)

$$x_i \circ_\gamma y \leqslant z$$

for each $x \in P(y, z)$. Since $t \in P(y, z)$, we have $t_i \circ_\gamma y \leqslant z$, hence $t_i \in P(y, z)$. Clearly $t_i \geqslant t$. Therefore we must have $t_i = t$. This yields $t \in P_i$. □

Analogously, we have (by applying (6b))

Lemma 4.6.1. *Let* (1) *and* (5) *be valid. Let* $i \in I$ *and* $y, z \in P'_i$. *Then* $z/y = P'_i$.

Lemma 4.7. *Let* (1) *and* (5) *be valid. Let* $i \in I$ *and* $y, z \in P$. *Then* $(z/y)_i = z_i/y_i$.

Proof. As above, let $z/y = t$. Further, put $z_i/y_i = q$. In view of Lemma 4.6, $q \in P_i$.

Since $t \in P(y, z)$, (6b) yields

$$t_i \circ_\gamma y_i \leqslant z_i,$$

hence $t_i \in P(y_i, z_i)$. Since $q = \max P(y_i, z_i)$, we obtain $q \geqslant t_i$.

Denote $q \wedge t'_i = q_1$. According to Lemma 4.4,

$$(q_1)_i = q, \quad (q_1)'_i = t'_i.$$

From this and from (6a), (6b) we conclude that q_1 belongs to $P(y, z)$. Therefore $q_1 \leqslant t$. Hence $(q_1)_i \leqslant t_i$. Thus $q = t_i$. This completes the proof. $\square$

Similarly, we have

Lemma 4.7.1. *Under the assumptions as in Lemma* 4.7, $(z/y)'_i = z'_i/y'_i$.

The results analogous to Lemma 4.7 and Lemma 4.7.1 are valid for the operation $y \setminus z$.

Summarizing, from the previous lemmas of the present section we obtain

Proposition 4.8. *Let* $\mathbf{G}$ *be a lattice ordered group and let us use the notation as in* (5). *Suppose that*

$$\varphi\colon\ P \to P_i \times P'_i$$

is an internal direct product decomposition of the lattice P *with respect to the element* e.

(i) *Both* P_i *and* P'_i *are closed with respect to the operations* $\wedge, \vee_\gamma, \circ_\gamma, \setminus$ *and* $/$; *also,* $e \in P_i \cap P'_i$. *Thus the algebras* $\mathbf{P}_i = (P_i, \vee, \wedge_\gamma, \circ_\gamma, \setminus, /, e)$ *and* $\mathbf{P}'_i = (P'_i, \wedge, \vee_\gamma, \circ_\gamma, \setminus, /, e)$ *are subalgebras of the* GMV*-algebra* $\mathbf{L}_\gamma$.

(ii) *The mapping* φ *determines a direct sum decomposition*

$$\varphi\colon\ \mathbf{L}_\gamma = \mathbf{P}_i \oplus \mathbf{P}'_i.$$

It is obvious that if $\mathbf{L}_\gamma$ is represented as a direct sum

$$\mathbf{L}_\gamma = \mathbf{X} \oplus \mathbf{Y}$$

and if for $z \in L_\gamma$ we have $z = x \cdot y$ with $x \in X$ and $y \in Y$, then the mapping $\varphi(z) = (x, y)$ determines an internal direct product of the corresponding lattices

$$\varphi\colon \ \ell(\mathbf{L}_\gamma) \to \ell(\mathbf{X}) \times \ell(\mathbf{Y}).$$

From this and from Proposition 2.10 we obtain the following.

Corollary 4.9. *Let us use the notation as in Proposition* 4.8. *Let $F(P)$ be the set of all internal direct factors of P taken with respect to the element e. Further, let $S(\mathbf{L}_\gamma)$ be the set of all direct summands of $\mathbf{L}_\gamma$. For each $\mathbf{A} \in S(\mathbf{L}_\gamma)$ put $\psi(\mathbf{A}) = \ell(\mathbf{A})$. Then ψ is a one-to-one mapping of the set $S(\mathbf{L}_\gamma)$ onto the set $F(P)$.*

From the mentioned relations between elements of $F(P)$ and $S(\mathbf{L}_\gamma)$ and from the well-known properties of internal direct factors of partially ordered sets we immediately obtain the following facts:

