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EXCHANGE RINGS WITH STABLE RANGE ONE

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Abstract. We characterize exchange rings having stable range one. An exchange ring R has stable range one if and only if for any regular $a \in R$, there exist an $e \in E(R)$ and a $u \in U(R)$ such that $a = e + u$ and $aR \cap eR = 0$ if and only if for any regular $a \in R$, there exist $e \in r.ann(a^{+})$ and $u \in U(R)$ such that $a = e + u$ if and only if for any $a, b \in R$, $R/aR \cong R/bR \Longrightarrow aR \cong bR$.

Keywords: exchange ring, stable range one, idempotent, unit

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1. Introduction

A right R-module A has the finite exchange property if for every right R-module Q and two decompositions $Q = M \oplus N = \bigoplus\limits_{i\in I} A_i$, where $M_R \cong A$ and the index set I is finite, there exist submodules $A_i' \subseteq A_i$ such that $Q = M \oplus \Big(\bigoplus\limits_{i\in I} A_i'\Big)$. We say that R is an exchange ring provided that R has the finite exchange property as a right R-module. By [14, Theorem 2.1], a ring R is an exchange ring if and only if for any $x \in R$ there exists an idempotent $e \in Rx$ such that $1 - e \in R(1 - x)$. It is well known in the literature that regular rings, π-regular rings, semi-perfect rings, left or right continuous rings, clean rings and unit C^*-algebras of real rank zero (cf. [3, Theorem 7.2]) are all exchange rings. In [1, Theorem 1.1], Ara proved that every purely infinite simple ring is an exchange ring.

Recall that a ring R has stable range one provided $aR + bR = R$ with $a, b \in R$ implies that there exists $y \in R$ such that $a + by \in U(R)$. This definition is right-left symmetric. Moreover, we know that a right R-module M can be cancelled from direct sums if and only if $\mathrm{End}_R M$ has stable range one. In this paper, we will characterize exchange rings having stable range one by various equivalent conditions.

An element $a \in R$ is regular if there exists an $x \in R$ such that $a = axa$. We say that $a \in R$ is unit-regular if it is the product of an idempotent and a unit. In [5, Theorem 3], Camillo and Yu proved that an exchange ring has stable range one if and only if every regular element in R is unit-regular. Further, Yu proved that every exchange ring with artinian primitive factors has stable range one (cf. [17, Theorem 1]). In [2, Theorem 4], Ara proved that every strongly π-regular ring is an exchange ring having stable range one.

In parallel, $a \in R$ is clean if it is the sum of an idempotent and a unit. Camillo and Khurana (cf. [4, Theorem 1]) gave a characterization of unit regular rings. They showed that a ring R is unit-regular if and only if for any $a \in R$ there exist an idempotent $e \in E(R)$ and a $u \in U(R)$ such that $a = e + u$ and $aR \cap eR = 0$.

Let $\mathbb{Z}$ be the ring of all integers. In [12, Example 4.5], Khurana and Lam showed that $\left(\begin{smallmatrix} 12 & 5 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(\mathbb{Z})$ is not clean although it is unit-regular. In other words, a single unit-regular element in a ring may be not clean. This has inspired us to investigate clean property of unit-regular elements in an exchange ring having stable range one. In this paper, we prove that an exchange ring R has stable range one if and only if for any regular $a \in R$, there exist an $e \in E(R)$ and a $u \in U(R)$ such that $a = e + u$ and $aR \cap eR = 0$. This gives an affirmative answer to the problem in [8]. Furthermore, we prove that an exchange ring R has stable range one if and only if for any regular $a \in R$, there exist $e \in r.ann(a^+)$ and $u \in U(R)$ such that $a = e + u$. Additionally, we prove that an exchange ring R has stable range one if and only if for any $a, b \in R$, $R/aR \cong R/bR \Longrightarrow aR \cong bR$.

Throughout the paper, every ring is associative with an identity. A ring R is (unit) regular provided every element in R is (unit) regular. Let $r.ann(a) = \{r \in R;\ ar = 0\}$ and $l.ann(a) = \{r \in R;\ ra = 0\}$. We use $E(R)$ to denote the set of all idempotents in R and $U(R)$ to denote the set of all units in R.

