Czechoslovak Mathematical Journal, 57 (132) (2007), 591–605

ULTRA LI-IDEALS IN LATTICE IMPLICATION ALGEBRAS AND MTL-ALGEBRAS

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(Received February 14, 2005)

Abstract. A mistake concerning the ultra LI-ideal of a lattice implication algebra is pointed out, and some new sufficient and necessary conditions for an LI-ideal to be an ultra LI-ideal are given. Moreover, the notion of an LI-ideal is extended to MTL-algebras, the notions of a (prime, ultra, obstinate, Boolean) LI-ideal and an ILI-ideal of an MTL-algebra are introduced, some important examples are given, and the following notions are proved to be equivalent in MTL-algebra: (1) prime proper LI-ideal and Boolean LI-ideal, (2) prime proper LI-ideal and ILI-ideal, (3) proper obstinate LI-ideal, (4) ultra LI-ideal.

Keywords: lattice implication algebra, MTL-algebra, (prime, ultra, obstinate, Boolean) LI-ideal, ILI-ideal

MSC 2000: 03G10, 06B10, 54E15

1. Introduction

In order to research a logical system whose propositional value is given in a lattice, Y. Xu proposed the concept of lattice implication algebras, and some researchers have studied their properties and the corresponding logic systems (see [15], [17]). In [7], Y. B. Jun et al. proposed the concept of an LI-ideal of a lattice implication algebra, discussed the relationship between filters and LI-ideals, and studied how to generate an LI-ideal by a set. In [11], K. Y. Qin et al. introduced the notion of ultra LI-ideals in lattice implication algebras, and gave some sufficient and necessary conditions for an LI-ideal to be ultra LI-ideal.

The interest in the foundations of fuzzy logic has been rapidly growing recently and several new algebras playing the role of the structures of truth-values have been

This work was supported by the Zhejiang Provincial Natural Science Foundation of China (Grant No. Y605389) and K. C. Wong Magna Fund in Ningbo University.

introduced. P. Hájek introduced the system of basic logic (BL) axioms for the fuzzy propositional logic and defined the class of BL-algebras (see [4]). G. J. Wang proposed a formal deductive system L^* for fuzzy propositional calculus, and a kind of new algebraic structures, called R_0-algebras (see [13], [14]). F. Esteva and L. Godo proposed a new formal deductive system MTL, called the monoidal t-norm-based logic, intended to cope with left-continuous t-norms and their residual. The algebraic semantics for MTL is based on MTL-algebras (see [3], [5]). It is easy to verify that a lattice implication algebra is an MTL-algebra. Varieties of MTL-algebras are described in [10], and some other results concerning MTL-algebras are presented in [19] and [20].

This paper is devoted to a discussion of the ultra LI-ideals, we correct a mistake in [11] and give some new equivalent conditions for an LI-ideal to be ultra. We also generalize the notion of an LI-ideal to MTL-algebras, introduce the notions of a (prime, ultra, obstinate, Boolean) LI-ideal and an ILI-ideal of MTL- algebra, give some important examples, and prove that the following notions are equivalent in an MTL-algebra: (1) prime proper LI-ideal and Boolean LI-ideal, (2) prime proper LI-ideal and ILI-ideal, (3) proper obstinate LI-ideal, (4) ultra LI-ideal.

2. Preliminaries

Definition 2.1 ([17]). By a *lattice implication algebra* L we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with an order-reversing involution $'$ and a binary operation $\rightarrow$ satisfying the following axioms:

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(I2) $x \rightarrow x = 1$,
(I3) $x \rightarrow y = y' \rightarrow x'$,
(I4) $x \rightarrow y = y \rightarrow x = 1 \Longrightarrow x = y$,
(I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
(L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
(L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$ for all $x, y, z \in L$.

We can define a partial ordering $\leqslant$ on a lattice implication algebra L by

$$x \leqslant y \quad \text{if and only if} \quad x \rightarrow y = 1.$$

For any lattice implication algebra L, $(L, \vee, \wedge)$ is a distributive lattice and the De Morgan law holds, that is

(L3) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
(L4) $(x \wedge y)' = x' \vee y'$, $(x \vee y)' = x' \wedge y'$ for all $x, y, z \in L$.

