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ON A THEOREM OF CANTOR-BERNSTEIN TYPE FOR ALGEBRAS

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Abstract. Freytes proved a theorem of Cantor-Bernstein type for algbras; he applied certain sequences of central elements of bounded lattices. The aim of the present paper is to extend the mentioned result to the case when the lattices under consideration need not be bounded; instead of sequences of central elements we deal with sequences of internal direct factors of lattices.

Keywords: lattice, $\mathcal{L}^*$-variety, center, internal direct factor

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1. Introduction

In a forthcoming paper [5], Freytes defines the notion of the $\mathcal{L}$-variety of algebras. He proved a theorem of Cantor-Bernstein type for an algebra belonging to an $\mathcal{L}$-variety. The idea and the method are based on those used by investigating the validity of Cantor-Bernstein theorem for MV-algebras; cf. De Simone, Mundici and Navara [2].

If V is an $\mathcal{L}$-variety, then to each $A \in V$ there corresponds a bounded lattice $L(A)$ (for detailed definitions, cf. Section 2 below). The core of the proofs in [5] essentially applies the properties of bounded lattices.

The class of all MV-algebras (cf. [1]) and, more generally, the class of all pseudo MV-algebras (cf. Georgescu and Iorgulescu [6], [7], and Rachůnek [22]; in [22] the term 'generalized MV-algebra' was applied) are examples of $\mathcal{L}$-varieties.

In the present paper we introduce the notion of the $\mathcal{L}^*$-variety of algebras. If V is an $\mathcal{L}^*$-variety, then to each $A \in V$ there corresponds a lattice $L(A)$ which need not

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be bounded. Each $\mathcal{L}$-variety is an $\mathcal{L}^*$-variety, but not conversely. The class LG of all lattice ordered groups is an example of an $\mathcal{L}^*$-variety; LG fails to be an $\mathcal{L}$-variety.

We extend the result of [5] to algebras belonging to an $\mathcal{L}^*$-variety. Our method is analogous to that of [5] with the distinction that instead of dealing with elements belonging to the center of a bounded lattice we deal with internal direct factors of a lattice which need not be bounded. We remark that if L is a bounded lattice then there is a one-to-one correspondence between central elements of L and internal direct factors of L.

Theorems of Cantor-Bernstein type (called also theorems of Cantor-Bernstein-Schröder type) were proved for Boolean algebras (Sikorski [23], Tarski [24]), lattice ordered groups (the author [10], [12], [13]), MV-algebras and pseudo MV-algebras (De Simone, Mundici and Navara [2], the author [14], [16], [19]), effect algebras and pseudo-effect algebras (Dvurečenskij [4], Jenča [20]), orthomodular lattices (de Simone, Navara and Pták [3]) and lattices (the author [15], [18]).

We remark that the results of [2], [7]–[11], [14], [15], [18] and [19] generalize the theorem proved by Sikorski and Tarski.

2. $\mathcal{L}$-varieties and $\mathcal{L}^*$-varieties

For an indexed system $(A_i)_{i\in I}$ of algebras belonging to a variety V we denote by $\prod\limits_{i\in I} A_i$ the direct product of this system; if $I = \{1,2,\dots,n\}$, then we write $A_1 \times \ldots \times A_n$. Let $A \in V$ and let

$$\varphi\colon\ A \to \prod_{i\in I} A_i \tag{1}$$

be an isomorphism. Then A_i are called direct factors of A.

Suppose that there is an element v_0 of A such that $\{v_0\}$ is a subalgebra of A. Let v_0 be fixed. For $a \in A$ and $i \in I$ let a_i be the component of $\varphi(a)$ in A_i.

Applying v_0, we can define the notion of *internal direct decomposition* and *internal direct factor* of A similarly to the case of groups (cf., e.g., Kurosh [21], p. 106).

Namely, we assume that (1) is valid and that

1) all A_i are subalgebras of A with $v_0 \in A_i$,
2) if $i \in I$ and $a \in A_i$, then $a_i = a$ and $a_j = v_0$ for each $j \in I$, $j \neq i$.