4.10.1. Let $\mathbf{P}_i \in S(\mathbf{L}_\gamma)$ and $x \in L_\gamma$. Then the component x_L of x in $\mathbf{P}_i$ is uniquely determined; namely

$$x_i = \min\{t \in P_i\colon \ t \geqslant x\}.$$

4.10.2. Let $\mathbf{P}_i \in S(\mathbf{L}_\gamma)$. Then the corresponding $\mathbf{P}'_i$ (under the notation as above) is uniquely determined; namely,

$$P'_i = \{t \in L_\gamma\colon \ t \vee p = e \quad \text{for each } p \in P_i\}.$$

4.10.3. The system $S(\mathbf{L}_\gamma)$ partially ordered by the set-theoretical inclusion is a Boolean algebra. If $\mathbf{X}, \mathbf{Y} \in S(\mathbf{L}_\gamma)$, then the underlying set of $\mathbf{X} \wedge \mathbf{Y}$ is $X \cap Y$. Under the notation as above, $\mathbf{P}'_i$ is the complement of $\mathbf{P}_i$ in the Boolean algebra $S(\mathbf{L}_\gamma)$.

5. On the GMV-algebra $\mathbf{G}^*$

Assume that $\mathbf{M}$ is a GMV-algebra and that

$$\mathbf{M} = \mathbf{G}^* \oplus \mathbf{L}_\gamma,$$

where $\mathbf{G}^*$ and $\mathbf{L}_\gamma$ are as in Theorem 2.1.

In this section we investigate the GMV-algebra $\mathbf{G}^*$. Put $\ell(\mathbf{G}^*) = Q$. We have $\ell(\mathbf{G}^*) = \ell(\mathbf{G})$.

Suppose that

$$\psi\colon\ Q\to\prod_{j\in J}Q_j$$

is an internal direct product decomposition of the lattice Q with respect to the element e.

Let $j\in J, x\in Q$. We denote by x_j or $x(Q_j)$ the component of x in Q_j. Further, let Q'_j be defined analogously as P'_i in Section 3. Then we have an internal direct product decomposition

$$\psi^j:\ Q\to Q_j\times Q'_j$$

where $\psi^j(x)=(x_j,x'_j)$ for each $x\in Q$. Then in view of Proposition 4.8 (applied for ψ^j) we conclude that Q_j is the underlying sublattice of an ℓ-subgroup $\mathbf{Q}_j$ of $\mathbf{G}$ (the meaning of $\mathbf{Q}'_j$ is analogous) and that ψ^j yields also a direct product decomposition of the lattice ordered group $\mathbf{G}$, i.e.,

$$\psi^j:\ \mathbf{G}\to\mathbf{Q}_j\times\mathbf{Q}'_j \tag{7}$$

is a direct product decomposition of the lattice ordered group $\mathbf{G}$.

Let us consider the GMV-algebras $\mathbf{G}^*$, $\mathbf{Q}^*_j$ and $(\mathbf{Q}'_j)^*$. For $x,y\in G$ we have

$$x/y=xy^{-1},\quad y\setminus x=y^{-1}x.$$

Thus in view of (7) we obtain that Q_j and Q'_j are closed with respect to the operations $/$ and $\setminus$; therefore

$$(x/y)_j=x_jy_j^{-1}=x_j/y_j^{-1},$$
$$(x/y)'_j=x'_j(y^{-1})'_j=x'_j/(y^{-1})'_j$$

and analogously for the operation $\setminus$. Hence

$$\mathbf{G}^*=\mathbf{Q}^*_j\oplus\mathbf{Q}'_j.$$

We verified that if Q_j is an internal direct factor of the lattice Q with respect to the element e, then $\mathbf{Q}^*_j$ is a direct summand of the GMV-algebra $\mathbf{G}^*$. Clearly, $\ell(\mathbf{Q}^*_j)=Q_j$.

Conversely, it is obvious that if $\mathbf{X}$ is a direct summand of the GMV-algebra $\mathbf{G}^*$, then the lattice $\ell(\mathbf{X})$ is a direct summand of the lattice $\ell(\mathbf{G}^*)$ with respect to the element e.

We denote by $F(Q)$ the system of all internal direct factors of the lattice Q taken with respect to the element e. Further, let $S(\mathbf{G}^*)$ be the system of all direct summands of the GMV-algebra G^*. In view of the above argument we have proved

Lemma 5.1. *For each* $\mathbf{X} \in S(\mathbf{G}^*)$ *let* $\chi(\mathbf{X}) = \ell(\mathbf{X})$. *Then* χ *is a one-to-one mapping of the set* $S(\mathbf{G}^*)$ *onto the set* $F(\ell(\mathbf{G}^*))$.