Theorem 2.2 ([17]). *In a lattice implication algebra L, the following relations hold:*

(1) $0 \to x = 1$, $1 \to x = x$ *and* $x \to 1 = 1$,

(2) $x' = x \to 0$,

(3) $x \to y \leqslant (y \to z) \to (x \to z)$,

(4) $x \vee y = (x \to y) \to y$,

(5) $x \leqslant y$ *implies* $y \to z \leqslant x \to z$ *and* $z \to x \leqslant z \to y$,

(6) $x \to (y \vee z) = (x \to y) \vee (x \to z)$,

(7) $x \to (y \wedge z) = (x \to y) \wedge (y \to z)$,

(8) $(x \to y) \vee (y \to x) = 1$,

(9) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$,

(10) $x \to (y \to z) = y \to (x \to z)$,

(11) $((x \to y) \to y) \to y = x \to y$.

From the above theorem it follows that lattice implication algebras are strictly connected with BCC-algebras and BCK-algebras of the form $(L, \to, 1)$ [2, 21].

For shortness, in the sequel the formula $(x \to y')'$ will be denoted by $x \otimes y$, the formula $x' \to y$ by $x \oplus y$.

Theorem 2.3 ([17]). *In a lattice implication algebra L, the relations*

(12) $x \otimes y = y \otimes x$, $x \oplus y = y \oplus x$,

(13) $x \otimes (y \otimes z) = (x \otimes y) \otimes z$, $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(14) $x \otimes x' = 0$, $x \oplus x' = 1$,

(15) $x \otimes (x \to y) = x \wedge y$,

(16) $x \to (y \to z) = (x \otimes y) \to z$,

(17) $x \leqslant y \to z \iff x \otimes y \leqslant z$,

(18) $x \leqslant a$ *and* $y \leqslant b$ *imply* $x \otimes y \leqslant a \otimes b$ *and* $x \oplus y \leqslant a \oplus b$

hold for all $x, y, z \in L$.

Definition 2.4 ([7]). A subset A of a lattice implication algebra L is called an *LI-ideal* of L if

(LI1) $0 \in A$,

(LI2) $(x \to y)' \in A$ and $y \in A$ imply $x \in A$ for all $x, y \in L$.

An LI-ideal A of a lattice implication algebra L is said to be *proper* if $A \neq L$.

Theorem 2.5 ([7], [17]). *Let A be an LI-ideal of a lattice implication algebra L, then*

(LI3) $x \in A$, $y \leqslant x$ *imply* $y \in A$,

(LI4) $x, y \in A$ *imply* $x \vee y \in A$.

The least LI-ideal containing a subset A is called the *LI-ideal generated by A* and is denoted by $\langle A \rangle$.

Theorem 2.6 ([7], [17]). *If A is a non-empty subset of a lattice implication algebra L, then*

$$\langle A \rangle = \{x \in L\colon\ a_n' \to (\ldots \to (a_1' \to x') \ldots) = 1 \ \text{ for some } \ a_1, \ldots, a_n \in A\}.$$

Theorem 2.7 ([11]). *Let A be a subset of a lattice implication algebra L. Then A is an LI-ideal of L if and only if it satisfies* (LI3) *and*

(LI5) $x \in A$ *and* $y \in A$ *imply* $x \oplus y \in A$.

Theorem 2.8 ([11]). *If A is a non-empty subset of a lattice implication algebra L, then*

$$\langle A \rangle = \{x \in L\colon\ x \leqslant a_1 \oplus a_2 \oplus \ldots \oplus a_n \ \text{ for some } \ a_1, \ldots, a_n \in A\}.$$

Definition 2.9 ([9]). An LI-ideal A of a lattice implication algebra L is said to be *ultra* if for every $x \in L$, the following equivalence holds:

(LI6) $x \in A \Longleftrightarrow x' \notin A$.

Definition 2.10 ([9]). A non-empty subset A of a lattice implication algebra L is said to be an *ILI-ideal* of L if it satisfies (LI1) and

(LI7) $(((x \to y)' \to y)' \to z)' \in A$ and $z \in A$ imply $(x \to y)' \in A$ for all $x, y, z \in L$.