Under these assumptions, (1) is defined to be an internal direct decomposition of A, and A_i are internal direct factors of A.

To each direct decomposition (1) of A there corresponds an internal direct decomposition determined by an isomorphism φ_0 and by direct factors A_i^0 which are defined as follows:

Let $i \in I$. We denote by A_i^0 the set of all $a \in A$ such that $a_j = v_0$ for each $j \in I$, $j \neq i$.

Further, for $a \in A$ and $i \in I$ let a_i^0 be the element of A_i^0 such that $(a_i^0)_i = a_i$. Put

$$\varphi_0(a) = (a_i^0)_{i \in I}.$$

Then

$$\varphi_0\colon\ A \to \prod_{i \in I} A_i^0 \tag{2}$$

is an internal direct decomposition of A with internal direct factors A_i^0. For each $i \in I$ we have $A_i \simeq A_i^0$.

For lattices and lattice ordered groups we apply the standard terminology and notation; the group operation in a lattice ordered group will be written additively.

A lattice L is called bounded if it has the least element 0_L and the greatest element 1_L. When no misunderstanding can occur then we write 0 and 1 instead of 0_L and 1_L. The system of all elements z of a bounded lattice L such that z is neutral and has a complement is denoted by $Z(L)$; the elements of $Z(L)$ are *central* and $Z(L)$ is the *center* of L. Each element $z \in Z(L)$ has a unique complement which will be denoted by $\neg z$. The system $Z(L)$ is a sublattice of L and with respect to the induced partial order, $Z(L)$ is a Boolean algebra.

If $Z_0 = \{z_i\}_{i \in I}$ is a nonempty subset of $Z(L)$ then we have to distinguish between the supremum of Z_0 in L (denoted by $\bigvee_{i \in I} z_i$) and the supremum of Z_0 in $Z(L)$ (denoted by $\bigsqcup_{i \in I} z_i$); in fact, these suprema need not exist in general.

Definition 2.1 (Cf. [5], Definition 1.2)**.** A variety V of algebras is an $\mathcal{L}$-variety iff

(1) there are terms in the language of V defining on each $A \in V$ operations $\vee, \wedge, 0, 1$ such that $L(A) = (A; \vee, \wedge, 0, 1)$ is a bounded lattice;

(2) for all $A \in V$ and all $z \in Z(L(A))$, the binary relation Θ_z on A defined by $a\Theta_z b$ iff $a \wedge z = b \wedge z$ is a congruence on A such that $A \simeq A/\Theta_z \times A/\Theta_{\neg z}$.

From the definition of the center of a lattice we immediately obtain

Lemma 2.2. *Let L be a bounded lattice and let $z \in L$. Then the following conditions are equivalent:*

(i) *z is a central element of L;*

(ii) *the interval $[0, z]$ of L is an internal direct factor of L with respect to the element $v_0 = 0$.*

Moreover, if (i) holds, then the mapping $a \to (a \wedge z, a \wedge \neg z)$ is an internal direct decomposition of L with respect to the element $v_0 = 0$. Conversely, if $L \to L_1 \times L_2$ is an internal direct decomposition of L with respect to the element 0 and if 1_{L_i} is the component of 1 in L_i $(i = 1, 2)$, then 1_{L_1}, 1_{L_2} are central elements of L and $1_{L_2} = \neg 1_{L_1}$.

As a consequence of 2.2 and of Definition 2.1 we get

Lemma 2.3. *A variety V is an $\mathcal{L}$-variety iff the condition* (1) *from* 2.1 *is valid and*

(2′) *for all $A \in V$, each internal direct decomposition of the lattice $L(A)$ with two internal direct factors with respect to the element $v_0 = 0$ is, at the same time, an internal direct decomposition of A.*

In view of 2.3, if V is an $\mathcal{L}$-variety and $A \in V$, then for each $z \in Z(L(A))$, the interval $[0, z]$ of A is a subalgebra of A; we emphasize this fact by writing $[0, z]_A$ for denoting this subalgebra. In particular, $\{0\}$ is a subalgebra of A.