2. Clean property

Theorem 2.1. *Let R be an exchange ring. Then the following assertions are equivalent:*

(1) *R has stable range one.*

(2) *For any regular $a \in R$, there exist an $e \in E(R)$ and a $u \in U(R)$ such that $a = e + u$ and $aR \cap eR = 0$.*

(3) *For any regular $a \in R$, there exist an $e \in E(R)$ and a $u \in U(R)$ such that $a = e + u$ and $Ra \cap Re = 0$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R$ be regular. Then we have $x \in R$ such that $a = axa$, and so $R = aR \oplus (1 - ax)R = xR \oplus r.ann(a)$. Clearly, $aR \cong axR$

has the finite exchange property. So there exist right R-modules X_1, Y_1 such that $R = aR \oplus X_1 \oplus Y_1$ with $X_1 \subseteq r.ann(a)$ and $Y_1 \subseteq xR$. It is easy to verify that $r.ann(a) = r.ann(a) \cap (X_1 \oplus aR \oplus Y_1) = X_1 \oplus X_2$, where $X_2 = r.ann(a) \cap (aR \oplus Y_1)$. Likewise, we have a right R-module Y_2 such that $xR = Y_1 \oplus Y_2$. Obviously, $a \in R$ is unit-regular; hence, $r.ann(a) \cong R/aR$. Thus $X_1 \oplus X_2 = r.ann(a) \cong R/aR \cong X_1 \oplus Y_1$, and so we have an isomorphism $k\colon X_1 \oplus X_2 \to X_1 \oplus Y_1$. Furthermore, X_1 can be cancelled from direct sums, and hence we get a right R-module isomorphism $\psi\colon X_2 \to Y_1$.

Let $h\colon R = X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 \to X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 = R$ be given by $h(x_1 + x_2 + y_1 + y_2) = k(x_1 + x_2) + y_1$ for any $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$. Let $v\colon R = X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 \to X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 = R$ be given by $v(x_1 + y_1 + x_2 + y_2) = k^{-1}(x_1 + y_1) + \psi(x_2)$ for any $x_1 \in X_1, y_1 \in Y_1, x_2 \in X_2, y_2 \in Y_2$. For any $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$, we have

$$
\begin{aligned}
hvh(x_1 + x_2 + y_1 + y_2) &= hv\big(k(x_1 + x_2) + y_1\big) \\
&= h\big(x_1 + x_2 + k^{-1}(y_1)\big) = k(x_1 + x_2) + y_1 \\
&= h(x_1 + x_2 + y_1 + y_2);
\end{aligned}
$$

hence $h = hvh$. Set $e = hv$. Then $e \in \mathrm{End}_R(R)$ is an idempotent.

Assume that $(a - hv)(x_1 + y_1 + x_2 + y_2) = 0$ for any $x_1 \in X_1, y_1 \in Y_1, x_2 \in X_2, y_2 \in Y_2$. Then

$$a(y_1 + y_2) = x_1 + y_1 + \psi(x_2) \in aR \cap (X_1 \oplus Y_1) = 0,$$

and consequently $x_1 = -y_1 - \psi(x_2) \in X_1 \cap Y_1 = 0$. It follows from $a(y_1 + y_2) = 0$ that $y_1 + y_2 \in (X_1 \oplus X_2) \cap (Y_1 \oplus Y_2) = 0$; hence $y_1 + y_2 = 0$. This infers that $y_1 = -y_2 \in Y_1 \cap Y_2 = 0$, and so $y_1 = y_2 = 0$. Furthermore, we get $\psi(x_2) = -y_1 = 0$. As ψ is an isomorphism, we have $x_2 = 0$. Thus $x_1 + y_1 + x_2 + y_2 = 0$. This means that $a - e \in R$ is a monomorphism.

Given any $t \in aR, x_1 \in X_1, y_1 \in Y_1$, we have $t \in aR = a(Y_1 \oplus Y_2)$. So we can find $y_1' \in Y_1$ and $y_2' \in Y_2$ such that $t = a(y_1' + y_2')$. Choose $x_1' = -x_1$ and $x_2' = -\psi^{-1}(y_1 + y_1')$. It is easy to verify that

$$
\begin{aligned}
(a - hv)(x_1' + x_2' + y_1' + y_2') &= a(y_1' + y_2') - \big(x_1' + y_1' + \psi(x_2')\big) \\
&= t - (-x_1 + y_1' - y_1 - y_1') = t + x_1 + y_1.
\end{aligned}
$$

This means that $a - hv\colon R \to R$ is an epimorphism, and then $a - hv$ is an isomorphism. Let $e = hv$ and $u = a - e$. Then $a = e + u$. In addition, we have $aR \cap eR \subseteq aR \cap (X_1 \oplus Y_1) = 0$. Hence $aR \cap eR = 0$.