Theorem 2.11 ([9]). *If A is an LI-ideal of a lattice implication algebra L, then the following assertions are equivalent:*

(i) *A is an ILI-ideal of L,*

(ii) $((x \to y)' \to y)' \in A$ *implies* $(x \to y)' \in A$ *for all* $x, y, z \in L$,

(iii) $((x \to y)' \to z)' \in A$ *implies* $((x \to z)' \to (y \to z)')' \in A$ *for all* $x, y, z \in L$,

(iv) $(x \to (y \to x)')' \in A$ *implies* $x \in A$ *for all* $x, y, z \in L$.

Definition 2.12 ([6]). A proper LI-ideal A of a lattice implication algebra L is said to be a *prime* LI-ideal of L if $x \wedge y \in A$ implies $x \in A$ or $y \in A$ for any $x, y \in L$.

Theorem 2.13 ([9]). *Let A be a proper LI-ideal of a lattice implication algebra L. The following assertions are equivalent:*
(i) *A is a prime LI-ideals of L,*
(ii) *$x \wedge y = 0$ implies $x \in A$ or $y \in A$ for any $x, y \in L$.*

An LI-ideal of a lattice implication algebra L is called *maximal*, if it is proper and not a proper subset of any proper LI-ideal of L.

Theorem 2.14 ([9]). *In a lattice implication algebra L, any maximal LI-ideal must be prime.*

Theorem 2.15 ([9]). *Let L be a lattice implication algebra and A a proper LI-ideal of L. Then A is both a prime LI-ideal and an ILI-ideal of L if and only if $x \in A$ or $x' \in A$ for any $x \in L$.*

Theorem 2.16 ([9]). *Let L be a lattice implication algebra and A a proper LI-ideal. Then A is both a maximal LI-ideal and an ILI-ideal if and only if for any $x, y \in L$, $x \notin A$ and $y \notin A$ imply $(x \to y)' \in A$ and $(y \to x)' \in A$.*

Definition 2.17 ([1], [3]). A *residuated lattice* is an algebra $(L, \wedge, \vee, \otimes, \to, 0, 1)$ with four binary operations and two constants such that
(i) $(L, \wedge, \vee, 0, 1)$ is a lattice with the largest element 1 and the least element 0 (with respect to the lattice ordering $\leqslant$),
(ii) $(L, \otimes, 1)$ is a commutative semigroup with the unit element 1, i.e., $\otimes$ is commutative, associative, $1 \otimes x = x$ for all x,
(iii) $\otimes$ and $\to$ form an adjoint pair, i.e., $z \leqslant x \to y$ if and only if $z \otimes x \leqslant y$ for all $x, y, z \in L$.

Definition 2.18 ([3]). A residuated lattice L is called an *MTL-algebra*, if it satisfies the pre-linearity equation: $(x \to y) \vee (y \to x) = 1$ for all $x, y \in L$. An MTL-algebra L is called an *$IMTL$-algebra*, if $(a \to 0) \to 0 = a$ for any $a \in L$.

In the sequel x' will be reserved for $x \to 0$, L for $(L, \wedge, \vee, \otimes, \to, 0, 1)$.

Proposition 2.19 ([3], [12]). *Let L be a residuated lattice. Then for all $x, y, z \in L$,*
(R1) $x \leqslant y \Longleftrightarrow x \to y = 1$,
(R2) $x = 1 \to x$, $x \to (y \to x) = 1$, $y \leqslant (y \to x) \to x$,
(R3) $x \leqslant y \to z \Longleftrightarrow y \leqslant x \to z$,
(R4) $x \to (y \to z) = (x \otimes y) \to z = y \to (x \to z)$,
(R5) $x \leqslant y$ *implies* $z \to x \leqslant z \to y$ *and* $y \to z \leqslant x \to z$,

(R6) $z \to y \leqslant (x \to z) \to (x \to y)$, $z \to y \leqslant (y \to x) \to (z \to x)$,

(R7) $(x \to y) \otimes (y \to z) \leqslant x \to z$,

(R8) $x' = x'''$, $x \leqslant x''$,

(R9) $x' \wedge y' = (x \vee y)'$,

(R10) $x \vee x' = 1$ *implies* $x \wedge x' = 0$,

(R11) $\big(\bigvee_{i\in\Gamma} y_i\big) \to x = \bigwedge_{i\in\Gamma} (y_i \to x)$,

(R12) $x \otimes \big(\bigvee_{i\in\Gamma} y_i\big) = \bigvee_{i\in\Gamma} (x \otimes y_i)$,

(R13) $x \to \big(\bigwedge_{i\in\Gamma} y_i\big) = \bigwedge_{i\in\Gamma} (x \to y_i)$,

(R14) $\bigvee_{i\in\Gamma} (y_i \to x) \leqslant \big(\bigwedge_{i\in\Gamma} y_i\big) \to x$,

where Γ *is a finite or infinite index set and we assume that the corresponding infinite meets and joints exist in* L.