Corollary 2.3.1. *Let V be an $\mathcal{L}$-variety and $A \in V$. Put $v_0 = 0$. For each $z \in Z(L(A))$ we set $\chi(z) = [0, z]_A$. Then χ is a bijection of $Z(A)$ onto the set of all internal direct factors of A (with respect to the element v_0). For any $z_1, z_2 \in Z(L(A))$ we have*

$$z_1 \leqslant z_1 \Leftrightarrow \chi(z_1) \subseteq \chi(z_2).$$

In view of [5] we obtain

Proposition 2.4. *The variety PMV of all pseudo MV-algebras is an $\mathcal{L}$-variety.*

Internal direct product decompositions of pseudo MV-algebras were investigated in [17].

Proposition 2.5. *The variety LG of all lattice ordered groups fails to be an $\mathcal{L}$-variety.*

P r o o f. By way of contradiction, assume that LG is an $\mathcal{L}$-variety. Hence the conditions (1) and (2) from 2.1 are satisfied for $V = LG$. Let us write now 0^* and 1^* (instead of 0 and 1 as used in 2.1) since the symbol 0 is used for the neutral element of a lattice ordered group. Let $G \in V$ with $G \neq \{0\}$. If $0^* = 1^*$ then from the condition (2) of 2.1 and from Proposition 1.4 of [5] we conclude that G is a one-element set, which is a contradiction. Hence $0^* \neq 1^*$. In view of 2.1, there exist terms $f(x_1, \ldots, x_n)$ and $g(y_1, \ldots, y_m)$ in the language of LG such that $f(x_1, \ldots, x_n) = 0^*$ and $g(y_1, \ldots, y_m) = 1^*$. Thus 0^* and 1^* are elements of G and for all $a_1, \dots, a_n, b_1, \dots, b_m \in G$ we have $f(a_1, \dots, a_n) = 0^*$, $g(b_1, \dots, b_m) = 1^*$. Take $a_1 = \dots = a_n = b_1 = \dots = b_m = 0$. Since $\{0\}$ is an ℓ-subgroup of G we obtain $f(a_1, \dots, a_n) = 0 = g(b_1, \dots, b_m)$ whence $0^* = 1^*$; again, we have arrived at a contradiction. $\square$

Definition 2.6. A variety V of algebras is an $\mathcal{L}^*$-variety if the following conditions are satisfied:

(1′) There are terms of the language V defining on each $A \in V$ operations $\vee, \wedge$ and v_0 such that $L(A) = (A; \vee, \wedge)$ is a lattice and v_0 is a constant on A.

(2′) = condition (2) of 2.3.

Under the notation as in 2.6 we say that v_0 is the distinguished element of A.

Proposition 2.7. *The variety LG of all lattice ordered groups is an $\mathcal{L}^*$-variety.*

Proof. It suffices to take $v_0 = 0$ and apply the results of [9]. $\square$

Corollary 2.8. *Let V be a variety of algebras. If V is an $\mathcal{L}$-variety, then it is an $\mathcal{L}^*$-variety. The converse statement does not hold.*

Proof. The first assertion follows from Definitions 2.1 and 2.6 and from Lemma 2.3. The second assertion is a consequence of 2.5 and 2.7. $\square$

We remark that if V is an $\mathcal{L}$-variety and $A \in V$, then we always consider 0 to be the distinguished element of A.

3. Auxiliary results

In this section we deal with the system $D(L)$ of all internal direct factors of a lattice L with respect to a fixed element v_0 of L. All internal direct product decompositions of L under consideration will be taken with respect to v_0. The system $D(L)$ is partially ordered by the set-theoretical inclusion. Then L is the greatest element and $\{v_0\}$ is the least element of $D(L)$.

Assume that

$$\varphi\colon\ L \to \prod_{i \in I} A_i \tag{1}$$

is an internal direct product decomposition of L. For $x \in L$ and $i \in I$, the element $(\varphi(x))_i$ is said to be the component of x in A_i with respect to φ.