(2) $\Rightarrow$ (1) Given any regular $a \in R$, there exist an $e \in E(R)$ and a $u \in U(R)$ such that $a = e + u$ and $aR \cap eR = 0$. As a result, $(au^{-1} - 1)a = eu^{-1}a \in aR \cap eR = 0$; hence, $a = au^{-1}a$. According to [5, Theorem 3], we complete the proof.

(1) $\Leftrightarrow$ (3) As stable range one condition is symmetric, we obtain the result by applying (1) $\Leftrightarrow$ (2) to the opposite ring R^{op}. $\square$

A ring R is said to have bounded index if there exists an integer n such that $x^n = 0$ for every nilpotent $x \in R$. Let R be an exchange ring of bounded index. We claim that for any regular $a \in R$, there exist an $e \in E(R)$ and a $u \in U(R)$ such that $a = e + u$ and $aR \cap eR = 0$. In view of [17, Theorem Corollary 4], R has stable range one, and we are done by Theorem 2.1.

Recall that a ring R is clean provided that every element in R is clean. It is well known that every clean ring is an exchange ring. Now we give a new proof of [14, Proposition 1.8] as follows.

Corollary 2.2. *Every exchange ring with all idempotents central is a clean ring.*

Proof. Let R be an exchange ring with all idempotents central, and let $a \in R$. By [14, Theorem 2.1], there exists an idempotent $e \in R$ such that $e = as$ and $1 - e = (1-a)t$ for some $s, t \in R$. Clearly, $ea = (ea)s(ea)$ and $(1-e)a = \big((1-e)a\big)t\big((1-e)a\big)$. In view of Theorem 2.1, we can find idempotents $f_1, f_2 \in R$ and units $u_1, u_2 \in R$ such that $ea = f_1 + u_1$ and $(1 - e)(1 - a) = f_2 + u_2$. It follows that

$$\begin{aligned} a &= ea + (1 - e)a \\ &= (ef_1 + eu_1) + ((1 - e) - (1 - e)f_2 - (1 - e)u_2) \\ &= (ef_1 + (1 - e)(1 - f_2)) + (eu_1 - (1 - e)u_2). \end{aligned}$$

Let $f = ef_1 + (1 - e)(1 - f_2)$ and $u = eu_1 - (1 - e)u_2$. Then $f = f^2$ and $u^{-1} = eu_1^{-1} - (1 - e)u_2^{-1}$, and therefore R is a clean ring. $\square$

We note that an exchange ring plus stable range one is a Morita invariant. Using this fact, we derive

Corollary 2.3. *Let R be an exchange ring and $\frac{1}{2} \in R$. If R has stable range one, then every regular square matrix over R is the sum of three invertible matrices.*

Proof. Since R is an exchange ring having stable range one, so is $M_n(R)$. Let $A \in M_n(R)$ be regular. In view of Theorem 2.1, there exist an idempotent $E \in M_n(R)$ and an invertible $U \in M_n(R)$ such that $A = E + U$. As $\frac{1}{2} \in R$, it follows that $E = \mathrm{diag}(\frac{1}{2}, \ldots, \frac{1}{2})_{n\times n} + \big(E - \mathrm{diag}(\frac{1}{2}, \ldots, \frac{1}{2})_{n\times n}\big)$. One easily checks that $\big(E - \mathrm{diag}(\frac{1}{2}, \ldots, \frac{1}{2})_{n\times n}\big)\big(4E - \mathrm{diag}(2, \ldots, 2)_{n\times n}\big) = I_n = \big(4E - \mathrm{diag}(2, \ldots, 2)_{n\times n}\big)\big(E - \mathrm{diag}(\frac{1}{2}, \ldots, \frac{1}{2})_{n\times n}\big)$. That is, $E - \mathrm{diag}(\frac{1}{2}, \ldots, \frac{1}{2})_{n\times n} \in M_n(R)$ is invertible. Therefore A is the sum of three invertible matrices, as asserted. $\square$

Example 2.4. Let R be a 2×2 matrix over $\mathbb{F}/(x^2)$, where $\mathbb{F}$ is a field. Clearly, R is a strongly π-regular ring; hence, it is an exchange ring having stable range one. Take $a = \left(\begin{smallmatrix} 0 & 1 \\ 0 & x \end{smallmatrix}\right)$. Then $a \in R$ is regular, while $a^2 \in R$ is not regular. In view of Theorem 2.1, there exist an $e \in E(R)$ and a $u \in U(R)$ such that $a = e + u$ with $aR \cap eR = 0$, while a^2 can not be written in this form.