Proposition 2.20 ([3], [18]). *Let* L *be an* MTL*-algebra. Then for all* $x, y, z \in L$,

(M1) $x \otimes y \leqslant x \wedge y$,

(M2) $x \leqslant y$ *implies* $x \otimes z \leqslant y \otimes z$,

(M3) $y \to z \leqslant x \vee y \to x \vee z$,

(M4) $x' \vee y' = (x \wedge y)'$,

(M5) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$,

(M6) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,

(M7) $x \to (y \vee z) = (x \to y) \vee (x \to z)$,

(M8) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, *i.e., the lattice structure of* L *is distributive.*

Definition 2.21 ([3]). Let L be an MTL-algebra. A *filter* is a nonempty subset F of L such that

(F1) $x \otimes y \in F$ for any $x, y \in F$,

(F2) for any $x \in F$, $x \leqslant y$ implies $y \in F$.

Proposition 2.22 ([3]). *A subset* F *of an* MTL*-algebra* L *is a filter of* L *if and only if*

(F3) $1 \in F$,

(F4) $x \in F$ *and* $x \to y \in F$ *imply* $y \in F$.

In [11], the following result is presented: *Let A be a subset of a lattice implication algebra L, then A is an ultra LI-ideal of L if and only if A is a maximal proper LI-ideal of L.* The following example shows that this result is not true.

Example 3.1. Let $L = \{0, a, b, 1\}$ be a set with the Cayley table

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

For any $x \in L$, we have $x' = x \to 0$. The operations $\wedge$ and $\vee$ on L are defined as follows:

$$x \vee y = (x \to y) \to y, \quad x \wedge y = ((x' \to y') \to y')'.$$

Then $(L, \vee, \wedge, 0, 1)$ is a lattice implication algebra. It is easy to check that $\{0\}$ is a maximal proper LI-ideal of L, but not an ultra LI-ideal of L, because $a' = b \notin \{0\}$, but $a \notin \{0\}$.

Below, we give some new sufficient and necessary conditions for an LI-ideal to be an ultra LI-ideal.

Theorem 3.2. *Let L be a lattice implication algebra and A an LI-ideal of L. Then the following assertions are equivalent:*

(i) *A is an ultra LI-ideal,*

(ii) *A is a prime proper LI-ideal and an ILI-ideal,*

(iii) *A is a proper LI-ideal and $x \in A$ or $x' \in A$ for any $x \in L$,*

(iv) *A is a maximal ILI-ideal,*

(v) *A is a proper LI-ideal and for any $x, y \in L$, $x \notin A$ and $y \notin A$ imply $(x \to y)' \in A$ and $(y \to x)' \in A$.*

Proof. (i) $\Longrightarrow$ (ii): A is a proper LI-ideal, because $0 \in A$, and so $1 = 0' \notin A$.

We show that A is a prime LI-ideal. Assume $x \wedge y = 0$ for some $x, y \in L$. We prove that $x \in A$ or $y \in A$. If $x \notin A$ and $y \notin A$, then $x' \in A$ and $y' \in A$, by the definition of an ultra LI-ideal. So, by Theorem 2.5 (LI4), we have $x' \vee y' \in A$, thus $1 = 0' = (x \wedge y)' = x' \vee y' \in A$. This means that $A = L$, a contradiction. Therefore $x \wedge y = 0$ implies $x \in A$ or $y \in A$. So, by Theorem 2.13, A is a prime proper LI-ideal.

Now we show that A is an ILI-ideal. Let $((x \to y)' \to y)' \in A$. If $(x \to y)' \notin A$, then $x \to y \in A$ by the definition of an ultra LI-ideal. Since $y \leqslant x \to y$, we have

$y \in A$. From $((x \to y)' \to y)' \in A$ and $y \in A$, we conclude $(x \to y)' \in A$, by Definition 2.4 (LI2). This is a contradiction. Thus, $(x \to y)' \in A$. By Theorem 2.11 (ii), A is an ILI-ideal. This means that (ii) holds.