Proposition 3.1 (Cf. Theorem (A), [11]). *Assume that* (1) *is valid and that, moreover,*

$$(1') \qquad \psi\colon\ L \to \prod_{i\in I} B_i$$

is an internal direct product decomposition of L such that $A_i = B_i$ for each $i \in I$. Then $(\varphi(x))_i = (\psi(x))_i$ for each $x \in L$ and each $i \in I$.

In other words, if a system $\{A_i\}_{i\in I}$ yielding an internal direct product decomposition of L is given, then the mapping φ is uniquely determined. In view of 3.1, instead of (1) we write

$$(2) \qquad L = (\text{int}) \prod_{i\in I} A_i.$$

If $I = \{1, 2, \ldots, n\}$, then we apply the notation

$$(3) \qquad L = (\text{int}) A_1 \times A_2 \times \ldots \times A_n.$$

If (2) is valid and $x \in L$, $i \in I$ then the component of x in A_i will be denoted by x_i or by $x(A_i)$.

From the definition of the internal direct product decomposition we immediately obtain

Lemma 3.2. *Let* (2) *be valid and $\emptyset \neq I(1) \subseteq I$. We denote*

$$L_1 = \{x \in L\colon\ x_i = v_0 \ \text{ for each } \ i \in I \setminus I(1)\}.$$

Then $L_1 \in D(L)$ and $L_1 = (\text{int}) \prod_{i\in I(1)} A_i$. If, moreover, $I(2) = I \setminus I(1) \neq \emptyset$ and if L_2 is defined analogously to L_1, then $L = (\text{int}) L_1 \times L_2$.

From the result of Hashimoto [8] (cf. also Theorem (B) in [11]) we infer

Proposition 3.3. *Let* (2) *be valid. Further, suppose that the relation*

$$L = (\text{int}) \prod_{j\in J} B_j$$

holds. Then we have

$$L = (\text{int}) \prod_{i\in I,\, j\in J} (A_i \cap B_j),$$

$$A_i = (\text{int}) \prod_{j\in J} (A_i \cap B_j) \quad \text{for each } \ i \in I,$$

$$B_j = (\text{int}) \prod_{i\in I} (A_i \cap B_j) \quad \text{for each } \ j \in J.$$

Lemma 3.4. *Assume that the relations*

$$L = (\mathrm{int})A \times B, \tag{4}$$

$$L = (\mathrm{int})A \times C \tag{5}$$

are valid. Then $B = C$.

P r o o f. In view of 3.3 we have

$$B = (\mathrm{int})(B \cap A) \times (B \cap C).$$

According to (4) we get $B \cap A = \{v_0\}$, whence

$$B = (\mathrm{int})\{v_0\} \times (B \cap C) = B \cap C.$$

Thus $B \subseteq C$. Analogously, by using (5) we obtain $C \subseteq B$. □

Hence if (4) is valid, then A uniquely determines B; we write $B = \neg A$ and $A = \neg B$.

Lemma 3.5 (Cf. [15]). *The system $D(L)$ is a Boolean algebra.*

Let $A, B \in D(L)$. Then we have

$$L = (\mathrm{int})A \times \neg A, \quad L = (\mathrm{int})B \times \neg B.$$

In view of 3.3 we obtain

$$L = (\mathrm{int})(A \cap B) \times (A \cap \neg B) \times (\neg A \cap B) \times (\neg A \cap \neg B). \tag{6}$$

From (6) and from 3.2 we infer that there exist $P, Q \in D(L)$ such that

$$P = A \cap B, \quad Q = (\mathrm{int})(A \cap B) \times (A \cap \neg B) \times (\neg A \cap B).$$

Lemma 3.5.1. *Let A, B, P and Q be as above. Then $P = A \wedge B$ and $Q = A \vee B$.*