Now, let us assume that $\mathbf{M}$ is any GMV-algebra. Further, suppose that I is a nonempty set and that for each $i \in I$, $\mathbf{M}_i$ is a direct summand of M. If $x \in M$, then, as above, $x(\mathbf{M}_i)$ will denote the component of x in $\mathbf{M}_i$. Consider the mapping

$$\alpha\colon\ M \to \prod_{i\in I} M_i$$

defined by $\alpha(x) = (x(\mathbf{M}_i))_{i\in I}$ for each $x \in M$. If α is bijective then we say that $\mathbf{M}$ is a *complete direct sum* of the system $(\mathbf{M}_i)_{i\in I}$ and we express this fact by writing

$$\mathbf{M} = \sum_{i\in I}^{*} \mathbf{M}_i.$$

Proposition 5.2. *Let* $\mathbf{G}$ *be a lattice ordered group. Assume that*

$$\varphi\colon\ \ell(\mathbf{G}) \to \prod_{i\in I} P_i$$

is an internal direct product decomposition of the lattice $\ell(\mathbf{G})$. *Let* χ *be as in Lemma* 5.1*; for each* $i \in I$ *put* $\mathbf{T}_i = \chi^{-1}(P_i)$. *Then*

$$\mathbf{G}^* = \sum_{i\in I}^{*} \mathbf{T}_i.$$

Proof. Since $\ell(\mathbf{G}) = \ell(\mathbf{G}^*)$, the assertion follows from Lemma 5.1. □

Analogously, from Corollary 4.9 we obtain

Proposition 5.3. *Let* $\mathbf{L}_\gamma$ *be as in Corollary* 4.9. *Assume that*

$$\varphi\colon\ \ell(\mathbf{L}_\gamma) \to \prod_{i\in I} P_i$$

is an internal direct product decomposition of the lattice $\ell(\mathbf{L}_\gamma)$ *with respect to the element* e. *Put* $\psi^{-1}(P_i) = \mathbf{Q}_i$ *for each* $i \in I$. *Then*

$$\mathbf{L}_\gamma = \sum_{i\in I}^{*} \mathbf{Q}_i.$$

6. Direct summands of $\mathbf{M}$

Again, let $\mathbf{M}$ be a GMV-algebra and let $S(\mathbf{M})$ be the system of all direct summands of $\mathbf{M}$. Further, we denote by $F(\ell(\mathbf{M}))$ the system of all internal direct factors of the lattice $\ell(\mathbf{M})$ with respect to the element e.

If $\mathbf{X} \in S(\mathbf{M})$, then, obviously, the lattice $\ell(\mathbf{X})$ belongs to $F(\ell(\mathbf{M}))$.

Conversely, assume that X is an element of $F(\ell(\mathbf{M}))$. Then there exists $Y \in F(\ell(\mathbf{M}))$ and an internal direct product decomposition

$$\varphi_0\colon\ \ell(\mathbf{M}) \to X \times Y.$$

At the same time, in view of Theorem 2.1, we have an internal direct product decomposition with respect to the element e

$$\varphi_1\colon\ \ell(M) \to \ell(\mathbf{G}^*) \times \ell(\mathbf{L}_\gamma).$$

It is well-known that any two internal direct product decompositions of a lattice (taken with respect to the same element) have a common refinement; hence from φ_0 and φ_1 we can construct a new internal direct product decomposition

$$\varphi_2\colon\ \ell(\mathbf{M}) \to (X \cap \ell(\mathbf{G}^*)) \times (X \cap \ell(\mathbf{L}_\gamma)) \times (Y \cap \ell(\mathbf{G}^*) \times (Y \cap \ell(\mathbf{L}_\gamma)).$$

At the same time, we have internal direct product decompositions with respect to the element e

$$\begin{aligned}
\varphi_{21}\colon\ \ell(\mathbf{G}^*) &\to (X \cap \ell(\mathbf{G}^*) \times (Y \cap \ell(\mathbf{G}^*)),\\
\varphi_{22}\colon\ \ell(\mathbf{L}_\gamma) &\to (X \cap \ell(\mathbf{L}_\gamma) \times (Y \cap \ell(\mathbf{L}_\gamma)),\\
\varphi_{23}\colon\ X &\to (X \cap \ell(\mathbf{G}^*) \times (X \cap \ell(\mathbf{L}_\gamma)),\\
\varphi_{24}\colon\ Y &\to (X \cap \ell(\mathbf{G}^*) \times (Y \cap \ell(\mathbf{L}_\gamma)).
\end{aligned}$$