(ii) $\Longleftrightarrow$ (iii): See Theorem 2.15.

(iii) $\Longrightarrow$ (i): For any $x \in L$, if $x' \notin A$ then $x \in A$ by (iii). If $x \in A$, we prove that $x' \notin A$. Indeed, if $x' \in A$, then $x \oplus x' = 1 \in A$ by Theorem 2.3 (14) and Theorem 2.7 (LI5). This is a contradiction with the fact that A is a proper LI-ideal. This means that A is an ultra LI-ideal.

(iv) $\Longleftrightarrow$ (v): See Theorem 2.16.

(i) $\Longrightarrow$ (v): A is a proper LI-ideal, because $0 \in A$, and so $1 = 0' \notin A$.

If $x \notin A$, from $x \leqslant y \to x$ and Theorem 2.7 (LI3), we have $y \to x \notin A$. Thus, by the definition of an ultra LI-ideal, $(y \to x)' \in A$. Similarly, from $y \notin A$ we obtain $(x \to y)' \in A$. That is, (v) holds.

(v) $\Longrightarrow$ (i): By (v), $1 \notin A$. If $x' \notin A$, by (v) we have $(1 \to x')' \in A$, that is $x \in A$. If $x \in A$, then $x' \notin A$ (see (iii) $\Longrightarrow$ (i)). This means that A is an ultra LI-ideal. The proof is complete. □

Remind [11] that a subset A of a lattice implication algebra L has the *finite additive property* if $a_1 \oplus a_2 \oplus \ldots \oplus a_n \neq 1$ for any finite members $a_1, \ldots, a_n \in A$. $\langle A \rangle$ is a proper LI-ideal of L if and only if A has the finite additive property.

Our theorem proves that the part results formulated in Theorem 3.7 and Corollary 3.8 in [11] is correct. Namely we have

Theorem 3.3. *If a subset A of a lattice implication algebra L has the finite additive property, then there exists a maximal LI-ideal of L containing A. Every proper LI-ideal of a lattice implication algebra can be extended to a maximal LI-ideal.*

4. LI-ideals of MTL-algebras

Definition 4.1. A subset A of an MTL-algebra L is called an *LI-ideal* of L if $0 \in A$ and

(LI8) $(x' \to y')' \in A$ and $x \in A$ imply $y \in A$ for all $x, y \in L$.

Obviously, for a lattice implication algebra L, (LI8) coincides with (LI2). For a MTL-algebra it is not true because $x = x''$ is not true.

An LI-ideal A of an MTL-algebra L is said to be *proper* if $A \neq L$.

Lemma 4.2 ([17], Theorem 4.1.3). *A non-empty subset A of a lattice implication algebra L is a filter of L if and only if $A' = \{a' \colon a \in A\}$ is an LI-ideal of L.*

For MTL-algebras the above lemma is not true.

Example 4.3. Consider the set $L = \{0, a, b, c, d, 1\}$, where $0 < a < b < c < d < 1$, and two operations $\otimes$, $\rightarrow$ defined by the following two tables:

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	a	a
b	0	0	0	b	b	b
c	0	0	b	c	c	c
d	0	a	b	c	d	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	1	1	1	1
b	b	b	1	1	1	1
c	a	a	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

If we define on L the operations $\wedge$ and $\vee$ as min and max, respectively, then $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ will be an MTL-algebra. Obviously, $A = \{0, a, b, c, d, 1\}$ is a filter of L, but $A' = \{0, a, b, c, 1\}$ is not an LI-ideal of L, since

$$(0' \rightarrow d')' = 1 \in A \text{ and } 0 \in A, \quad \text{but } d \notin A.$$

Moreover, $B = \{1, c\}$ is not a filter of L, because $c \rightarrow d = 1 \in B$ and $c \in B$, $d \notin B$. By the following MATHEMATICA program, we can verify that $B' = \{0, a\}$ is an LI-ideal of L:

```
M1={{6,6,6,6,6,6},{4,6,6,6,6,6},{3,3,6,6,6,6},{2,2,3,6,6,6},
     {1,2,3,4,6,6},{1,2,3,4,5,6}};
a1=0;
For[i=1, i<7, i++, For[j=1, j<7, j++,
    If[(i==1||i==2) &&(M1[[M1[[M1[[i,1]],M1[[j,1]]]],1]]==1||
      M1[[M1[[M1[[i,1]],M1[[j,1]]]],1]]==2)&&(j!=1&&j!=2), a1++]]];
If[a1==0, Print[‘‘true’’], Print[‘‘false’’]]
```

From Example 4.3 we see that LI-ideals have a proper meaning in MTL-algebras.