P r o o f. From $P \in D(L)$ we immediately obtain $P = A \wedge B$. Further, in view of 3.3 we have

$$A = (\mathrm{int})(A \cap B) \times (A \cap \neg B),$$
$$B = (\mathrm{int})(A \cap B) \times (\neg A \cap B),$$

thus $A \subseteq Q$ and $B \subseteq Q$. Let Y be an element of $D(L)$ such that $A \subseteq Y$ and $B \subseteq Y$. According to (6) and 3.3,

$$Y = (\mathrm{int})(A \cap B \cap Y) \times (A \cap \neg B \cap Y) \times (\neg A \cap B \cap Y) \times (\neg A \cap \neg B \cap Y),$$

whence

$$Y = (\mathrm{int})(A \cap B) \times (A \cap \neg B) \times (\neg A \cap B) \times (\neg A \cap \neg B \cap Y),$$
$$Y = (\mathrm{int})Q \times (\neg A \cap \neg B \cap Y).$$

Thus $Q \subseteq Y$. Therefore $Q = A \vee B$. □

Lemma 3.6. *Let $A \in D(L)$. Then $\neg A$ is the unique complement of A in $D(L)$.*

Proof. We have $L = (\mathrm{int})A \times \neg A$, whence $A \wedge \neg A = A \cap \neg A = \{v_0\}$. Further, in view of 3.5.1,

$$A \vee \neg A = (\mathrm{int})(A \cap \neg A) \times (A \cap \neg\neg A) \times (\neg A \cap \neg A) = (\mathrm{int})\{v_0\} \times A \times \neg A = L.$$

Hence $\neg A$ is a complement of A. The uniqueness is a consequence of 3.5. □

Lemma 3.7. *Let $A \in D(L)$. Then*
(i) *$D(A) = \{X \in D(L)\colon\ X \leqslant A\}$;*
(ii) *if $A_1 \in D(A)$, then the complement of A_1 in $D(A)$ is equal to $\neg A_1 \cap A$.*

Proof. a) Let $X \in D(A)$. Then $X \subseteq A$ and there exists $Y \in D(A)$ with $A = (\mathrm{int})X \times Y$. From the relation $L = (\mathrm{int})A \times \neg A$ and from the definition of the internal direct product decomposition we obtain $L = (\mathrm{int})X \times Y \times \neg A$, whence $X \in D(L)$.

b) Assume that $X \in D(L)$, $X \leqslant A$. Then $L = (\mathrm{int})X \times \neg X$, thus in view of 3.3,

$$A = (\mathrm{int})(A \cap X) \times (A \cap \neg X) = (\mathrm{int})X \times (A \cap \neg X),$$

hence $X \in D(A)$.

c) The assertion (ii) is a consequence of 3.5. □

From 3.7 and 2.3.1 we obtain

Corollary 3.7.1 (Cf. [5], Proposition 3.1)**.** *Let L be a bounded lattice and $z \in Z(L)$. Then*
(i) *$Z([0, z]) = Z(L) \cap [0, z]$;*
(ii) *if $x \in Z([0, z])$, then the complement of x relative to $[0, z]$ is $z \wedge \neg x$, where $\neg x$ is the complement of x in L.*

4. Conditions CBS and CBS^*

If V is an $\mathcal{L}$-variety and $A \in V$, then we write $Z(A)$ rather than $Z(L(A))$.

Dealing with the Cantor-Bernstein theorem for algebras we will apply the following definitions.

Definition 4.1 (Cf. [5]). Let V be an $\mathcal{L}$-variety and $A \in V$. We say that A possesses the CBS-property if, whenever $B \in V$, $a \in Z(A)$, $b \in Z(B)$ and

$$A \simeq [0, b]_B, \quad B \simeq [0, a]_A,$$

then $A \simeq B$.

Similarly to Section 3, when dealing with internal direct product decompositions of an algebra A belonging to an $\mathcal{L}^*$-variety V we always assume that the investigation is taken with respect to the distinguished element v_0 of A.

Definition 4.2. Let V be an $\mathcal{L}^*$-variety and $A \in V$. We say that A possesses the CBS^*-property if, whenever $B \in V$, such that A is isomorphic to a direct factor of B and B is isomorphic to a dirct factor of A, then $A \simeq B$.