In view of φ_{21} and of Proposition 5.2 we conclude that there are GMV-algebras $\mathbf{M}_1$ and $\mathbf{M}_2$ such that

$$\ell(\mathbf{M}_1) = X \cap \ell(\mathbf{G}^*), \quad \ell(\mathbf{M}_2) = Y \cap \ell(G^*)$$

and

$$\mathbf{G}^* = \mathbf{M}_1 \oplus \mathbf{M}_2.$$

Analogously, according to the relation φ_{22} and Proposition 5.3, there are GMV-algebras $\mathbf{M}_3$ and $\mathbf{M}_4$ such that

$$\ell(\mathbf{M}_3) = X \cap \ell(\mathbf{L}_\gamma), \quad (\mathbf{M}_4) = Y \cap \ell(\mathbf{L}_\gamma)$$

and

$$\mathbf{L}_\gamma = \mathbf{M}_3 \oplus \mathbf{M}_4.$$

Therefore, in view of Theorem 2.1, we have

$$\mathbf{M} = (M_1 \oplus \mathbf{M}_2) \oplus (\mathbf{M}_3 \oplus \mathbf{M}_4).$$

It is obvious that the operation $\oplus$ is associative and commutative; hence

$$\mathbf{M} = (\mathbf{M}_1 \oplus \mathbf{M}_3) \oplus (\mathbf{M}_2 \oplus \mathbf{M}_4).$$

Thus $\mathbf{M}_1 \oplus \mathbf{M}_3$ is a direct summand of $\mathbf{M}$. Further, in view of φ_{23} we conclude that

$$\ell(M_1 \oplus M_3) = X.$$

Summarizing, we have

Theorem 6.1. *Let* $\mathbf{M}$ *be a GMV-algebra. For each* $\mathbf{M}_1 \in S(\mathbf{M})$ *put* $\varphi(\mathbf{M}_1) = \ell(\mathbf{M}_1)$. *Then* φ *is a bijection of* $S(\mathbf{M})$ *onto* $\mathbf{F}(\ell(\mathbf{M}))$.

Theorem 6.2. *Let* $\mathbf{M}$ *be a GMV-algebra. Assume that*

$$\varphi_1\colon\ \ell(\mathbf{M}) \to \prod_{i\in I} P_i$$

is an internal direct product decomposition of the lattice $\ell(\mathbf{M})$ *with respect to the element* e. *Put* $\varphi^{-1}(P_i) = \mathbf{Q}_i$ *for each* $i \in I$, *where* φ *is as in Theorem* 6.1. *Then*

$$\mathbf{M} = \sum_{i\in I}^{*} \mathbf{Q}_i. \tag{8}$$

P r o o f. In view of Theorem 6.1, each $\mathbf{Q}_i$ is a direct summand of $\mathbf{M}$. Moreover, for $x \in M$, the component of x in $\mathbf{Q}_i$ coincides with the component of x in P_i. Hence the mapping $x \mapsto x(P_i)$ is a homomorphism of $\mathbf{M}$ into $\mathbf{Q}_i$. From this and from the direct product decomposition φ_1 we infer that (8) holds. □

Since any two internal direct product decompositions of a lattice have a common refinement, we obtain

Corollary 6.3. *Any two complete direct sum decompositions of a GMV-algebra have a common refinement. Namely, if* (8) *is valid, and, at the same time,*

$$\mathbf{M} = \sum_{j\in J}^{*} \mathbf{T}_{\gamma}, \tag{9}$$

then

$$\mathbf{M} = \sum_{i\in I, j\in J}^{*} \mathbf{V}_{i,j},$$

where $V_{i,j} = Q_i \cap T_j$ for each $i \in I$ and $j \in J$, and $\mathbf{V}_{ij}$ is a subalgebra of $\mathbf{Q}_i$ and of $\mathbf{T}_j$.

A GMV-algebra is *directly irreducible* if, whenever $\mathbf{M} = \mathbf{M}_1 \oplus \mathbf{M}_2$, then either M_1 or M_2 is a one-element set. In the opposite case, $\mathbf{M}$ is *directly irreducible.*

For monoids, we define the notation of direct sum, direct summand, direct irreducibility and direct reducibility in the same way as for GMV-algebras.