Theorem 4.4. *Let A be an LI-ideal of an MTL-algebra L, then*

(LI3) *if $x \in A$, $y \leqslant x$, then $y \in A$,*
(LI9) *if $x \in A$, then $x'' \in A$,*
(LI4) *if $x, y \in A$, then $x \vee y \in A$.*

P r o o f. Assume $x \in A$, $y \leqslant x$. From $y \leqslant x$, by Proposition 2.19 (R5), we have $x \to 0 \leqslant y \to 0$, i.e., $x' \leqslant y'$. By Proposition 2.19 (R1), $x' \to y' = 1$. Then $(x' \to y')' = 1' = 0 \in A$ and $x \in A$, and by (LI8) we get $y \in A$. This means that (LI3) holds.

Suppose $x \in A$. By Proposition 2.19 (R8) we have $(x' \to (x'')')' = (x' \to x')' = 1' = 0 \in A$. Applying (LI8) we get $x'' \in A$, i.e., (LI9) holds.

Assume $x, y \in A$. By Proposition 2.19 (R2) we have $y' \leqslant x' \to y'$. So, $(x' \to y')' \leqslant y''$ by (R5). Whence, by $y \in A$ and (LI9), we obtain $y'' \in A$. From this and $(x' \to y')' \leqslant y''$, using (LI3) we get $(x' \to y')' \in A$. Thus

$$\begin{aligned}
(x' \to (x \vee y)')' &= (x' \to (x' \wedge y'))' && \text{(by (R9))}\\
&= ((x' \to x') \wedge (x' \to y'))' && \text{(by (R13))}\\
&= (1 \wedge (x' \to y'))' && \text{(by (R1))}\\
&= (x' \to y')' \in A.
\end{aligned}$$

From this and $x \in A$, using (LI8), we deduce $x \vee y \in A$, i.e., (LI4) holds.

The proof is complete. □

Definition 4.5. An LI-ideal A of an MTL-algebra L is said to be an *ILI-ideal* of L if it satisfies

(LI10) $(x \to (y \to x)')' \in A$ implies $x \in A$ for all $x, y, z \in L$.

Example 4.6. Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$, be a set with the Cayley tables:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

Defining the operations $\wedge$ and $\vee$ on L as min and max, respectively, we obtain an MTL-algebra $(L, \wedge, \vee, \otimes, \to, 0, 1)$ in which $A = \{0\}$ is an ILI-ideal of L.

In Example 4.3, $\{0, a\}$ is an LI-ideal, but it is not an ILI-ideal of L, because

$$(b \to (1 \to b)')' = 0 \in \{0, a\}, \quad \text{but} \quad b \notin \{0, a\}.$$

Theorem 4.7. *For each ILI-ideal A of an MTL-algebra L we have*

(LI11) $x \wedge x' \in A$ *for each* $x \in L$.

Proof. Indeed, for all $x \in L$ we get

$$\begin{aligned}
&((x \wedge x') \to (1 \to (x \wedge x'))')' \\
&\qquad = ((x \wedge x') \to (x \wedge x')')' \\
&\qquad = ((x \wedge x') \to (x' \vee x''))' && \text{(by Proposition 2.20 (M4))} \\
&\qquad = (((x \wedge x') \to x') \vee ((x \wedge x') \to x''))' && \text{(by Proposition 2.20 (M7))} \\
&\qquad = (1 \vee ((x \wedge x') \to x''))' && \text{(by Proposition 2.19 (R1))} \\
&\qquad = 1' = 0 \in A.
\end{aligned}$$

From this, applying (LI10), we deduce (LI11). □

Definition 4.8. An LI-ideal A satisfying (LI11) is called *Boolean.*

Theorem 4.9. *If A is a Boolean LI-ideal of an MTL-algebra L, then*
(LI12) $(x \to x')' \in A$ *implies* $x \in A$.