In view of Section 2, the term 'direct facotr' can be replaced by the term 'internal direct factor' in 4.2.

Lemma 4.3. *Let V be an $\mathcal{L}$-variety and $A \in V$. Then the properties CBS and CBS^* for A are equivalent.*

Proof. This is a consequence of 2.2. □

Lemma 4.4. *Let V be an $\mathcal{L}^*$-variety and $A \in V$. Then the following conditions are equivalent:*

- (i) *A has the CBS^*-property;*
- (ii) *whenever A_1 and A_2 are internal direct factors of A such that $A_1 \subseteq A_2$ and $A_1 \simeq A$, then $A_2 \simeq A$.*

Proof. Let (i) be valid. Assume that A_1 and A_2 are internal direct factors of A such that $A_1 \subseteq A_2$. Then A_1 is, at the same time, an internal direct factor of A_2. Put $B = A_2$. In view of CBS^*, we have $A \simeq A_2$. Hence (ii) holds.

Conversely, assume that (ii) is valid. Let $B \in V$. Suppose that B_1 is an internal direct factor of B and A_1 is an internal direct factor of A such that there exist isomorphisms

$$\varphi_1\colon A \to B_1, \quad \varphi_2\colon B \to A_2.$$

Put $\varphi_2(B_1) = A_1$. Then A_1 is an internal direct factor of A_2. Thus A_1 is also an internal direct factor of A. Since $\varphi_2(\varphi_1(A)) = A_1$, we have $A \simeq A_1$. Therefore in view of (ii) we obtain $A \simeq A_2$, hence $A \simeq B$. We have proved that A has the property CBS^*. $\square$

Let us recall the notions of an A-sequence, B-sequence and CBS-sequence as defined in [5].

Assume that A is an algebra belonging to an $\mathcal{L}$-variety V. Suppose that $b \in Z(A)$ and that there exists an isomorphism $\alpha\colon\ A \to [0,b]_A$. Further, let $z \in Z(A)$, $z \geqslant b$, $a \in Z(A)$. Put $B = [0,z]_A$ and let $\beta\colon\ B \to [0,a]_A$ be an isomorphism.

We define recursively sequences (a_n), (b_n) $(n = 0,1,2,\ldots)$ by putting

$$\begin{aligned} a_0 &= 1_A, & b_0 &= 1_B = z,\\ a_1 &= \beta(z) = a, & b_1 &= \alpha(a_0) = b,\\ \alpha_{n+1} &= \beta(b_n), & b_{n+1} &= \alpha(a_n). \end{aligned}$$

Further, we consider the sequence

$$(c_n)_{n\in\mathbb{N}} = (a_{2n} \wedge \neg a_{2n+1})_{n\in\mathbb{N}}$$

which is called a CBS-sequence and denoted by $\langle b, z, \alpha, \beta\rangle$.

Now assume that A is an algebra belonging to an $\mathcal{L}^*$-variety V. We denote by $D(A)$ the set of all internal direct factors of $L = L(A)$ with respect to the distinguished element v_0 of A. Then each element of $D(A)$ is also an internal direct factor of A.

Suppose that $B^* \in D(A)$ and that there exists an isomorphism $\alpha\colon\ A \to B^*$. Further, let $B_0 \in D(A)$, $B_0 \supseteq B^*$. Put $A_0 = A$. Assume that there exists $A_1 \in D(A)$ and an isomorphism $\beta\colon\ B_0 \to A_1$. Put $B_1 = B^*$. For $n \in \mathbb{N}$ we define by induction

$$A_{n+1} = \beta(B_n), \quad B_{n+1} = \alpha(A_n).$$

Further, under the notation as in Section 3, we consider the sequence

$$(C_n)_{n\in\mathbb{N}} = (A_{2n} \wedge \neg A_{2n+1})_{n\in\mathbb{N}};$$

this will be called a CBS^*-sequence and denoted by $\langle B^*, B_0, \alpha, \beta\rangle$.