Theorem 2.5. *If R is an exchange ring having stable range one, then every square matrix over R is an algebraic sum of idempotent matrices and invertible matrices.*

Proof. Let R be an exchange ring having stable range one, and let $S = M_n(R)$. Then S is an exchange ring having stable range one. Let $A \in S$. By [14, Theorem 2.1], there exists an idempotent $E \in S$ such that $E = AS$ and $I_n - E = (I_n - A)T$ for some $S, T \in S$. Analogously to Corollary 2.3, we see that both EA and $(I_n - E)A$ are regular. In view of Theorem 2.1, we can find idempotents $F_1, F_2 \in S$ and invertible $U_1, U_2 \in S$ such that $EA = F_1 + U_1$ and $(I_n - E)(I_n - A) = F_2 + U_2$. So we deduce that $A = EA + (I_n - E)A = F_1 + U_1 + (I_n - E) - F_2 - U_2$. This means that A is an algebraic sum of idempotent matrices and invertible matrices. $\square$

Let I be an ideal of a ring R. We say that I has stable range one provided $(1_R + a)R + bR = R$ with $a \in I, b \in R$ implies that there exists a $y \in R$ such that $1_R + a + by \in U(R)$.

Corollary 2.6. *Let R be a regular ring, and let $A = (a_{ij}) \in M_n(R)$. If each $Ra_{ij}R$ has stable range one, then A is an algebraic sum of idempotent matrices and invertible matrices.*

Proof. Let $I = \sum\limits_{1 \leqslant i,j \leqslant n} Ra_{ij}R$. Given $(1 + \sum\limits_{1 \leqslant i,j \leqslant n} r_{ij})x + b = 1$ with $x, b \in R$ and each $r_{ij} \in Ra_{ij}R$, then $(1 + r_{11})x + \Big(\sum\limits_{1 \leqslant i,j \leqslant n, i \neq 1} r_{ij}\Big)x + b = 1$. As $Ra_{11}R$ has stable range one, we can find $y_{11} \in R$ such that $x + y_{11}\Big(\sum\limits_{1 \leqslant i,j \leqslant n, i \neq 1} r_{ij}\Big)x + y_{11}b = u_1 \in U(R)$. Let $r'_{ij} = y_{11}r_{ij}$. Then $r'_{ij} \in Ra_{ij}R$ and $\Big(1 + \sum\limits_{1 \leqslant i,j \leqslant n, i \neq 1} r'_{ij}\Big)(xu_1) + bu_1 = 1$. Likewise, we prove that $(1 + r'_{nn})xu_1u_2 \ldots u_{nn} + bu_1u_2 \ldots u_{nn} = 1$ for some $u_2, \ldots, u_n \in U(R)$. As $Ra_{nn}R$ has stable range, we have $z \in R$ such that $xu_1u_2 \ldots u_{nn} + zbu_1u_2 \ldots u_{nn} \in U(R)$. Thus $x + zb \in U(R)$, and so I has stable range one. Clearly, each $a_{ij} \in I$. Furthermore, there exists an idempotent $e \in I$ such that each $a_{ij} \in eRe$; hence $A \in M_n(eRe)$. Clearly, eRe is unit-regular. It follows by Theorem 2.5 that A is an algebraic sum of idempotent matrices and invertible matrices over eRe. Let $U \in M_n(eRe)$ be invertible. Then we have $V \in M_n(eRe)$ such that $UV = \operatorname{diag}(e, e, \ldots, e)_{n \times n}$. Hence $\big(U + \operatorname{diag}(1 - e, 1 - e, \ldots,$

$1-e)_{n\times n}\big)\big(V+\mathrm{diag}(1-e,1-e,\dots,1-e)_{n\times n}\big)=I_n$. In other words, $U+\mathrm{diag}(1-e,1-e,\dots,1-e)_{n\times n}\in M_n(R)$ is invertible, and so U is an algebraic sum of an idempotent matrix and an invertible matrix over R. Therefore A is an algebraic sum of idempotent matrices and invertible matrices over R, as asserted. $\square$

Recall that an ideal I of a ring R is of bounded index if there is a positive integer n such that $x^n=0$ for any nilpotent $x\in I$.