Proof. According to the assumption $x \wedge x' \in A$ for all $x \in L$. Let $(x \to x')' \in A$. Then

$$\begin{aligned}
&((x \wedge x')' \to x')' \\
&\qquad = (x \to (x \wedge x')'')' && \text{(by Proposition 2.19 (R4))} \\
&\qquad = (x \to (x'' \wedge x'''))' && \text{(by Propositions 2.19 (R9) and 2.20 (M4))} \\
&\qquad = ((x \to x'') \wedge (x \to x'''))' && \text{(by Proposition 2.19 (R13))} \\
&\qquad = (1 \wedge (x \to x'))' && \text{(by Proposition 2.19 (R8), (R1))} \\
&\qquad = (x \to x')' \in A.
\end{aligned}$$

Now, applying (LI8) we get $x \in A$, which completes the proof. □

Theorem 4.10. *For LI-ideals of MTL-algebras the conditions* (LI10) *are equivalent* (LI11).

Proof. (LI10) $\Longrightarrow$ (LI11): See Theorem 4.7.
(LI11) $\Longrightarrow$ (LI10): Let $(x \to (y \to x)')' \in A$. Then

$$\begin{aligned}
&((x \to (y \to x)')'' \to (x \to x')'')' \\
&\qquad = ((x \to x')' \to (x \to (y \to x)')')' && \text{(by Proposition 2.19 (R4), (R8))} \\
&\qquad \leqslant ((x \to (y \to x)') \to (x \to x'))' && \text{(by Proposition 2.19 (R6))} \\
&\qquad \leqslant ((y \to x)' \to x')' && \text{(by Proposition 2.19 (R6))} \\
&\qquad \leqslant (x \to (y \to x))' && \text{(by Proposition 2.19 (R6))} \\
&\qquad = 1' = 0 \in A && \text{(by Proposition 2.19 (R2)).}
\end{aligned}$$

This, by (LI8), implies $(x \to x')' \in A$, whence, using (LI12), we obtain $x \in A$. So, (LI11) implies (LI10). $\square$

Definition 4.11. A proper LI-ideal A of an MTL-algebra L is said to be a *prime* if $x \wedge y \in A$ implies $x \in A$ or $y \in A$ for any $x, y \in L$.

Theorem 4.12. *A proper LI-ideal A of a MTL-algebra L is prime if and only if for all $x, y \in L$ we have $(x \to y)' \in A$ or $(y \to x)' \in A$.*

Proof. Assume that an LI-ideal A of L is prime. Since

$$(x \to y)' \wedge (y \to x)' = ((x \to y) \vee (y \to x))' = 1' = 0 \in A$$

for all $x, y \in L$, the assumption on A implies $(x \to y)' \in A$ or $(y \to x)' \in A$.

Conversely, let A be a proper LI-ideal of L and let $x \wedge y \in A$. Assume that $(x \to y)' \in A$ or $(y \to x)' \in A$ for $x, y \in L$. If $(x \to y)' \in A$, then

$$\begin{aligned}
((x \wedge y)' \to x')' &= ((x' \vee y') \to x')' && \text{(by Proposition 2.20 (M4))}\\
&= ((x' \to x') \wedge (y' \to x'))' && \text{(by Proposition 2.19 (R11))}\\
&= (1 \wedge (y' \to x'))' && \text{(by Proposition 2.19 (R1))}\\
&= (y' \to x')' \leqslant (x \to y)' \in A && \text{(by Proposition 2.19 (R6)).}
\end{aligned}$$

So, $((x \wedge y)' \to x')' \in A$ (Theorem 4.4 (LI3)), which together with $x \wedge y \in A$ and the definition of an LI-ideal, gives $x \in A$.

Similarly, from $(y \to x)' \in A$ we can obtain $y \in A$.

This means that A is a prime LI-ideal of L. The proof is complete. $\square$

Theorem 4.13. *Let A be an LI-ideal of an MTL-algebra L. Then A is both a prime LI-ideal and a Boolean LI-ideal of L if and only if $x \in A$ or $x' \in A$ for any $x \in L$.*

Proof. Assume that for all $x \in L$ we have $x \in A$ or $x' \in A$. At first we show that an LI-ideal A is prime. For this let $x \wedge y \in A$. If $x \notin A$, then $x' \in A$. Hence

$$\begin{aligned}
((x \wedge y)' \to y')' &= ((x' \vee y') \to y')' && \text{(by Proposition 2.20 (M4))}\\
&= ((x' \to y') \wedge (y' \to y'))' && \text{(by Proposition 2.19 (R11))}\\
&= ((x' \to y') \wedge 1)' && \text{(by Proposition 2.19 (R1))}\\
&= (x' \to y')' \leqslant (y \to x)' && \text{(by Proposition P2.19 (R6))}\\
&\leqslant x' \in A && \text{(by Proposition 2.19 (R2)).}
\end{aligned}$$