Let χ be as in 2.3.1.

Lemma 4.5. *Assume that V is an $\mathcal{L}$-variety and $A \in V$.*

(i) *Let $(c_n)_{n\in\mathbb{N}}$ be a CBS-sequence. Then $(\chi(c_n))_{n\in\mathbb{N}}$ is a CBS^*-sequence.*

(ii) *Let $(C_n)_{n\in\mathbb{N}}$ be a CBS^*-sequence. Then $(\chi^{-1}(C_n))_{n\in\mathbb{N}}$ is a CBS-sequence.*

Proof. This is a consequence of Lemma 2.3.1. $\square$

5. CBS^*-completeness

In this section we prove that the CBS^*-property is equivalent to a condition concerning CBS^*-sequences.

Definition 5.1 (Cf. [5]). Let V be an $\mathcal{L}$-variety and $A \in V$. We say that A is CBS-complete if for all $b \in Z(L(A))$ such that $A \simeq [0,b]_A$ and for all $z \in Z(L(A))$ such that $z \geqslant b$ there exists a CBS-sequence $\langle b, z, \alpha, \beta\rangle$ such that (under the notation as above) the supremum $\bigsqcup\limits_{n\in\mathbb{N}} (A_{2n} \wedge a_{2n+1})$ of this sequence exists in $Z(L(A))$.

Definition 5.2. Let V be an $\mathcal{L}^*$-variety. An algebra $A \in V$ is called CBS^*-complete if for all $B^* \in D(A)$ such that $A \simeq B^*$ and for all $B_0 \in D(A)$ with $B_0 \supseteq B^*$ there exists a CBS^*-sequence $\langle B^*, B_0, \alpha, \beta\rangle$ such that the supremum of this sequence exists in $D(A)$.

Lemma 5.3. *Let V be an $\mathcal{L}$-variety and $A \in V$. Then A is CBS-complete iff it is CBS^*-complete.*

Proof. This is a consequence of 4.5 and of 2.3.1. $\square$

Lemma 5.4. *Let V be an $\mathcal{L}^*$-variety and $A \in V$. Assume that A has the CBS^*-property. Then A is CBS^*-complete.*

Proof. Let $B^* \in D(A)$ be such that $A \simeq B^*$. Further, let $B_0 \in D(A)$ with $B_0 \supseteq B^*$.

There exists an isomorphism $\alpha\colon A \to B^*$. Since A has the CBS^*-property, in view of 4.4 there exists an isomorphism $\beta\colon B_0 \to A$. Consider the corresponding CBS^*-sequence $\langle B^*, B_0, \alpha, \beta\rangle$. An easy calculation shows that all elements of this sequence are equal to $\{v_0\}$, hence the join of the elements of this sequence is $\{v_0\}$ as well. Therefore A is CBS^*-complete. $\square$

Again, let V be an $\mathcal{L}^*$-variety and $A \in V$. Consider a CBS^*-sequence $(B^*, B_0, \alpha, \beta)$ in A. We apply the notation as in Section 4.

By induction we can verify that $A_n \in D(A_{n-1})$ for each $n \in \mathbb{N}$. We have

$$\beta(\alpha(A_n)) = \beta(B_{n+1}) = A_{n+2} \quad \text{for } n = 0, 1, 2, \ldots. \tag{1}$$

Lemma 5.5. *For each $n \in \mathbb{N}$, C_n is a relative complement of A_{2n+1} in the Boolean algebra $D(A_{2n})$.*

Proof. This is a consequence of 3.6 and 3.7. □

From (1) and 5.5 we obtain

$$\beta(\alpha(C_n)) = C_{n+1} \quad \text{for each } n \in \mathbb{N}. \tag{2}$$

Let $X_1, X_2, X_3 \in D(A)$. The notation $X_1 \vee^0 X_2 = X_3$ will mean that $X_1 \vee X_2 = X_3$ and $X_1 \wedge X_2 = \{v_0\}$ in $D(A)$.