Let $\mathbf{M} = (M; \wedge, \vee, \cdot, \backslash, /, e)$ be an GMV-algebra; we consider the monoid $\operatorname{mon}\mathbf{M} = (M; \cdot, e)$.

If $\mathbf{M} = \mathbf{M}_1 \oplus \mathbf{M}_2$, then we obviously have

$$\operatorname{mon}\mathbf{M} = \operatorname{mon}\mathbf{M}_1 \oplus \operatorname{mon}\mathbf{M}_2.$$

The natural question arises whether the situation here is analogous to the situation when we cosider direct summands of $\mathbf{M}$ and internal direct factors of $\ell(\mathbf{M})$; i.e., we ask whether there exists a one-to-one correspondence between direct summands of $\mathbf{M}$ and direct summands of $\operatorname{mon}\mathbf{M}$.

The answer is ‘No’. Moreover, it can happen that $\mathbf{M}$ is directly irreducible and $\operatorname{mon}\mathbf{M}$ is directly reducible.

Example. Let R be the additive group of all reals with the natural linear order and $\mathbf{G} = R\circ R$, where $\circ$ denotes the operation of lexicographic product. Put $\mathbf{M} = \mathbf{G}^*$. Since $\ell(G^*)$ is a chain, it is directly irreducible and hence $\mathbf{M}$ is directly irreducible as well. On the other hand, the monoid $\operatorname{mon}\mathbf{M}$ is directly reducible.

7. Retract mappings of GMV-algebras

A *retract mapping* of an algebra $\mathcal{A}$ is an endomorhpism of $\mathcal{A}$ such that $f^2 = f$.

Let $\mathbf{M}$ be a GMV-algebra; we apply the notation as in Theorem 2.1. Thus $\mathbf{M} = \mathbf{G}^* \oplus \mathbf{L}_\gamma$.

Let $z \in M$. As above, we denote by $z(\mathbf{G}^*)$ the component of z in the direct summand $\mathbf{G}^*$. The meaning of $z(\mathbf{L}_\gamma)$ is analogous. If $x = z(\mathbf{G}^*)$ and $y = z(\mathbf{L}_\gamma)$, then $z = xy$. Conversely, if $z = x_1y_1$ and $x_1 \in G$, $y_1 \in L_\gamma$, then $x_1 = z(\mathbf{G}^*)$ and $y_1 = z(\mathbf{L}_\gamma)$.

In the present section we prove that each retract mapping f of $\mathbf{M}$ is determined by a pair $(f_1, f_2,)$ of mappings such that

(i_0) f_1 is a retract mapping of $\mathbf{G}^*$;

(ii_0) f_2 is a retract mapping of $\mathbf{L}_\gamma$.

We denote by $\mathcal{R}(\mathbf{M})$ the set of all retract mappings of $\mathbf{M}$. Further, let $\mathcal{T}(\mathbf{M})$ be the system of all pairs of mappings $(f_1, f_2,)$ such that the conditions (i_0) and (ii_0) are valid.

Our aim is to construct a bijection

$$\psi\colon\ \mathcal{R}(\mathbf{M}) \to \mathcal{T}(\mathbf{M}).$$

Lemma 7.1. *Let $z \in M$. The following conditions are equivalent:*

(i) $z \in G$;

(ii) *there exists $z_1 \in M$ such that $zz_1 = e$;*

(iii) *there exists $z_2 \in M$ such that $z_2z = e$.*

Proof. In view of Theorem 2.1, $(G; \cdot, e)$ is a group with neutral element e. Thus (i)⇒(ii) and (i)⇒(iii).

Assume that (ii) holds. We express z in the form $z = xy$, where $x = z(\mathbf{G}^*)$ and $y = z(\mathbf{L}_\gamma)$. Under analogous notation, let $z_1 = x_1y_1$. By way of contradiction, suppose that z does not belong to G. Hence $y \neq e$. Thus $e > y$ and $y \geqslant yy_1$. We obtain $yy_1 \neq e$, whence zz_1 does not belong to G, which is a contradiction. Therefore (ii)⇒(i). Analogously, (iii)⇒(i). □

Corollary 7.2. *Let $z, z_1 \in M$, $zz_1 = e$. Then both z and z_1 belong to G.*

Lemma 7.3. *Let f be a retract mapping of* $\mathbf{M}$. *Let $z \in G$. Then $f(z) \in G$.*