Corollary 2.7. *Let R be a regular ring, and let $A=(a_{ij})\in M_n(R)$. If each $Ra_{ij}R$ is of bounded index, then A is an algebraic sum of idempotent matrices and invertible matrices.*

Proof. For any idempotent $e\in Ra_{ij}R$ we have $eRe\subseteq Ra_{ij}R$. Hence eRe is a regular ring of bounded index. In view of [9, Corollary 7.11], eRe is unit-regular. This shows that $Ra_{ij}R$ has stable range one, and therefore we complete the proof by Corollary 2.6. $\square$

3. Extensions

Let I be a right ideal of a ring R. We say that $a\in R$ is a right unit modulo I provided $ab\equiv 1(\mathrm{mod}\, I)$. Now we extend this result as follows.

Lemma 3.1. *Let R be an exchange ring. Then the following conditions are equivalent:*
(1) *R has stable range one.*
(2) *Every right unit lifts modulo I any right ideal of R.*
(3) *Every left unit lifts modulo I any left ideal of R.*

Proof. (1) $\Rightarrow$ (2) Let I be a right ideal of R, and let $a\in R$ be a right unit modulo I. Then there exists $b\in R$ such that $ab\equiv 1(\mathrm{mod}\, I)$. Hence we can find an $r\in I$ such that $ab+r=1$. Since R has stable range one, we can find $c\in R$ such that $a+rc\in U(R)$. Set $u=a+rc$. Then $a-u=r(-c)\in I$. That is, $a\equiv u(\mathrm{mod}\, I)$, as desired.

(2) $\Rightarrow$ (1) Given $ab+c=1$ in R, then $ab-1\in cR$. This means that $ab\equiv 1(\mathrm{mod}\, cR)$. By hypothesis, there exists a right unit $u\in R$ such that $a-u\in cR$. So we can find an $r\in R$ such that $a+cr=u\in R$. As $u\in R$ is a right unit, there is $v\in R$ such that $uv=1$. Since $vu+(1-vu)=1$, by the above consideration we have $s\in R$ such that $v+(1-vu)s=t\in U(R)$ is a right unit. Clearly, $ut=u\big(v+(1-vu)s\big)=1$; hence, $t\in R$ is a left unit. Thus $t\in U(R)$. This implies that $u\in U(R)$. That is, $a+cr\in U(R)$. Therefore R has stable range one.

(1) $\Leftrightarrow$ (3) is symmetric. $\square$

We say that $b \in R$ is a reflexive inverse of $a \in R$ if $a = aba$ and $b = bab$, and denote b by a^+. Clearly, every regular element has a reflexive element. Using such elements, we give a new characterization of exchange rings having stable range one.

Theorem 3.2. *Let R be an exchange ring. Then the following conditions are equivalent:*
(1) *R has stable range one.*
(2) *For any regular $a \in R$, there exist $e \in r.ann(a^+)$ and $u \in U(R)$ such that $a = e + u$.*
(3) *For any regular $a \in R$, there exist $e \in l.ann(a^+)$ and $u \in U(R)$ such that $a = e + u$.*

P r o o f. (1) $\Rightarrow$ (2) Given any regular $a \in R$, there exists a^+ such that $a = aa^+a$ and $a^+ = a^+aa^+$. Hence $a^+(aa^+ - 1) = 0$. That is, $aa^+ \equiv 1(\operatorname{mod} r.ann(a^+))$. By virtue of Lemma 3.1, we can find a right unit $u \in R$ such that $a - u \in r.ann(a^+)$. Thus there exists $e \in r.ann(a^+)$ such that $a = e + u$. As R has stable range one, it is directly finite. This infers that $u \in U(R)$, as required.

(2) $\Rightarrow$ (1) Let $a \in R$ be regular. Then $a = aa^+a$ and $a^+ = a^+aa^+$. By assumption, there exist $e \in r.ann(a)$ and $u \in U(R)$ such that $a^+ = e+u$; hence, $a^+-u \in r.ann(a)$. As a result, $a(a^+ - u) = 0$. This implies that $a = aa^+a = aua$. That is, $a \in R$ is unit-regular. Consequently, R has stable range one by [5, Theorem 3].

(1) $\Leftrightarrow$ (3) Since R is an exchange ring having stable range one if and only if so is the opposite ring R^{op}, the result follows by symmetry. □

Corollary 3.3. *Let R be an exchange ring of bounded index. Then the following assertions hold:*
(1) *For any regular $a \in R$, there exist $e \in r.ann(a^+)$ and $u \in U(R)$ such that $a = e + u$.*
(2) *For any regular $a \in R$, there exist $e \in l.ann(a^+)$ and $u \in U(R)$ such that $a = e + u$.*

P r o o f. In view of [17, Corollary 4], R has stable range one. So the proof follows by Theorem 3.2. □

Recall that a ring R is strongly π-regular provided that for any $a \in R$ there exists a positive integer $n(a)$ such that $a^{n(a)} \in a^{n(a)+1}R$.