So, $((x \wedge y)' \to y')' \in A$, by Theorem 4.4 (LI3). From this, $x \wedge y \in A$ and Definition 4.1 we get $y \in A$. This proves that an LI-ideal A is prime. To prove that it is Boolean observe that $x \wedge x' \leqslant x'$ implies $x \wedge x' \leqslant x$, whence, by Theorem 4.4 (LI3), we obtain $x \wedge x' \in A$. Thus A is Boolean.

Conversely, if an LI-ideal A is both prime and Boolean, then by Definition 4.8, for all $x \in L$ we have $x \wedge x' \in A$. Hence $x \in A$ or $x' \in A$, by Definition 4.11. This completes the proof. □

Definition 4.14. An LI-ideal A of an MTL-algebra L is said to be *ultra* if for every $x \in L$

(LI6) $x \in A \Longleftrightarrow x' \notin A$.

It is easy to verify the following proposition is true.

Proposition 4.15. *Each ultra LI-ideal of an MTL-algebra is a proper LI-ideal.*

Definition 4.16. An LI-ideal A of an MTL-algebra L is said to be *obstinate* if for all $x, y \in L$

(LI13) $x \notin A$ and $y \notin A$ imply $(x \to y)' \in A$ and $(y \to x)' \in A$.

Theorem 4.17. *For an LI-ideal A of an MTL-algebra L the following conditions are equivalent:*

(i) *A is an ultra LI-ideal,*

(ii) *A is a proper LI-ideal and for any $x \in L$, $x \in A$ or $x' \in A$,*

(iii) *A is a prime proper LI-ideal and a Boolean LI-ideal,*

(iv) *A is a prime proper LI-ideal and an ILI-ideal,*

(v) *A is a proper obstinate LI-ideal.*

Proof. (i) $\Longrightarrow$ (ii): Obvious.

(ii) $\Longrightarrow$ (i): If $x' \notin A$, then $x \in A$, by (ii). Similarly, if $x \in A$, that must be $x' \notin A$. If not, i.e., if $x' \in A$, then, by Proposition 2.19 (R8), we have

$$(x' \to 1')' = (x' \to 0)' = x''' = x' \in A,$$

which together with $x \in A$ and (LI8) implies $1 \in A$. This, by Theorem 4.4 (LI3), gives $A = L$. This is a contradiction, because an LI-ideal A is proper. Obtained contradiction proves that $x \in A$ implies $x' \notin A$. So, A is an ultra LI-ideal.

(ii) $\Longleftrightarrow$ (iii): See Theorem 4.13.

(iv) $\Longrightarrow$ (iii): See Theorem 4.7.

(iii) $\Longrightarrow$ (iv): See Theorem 4.10.

(v) $\Longrightarrow$ (ii): Since A is a proper LI-ideal, $1 \notin A$. If $x \notin A$, then $(1 \to x)' = x' \in A$, by Definition 4.16. This means that (ii) holds.

(ii) $\Longrightarrow$ (v): It suffices to show that A is obstinate. Let $x \notin A$ and $y \notin A$. Then, according to (ii), we have $x' \in A$ and $y' \in A$. Thus

$$\begin{aligned}
(y'' \to (x \to y)'')' &= ((x \to y)' \to y''')' &&\text{(by Proposition 2.19 (R4))}\\
&= ((x \to y)' \to y')' &&\text{(by Proposition 2.19 (R8))}\\
&= (y \to (x \to y)'')' &&\text{(by Proposition 2.19 (R4))}\\
&\leqslant (y \to (x \to y))' &&\text{(by (R8), } x \to y \leqslant (x \to y)'' \text{ and (R5))}\\
&= 1' = 0 \in A &&\text{(by Proposition 2.19 (R2)).}
\end{aligned}$$

This together with $y' \in A$ and Definition 4.1 implies $(x \to y)' \in A$.

Similarly, we obtain $(y \to x)' \in A$. So, A is obstinate.

The proof is complete. □

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