Proof. In view of Lemma 7.1, there exists $z_1 \in M$ with $zz_1 = e$. We have $f(e) = e$ and $f(z_1) = f(z)f(z_1)$, hence $f(z)f(z_1) = e$. Then Corollary 7.2 yields that $f(z)$ belongs to G. □

Under the notation as in Lemma 7.3, we put $f|G = f_1$. In view of Lemma 7.3, we get

Lemma 7.4. *f_1 is a retract mapping of $\mathbf{G}^*$.*

Let $y \in L_\gamma$. We denote

$$x = f(y)(\mathbf{G}^*), \quad y_1 = f(y)(\mathbf{L}_\gamma).$$

Further, we put

$$f_2(y) = y_1, \quad f_3(y) = x.$$

We obtain mappings

$$f_2\colon\ L_\gamma \to L_\gamma, \quad f_3\colon\ L_\gamma \to G.$$

Lemma 7.5. *f_2 is a retract mapping of $\mathbf{L}_\gamma$.*

Proof. It is obvious that f_2 is an endomorphism of $\mathbf{L}_\gamma$. It remains to verify that $f_2(f_2(y)) = f_2(y)$ for each $y \in L_\gamma$.

Under the notation as above, we have $f_2(y) = y_1$ and $f(y) = xy_1$. Denote

$$x_1 = f(y_1)(\mathbf{G}^*), \quad y_2 = f(y_1)(\mathbf{L}_\gamma).$$

Thus, in view of the definition of f_2, we get $f_2(y_1) = y_2$. Further, we obtain

$$f(f(y)) = f(y) = xy_1,$$
$$f(f(y)) = f(xy_1) = f(x)f(y_1) = f(x)x_1y_2.$$

Since $x \in G$, in view of 7.3 we have $f(x)x_1 \in G$. Thus from

$$xy_1 = f(x)x_1y_2$$

we obtain $x = f(x)x_1$ and $y_1 = y_2$. We have verified that $f_2(y_1) = y_1$. □

Under the above notation we put

$$\psi(f) = (f_1, f_2).$$

From 7.4 and 7.5 we obtain

Theorem (A). *ψ is a mapping of the set $\mathcal{R}(\mathbf{M})$ into the set $\mathcal{T}(\mathbf{M})$.*

Lemma 7.6. *$f_3 = e$ for each $y \in L_\gamma$.*

P r o o f. Let $y \in L_\gamma$. If we view $\mathbf{M}$ as a direct product, we have

$$\begin{aligned} e = e_{\mathbf{M}} = f(e_{\mathbf{M}}) = f(e_{\mathbf{M}}/y) = f(e_{\mathbf{M}})/f(y) \\ = (e_{\mathbf{G}}, e_{\mathbf{L}_\gamma}/(x, y_1) = (e_{\mathbf{G}}/x, e_{\mathbf{L}_\gamma}) = (x^{-1}, e_{\mathbf{L}_\gamma}), \end{aligned}$$

so $x^{-1} = e_{\mathbf{G}}$ and $x = e_{\mathbf{G}}$. Therefore $f_3(y) = e_{\mathbf{G}}$. □

Corollary 7.7. *$f(y) = f_2(y)$ for each $y \in L_\gamma$.*

In view of Corollary 7.7 we conclude that the mapping ψ is a monomorphism.

Now let us suppose that f_1 and f_2 are as in conditions (i_0) and (ii_0). Further, let $z \in M$, $z = xy$, where $x \in G$ and $y \in L_\gamma$. We put

$$f_0(z) = f_1(x) f_2(y).$$

Then in view of Theorem 2.1 we obtain

Lemma 7.8. *f_0 is a retract mapping of $\mathbf{M}$.*

For a pair (f_1, f_2) belonging to the system $\mathcal{T}(\mathbf{M})$ we put

$$\chi((f_1, f_2)) = f_0,$$

where f_0 is as above.

According to Lemma 7.8 we get

Theorem (B). *χ is a mapping of the system $\mathcal{T}(\mathbf{M})$ into the set $\mathcal{R}(\mathbf{M})$.*

We have already noticed above that the mapping ψ is a monomorphism. Now from the definitions of ψ and χ we immediately obtain that $\chi = \psi^{-1}$. Hence we have

Theorem (C). *Let $\mathbf{M}$ be a GMV-algebra. The mapping ψ is a bijection of the set $\mathcal{R}(\mathbf{M})$ onto the system $\mathcal{T}(\mathbf{M})$.*

The author is indebted to the referee for valuable suggetions. The proof of Lemma 7.6 is due to the referee.

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