Corollary 3.4. *Let R be a strongly π-regular ring. Then the following assertions hold:*

(1) *For any regular $a \in R$, there exist $e \in r.ann(a^+)$ and $u \in U(R)$ such that $a = e + u$.*

(2) *For any regular $a \in R$, there exist $e \in l.ann(a^+)$ and $u \in U(R)$ such that $a = e + u$.*

Proof. In view of [2, Theorem 4], R is an exchange ring having stable range one. Therefore we complete the proof by Theorem 3.2. $\square$

A regular ring R is abelian provided that every idempotent in R is central.

Corollary 3.5. *Let R be a ring. Then the following assertions are equivalent:*

(1) *R is an abelian regular ring.*

(2) *For any $a \in R$, there exist $e \in r.ann(a)$ and $u \in U(R)$ such that $a = e + u$.*

(3) *For any $a \in R$, there exist $e \in l.ann(a)$ and $u \in U(R)$ such that $a = e + u$.*

Proof. (1) $\Rightarrow$ (2) Let R be an abelian regular ring. Then it is an exchange ring having stable range one by [17, Theorem 6]. For any $a \in R$, there exists $a^+ \in R$ such that $a = aa^+a$ and $a^+ = a^+aa^+$. As every idempotent in R is central, one checks that $r.ann(a^+) = r.ann(a)$. In view of Theorem 3.2, we can find $e \in r.ann(a)$ and $u \in U(R)$ such that $a = e + u$, as desired.

(2) $\Rightarrow$ (1) Given any $a \in R$, there exist $e \in r.ann(a)$ and $u \in U(R)$ such that $a = e + u$. Hence $a - u \in r.ann(a)$, and then $a(a - u) = 0$. This implies that $a = a^2u^{-1}$. According to [9, Theorem 3.5], R is an abelian regular ring.

(1) $\Leftrightarrow$ (3) is obtained by symmetry. $\square$

4. Cokernels

In [7, Theorem 14], the author proved that a regular ring R is unit-regular if and only if whenever $aR \cong bR$, then there exist $u, v \in R$ such that $a = ubv$. In this section, we characterize exchange rings having stable range one by cokernels of their elements, which is also a generalization of [10, Theorem 2.1].

Theorem 4.1. *Let R be an exchange ring. Then the following conditions are equivalent:*

(1) *R has stable range one.*

(2) *For any $a, b \in R$, $R/aR \cong R/bR$ implies that there exist $u, v \in U(R)$ such that $a = ubv$.*

(3) *For any $a, b \in R$, $R/Ra \cong R/Rb$ implies that there exist $u, v \in U(R)$ such that $a = ubv$.*

Proof. (1) $\Rightarrow$ (2) Since $\varphi\colon R/aR \cong R/bR$, there exists a $c \in R$ such that $\varphi(1 + aR) = c + bR$. So $R + bR = cR + bR$; hence, $R = cR + bR$. Since R has stable range one, there exists a $d \in R$ such that $c + bd = u \in U(R)$. Clearly, $bR = \varphi(aR) = \varphi(aR+aR) = caR+bR$, and then $caR \subseteq bR$. Furthermore, $uaR \subseteq bR$. On the other hand, we have $\varphi(1 + aR) = (c + bd) + bR = u + bR$. It follows that $\varphi^{-1}(1+bR) = u^{-1}+aR$. This implies that $u^{-1}b+aR = (u^{-1}+aR)b = \varphi^{-1}(1+bR)b = \varphi^{-1}(bR) = aR$. Hence $u^{-1}bR \subseteq aR$, and then $bR \subseteq uaR$. Thus we can find $x, y \in R$ such that $ua = bx$ and $b = uay$. Since R has stable range one, it follows from $xy + (1 - xy) = 1$ that there exists a $z \in R$ such that $x + (1 - xy)z = v \in U(R)$. Thus we deduce that $bx = b(x + (1 - xy)z) = bv$. As a result, we prove that $a = u^{-1}bx = u^{-1}bv$, as desired.