Lemma 5.6. $\beta(\alpha(\neg A_1)) = C_1$.

Proof. We have

$$\begin{aligned} \alpha(A) &= \alpha(A_0) = B_1, \\ \alpha(A) &= \alpha(A_1 \vee^0 \neg A_1) = \alpha(A_1) \vee^0 \alpha(\neg A_1), \end{aligned}$$

whence

$$\begin{aligned} B_1 &= B_2 \vee^0 \alpha(\neg A_1), \\ A_2 = \beta(B_1) &= \beta(B_2) \vee^0 \beta(\alpha(\neg A_1)), \\ A_2 &= A_3 \vee^0 \beta(\alpha(\neg A_1)). \end{aligned}$$

□

Therefore in view of 5.5 we get $\beta(\alpha(\neg A_1)) = C_1$.

Lemma 5.7. *Assume that there exists $C^0 = \bigvee_{n \in \mathbb{N}} C_{\mathbb{N}}$ in $D(A)$. Put $X = C^0 \vee \neg A_1$. Then $X \simeq C^0$.*

Proof. We have

$$X \simeq \beta(\alpha(X)) = \beta(\alpha(C^0)) \vee \beta(\alpha(\neg A_1)).$$

Further, in view of (2),

$$\beta(\alpha(C^0)) = \beta\Big(\alpha\Big(\bigvee_{n \in \mathbb{N}} C_n\Big)\Big) = \bigvee_{n \in \mathbb{N}} \beta(\alpha(C_n)) = \bigvee_{n=2,3,\dots} C_n.$$

Thus according to 5.6,

$$\beta(\alpha(X)) = \Big(\bigvee_{n=2,3,\dots} C_n\Big) \vee C_1 = \bigvee_{n \in \mathbb{N}} C_n = C^0.$$

Hence $X \simeq C^0$. □

Lemma 5.8. *Under the assumption as in* 5.7 *we have* $A \simeq B_0$.

P r o o f. Since $B_0 \simeq A_1$ it suffices to verify that $A \simeq A_1$. In view of 5.7 we have

$$A = (\mathrm{int})X \times \neg X \simeq C^0 \times \neg X.$$

From $X = C^0 \vee \neg A_1$ we obtain $\neg X = \neg C_0 \wedge A_1$. Further,

$$A = (\mathrm{int})C^0 \times \neg C^0,$$

whence

$$A_1 = (\mathrm{int})(A_1 \cap C^0) \times (A_1 \cap \neg C_0).$$

But $C_n \subseteq A_1$ for each $n \in \mathbb{N}$, whence $C^0 \subseteq A_1$ and thus

$$A_1 = (\mathrm{int})C^0 \times \neg X.$$

Therefore $A \simeq A_1$. □

In view of 4.4 and 5.8 we have

Corollary 5.9. *Let V be an $\mathcal{L}^*$-variety and $A \in V$. If A is CBS^*-complete, then it satisfies the CBS^*-property.*

Theorem 5.10. *Let V be an $\mathcal{L}^*$-variety and $A \in V$. Then A satisfies the CBS^*-property if and only if it is CBS^*-complete.*

P r o o f. This is a consequence of 5.4 and 5.9. □

The next theorem is the main result of [5]; it follows from 4.3, 5.3 and 5.10.

Theorem 5.11 (Cf. [5], Theorem 3.7)**.** *Let V be a variety and $A \in V$. Then A has the CBS-property iff it is CBS-complete.*

We conclude by remarking that the variety V_L of all lattices fails to be an $\mathcal{L}^*$-variety. In fact, if A is a lattice having more than one element then there does not exist any term in the language of V_L such that this term defines a constant on A.

Nevertheless, if $A \in V_L$ and if we take any (fixed) element v_0 of A then by considering internal direct decompositions of A with respect to v_0, the definitions of the CBS^*-property and the CBS^*-completeness can be applied for A; we can perform the same steps as above for the variety V_L and we obtain a result analogous to 5.10 saying that the CBS^*-property and the CBS^*-completeness are equivalent.

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