(2) $\Rightarrow$ (1) Given $eR \cong fR$ with idempotents $e, f \in R$, we have $R/(1 - e)R \cong R/(1 - f)R$. By assumption, there exist $u, v \in R$ such that $1 - e = u(1 - f)v$. Let $y = u(1 - f)u^{-1}$. Then $y(1 - e) = u(1 - f)u^{-1}(1 - e) = u(1 - f)v = 1 - e$ and $y = u(1 - f)u^{-1} = u(1 - f)v(v^{-1}u^{-1}) = (1 - e)v^{-1}u^{-1}$. Hence $(1 - e)y = y$. As a result, we prove that $(e + y)^{-1} = 2 - e - y$. Set $w = (e + y)u$. Then $w \in U(R)$. Furthermore, one easily checks that

$$\begin{aligned} w(1 - f)w^{-1} &= (e + y)u(1 - f)u^{-1}(2 - e - y) = (e + y)y(2 - e - y) \\ &= y(2 - e - y) = y - ye = 1 - e. \end{aligned}$$

This implies that $e = wfw^{-1}$. In view of [17, Theorem 10], we prove that R has stable range one.

(1) $\Leftrightarrow$ (3) is obtained by symmetry. $\square$

In the proof of Theorem 4.1, we prove that an exchange ring R has stable range one if and only if for any regular $a, b \in R$, $R/aR \cong R/bR$ implies that there exist $u, v \in U(R)$ such that $a = ubv$ if and only if for any regular $a, b \in R$, $R/Ra \cong R/Rb$ implies that there exist $u, v \in U(R)$ such that $a = ubv$. We note that the condition (1) and (2) above are not equivalent for some non-exchange rings. In [6, Example 6.7], Canfell supplied a principal ideal domain R which has elements a and b for which $R/aR \cong R/bR$ but $a \neq ubv$ for any $u, v \in U(R)$.

Corollary 4.2. *Let R be an exchange ring. Then the following assertions are equivalent:*

(1) *R has stable range one.*

(2) *For any regular $a, b \in R$, $r.ann(a) \cong r.ann(b)$ implies that there exist $u, v \in U(R)$ such that $a = ubv$.*

(3) *For any regular $a, b \in R$, $l.ann(a) \cong l.ann(b)$ implies that there exist $u, v \in U(R)$ such that $a = ubv$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $r.ann(a) \cong r.ann(b)$ with regular $a, b \in R$. Then there exist $x, y \in R$ such that $a = axa$ and $b = byb$. Hence $(1 - xa)R = r.ann(a) \cong r.ann(b) = (1 - yb)R$. As $1 - xa, 1 - ya \in R$ are idempotents, it follows that $R(1 - xa) \cong R(1 - yb)$. Clearly, $R(1 - xa) \cong R/Rxa$ and $R(1 - yb) \cong R/ybR$. As a result, $R/Ra \cong R/Rb$. In view of Theorem 4.1, we can find $u, v \in U(R)$ such that $a = ubv$.

(2) $\Rightarrow$ (1) Given $eR \cong fR$ with idempotents $e, f \in R$, we have $r.ann(1 - e) \cong r.ann(1 - f)$. By assumption, we can find $u, v \in U(R)$ such that $1 - e = u(1 - f)v$. Analogously to Theorem 4.1, we have a $w \in U(R)$ such that $1 - e = w(1 - f)w^{-1}$. Thus $1 - e = ab$ and $1 - f = ba$, where $a = (1 - e)w(1 - f) \in (1 - e)R(1 - f)$ and $b = (1 - f)w^{-1}(1 - e) \in (1 - f)R(1 - e)$. This implies that $(1 - e)R \cong (1 - f)R$. Using [17, Theorem 10], we prove that R is unit-regular.

(1) $\Leftrightarrow$ (3) is symmetric. □

A regular ring is unit-regular if and only if it has stable range one (cf. [9, Proposition 4.12]). It follows by Corollary 4.2 that a regular ring is unit-regular if and only if $r.ann(a) \cong r.ann(b)$ implies that there exist $u, v \in U(R)$ such that $a = ubv$ if and only if $l.ann(a) \cong l.ann(b)$ implies that there exist $u, v \in U(R)$ such that $a = ubv$.

Example 4.3. Let V be an infinite-dimensional vector space over a division ring D, and let $R = \mathrm{End}_D(V)$. Then R is an exchange ring but it has stable range ∞. Using Corollary 4.1, the condition (2) above doesn't hold. Let $\{x_1, x_2, \ldots\}$ be a basis of V. Define $\sigma\colon V \to V$ by $\sigma(x_i) = x_{i+1}$ for $i = 1, 2, 3, \ldots$. Let $\tau\colon V \to V$ be the identity map. Define $\varrho\colon V \to V$ given by $\tau(x_1) = 0$ and $\varrho(x_i) = x_{i-1} (i = 2, 3, \ldots, n, \ldots)$. Then $\varrho\sigma = 1_V$ and $\sigma\varrho \neq 1_V$. Thus σ and τ are both regular and $r.ann(\sigma) \cong r.ann(\tau)$, while $\sigma \neq u\tau v$ for any automorphisms u and v.

Corollary 4.4. *Let R be an exchange ring having stable range one, and let $a, b \in R$. Then the following conditions are equivalent:*

(1) *$\varphi\colon aR \cong bR$ and $\varphi(a) = ua$ for a $u \in U(R)$.*

(2) *There exist $v, w \in U(R)$ such that $a = vbw$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $\varphi\colon aR \cong bR$ and $\varphi(a) = ua$ for a $u \in U(R)$. Let $\psi\colon R \to R$ be given by $\psi(r) = ur$ for any $r \in R$. Then ψ is an automorphism.

So we have $\varphi\colon\ R/aR \to R/aR$ such that the following diagram commutates.

$$\begin{tikzcd}
0 \arrow[r] & aR \arrow[r, hook] \arrow[d, "\varphi"] & R \arrow[r] \arrow[d, "\psi"] & R/aR \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & bR \arrow[r, hook] & R \arrow[r] & R/bR \arrow[r] & 0.
\end{tikzcd}$$

Since both φ and ψ are isomorphic, so is φ. That is, $R/aR \cong R/bR$. According to Theorem 4.1, we prove that $a = vbw$ for some $v, w \in U(R)$.

(2) $\Rightarrow$ (1) Suppose that $a = vbw$ with $v, w \in U(R)$. Construct a map $\varphi\colon\ aR \to bR$ given by $\varphi(ar) = v^{-1}(ar)$ for any $r \in R$. It is easy to verify that $\varphi\colon\ aR \cong bR$. In addition, $\varphi(a) = v^{-1}a$, and thus we complete the proof. $\square$

It is easy to check that a regular ring R is unit-regular if and only if for any $a, b \in R$, $aR \cong bR \Longrightarrow R/aR \cong R/bR$. In contrast to this fact, we derive

Theorem 4.5. *Let R be an exchange ring. Then the following conditions are equivalent:*

(1) *R has stable range one.*
(2) *For any $a, b \in R$, $R/aR \cong R/bR \Longrightarrow aR \cong bR$.*
(3) *For any $a, b \in R$, $R/Ra \cong R/Rb \Longrightarrow Ra \cong Rb$.*

Proof. (1) $\Rightarrow$ (2) Given $R/aR \cong R/bR$, it follows by Theorem 4.1 that there exist $u, v \in U(R)$ such that $a = ubv$. Construct a map $\varphi\colon\ aR \to bR$ given by $\varphi(ar) = u^{-1}(ar)$ for any $r \in R$. Then $\varphi\colon\ aR \cong bR$, as asserted.

(2) $\Rightarrow$ (1) Given $eR \cong fR$ with idempotents $e, f \in R$, then $R/(1-e)R \cong R/(1-f)R$. By hypothesis, we get $(1-e)R \cong (1-f)R$. Using [17, Theorem 10], we prove that R is unit-regular.

(1) $\Leftrightarrow$ (3) is symmetric. $\square$

As an immediate consequence of Theorem 4.5, we deduce that an exchange ring R has stable range one if and only if for any regular $a, b \in R$, $R/aR \cong R/bR \Longrightarrow aR \cong bR$ if and only if for any regular $a, b \in R$, $R/Ra \cong R/Rb \Longrightarrow Ra \cong Rb$.

Corollary 4.6. *Let R be a regular ring. Then the following conditions are equivalent:*

(1) *R is unit-regular.*
(2) *For any $a, b \in R$, $R/aR \cong R/bR \Longleftrightarrow aR \cong bR$.*
(3) *For any $a, b \in R$, $R/Ra \cong R/Rb \Longleftrightarrow Ra \cong Rb$.*

Proof. (1) $\Rightarrow$ (2) For any $a, b \in R$, $R/aR \cong R/bR \Longrightarrow aR \cong bR$ by Theorem 4.5. Conversely, assume that $aR \cong bR$. Then we can find idempotents

$e, f \in R$ such that $aR = eR$ and $bR = fR$. In view of [9, Theorem 4.14], we have $(1-e)R \cong (1-f)R$; hence, $R/eR \cong R/fR$. As a result, we prove that $R/aR \cong R/bR$.

(2) $\Rightarrow$ (1) is clear by Theorem 4.5.

(1) $\Leftrightarrow$ (3) is symmetric. □

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