# SUBDIRECT PRODUCTS OF CERTAIN VARIETIES OF UNARY ALGEBRAS

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*Abstract.* J. Płonka in [12] noted that one could expect that the regularization $\mathscr{R}(\boldsymbol{K})$ of a variety $\boldsymbol{K}$ of unary algebras is a subdirect product of $\boldsymbol{K}$ and the variety $\boldsymbol{D}$ of all discrete algebras (unary semilattices), but is not the case. The purpose of this note is to show that his expectation is fulfilled for those and only those irregular varieties $\boldsymbol{K}$ which are contained in the generalized variety $\boldsymbol{TDir}$ of the so-called trap-directable algebras.

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The basic algebraic notions are defined here as for algebras in general (cf. [6], [9], for example), but we reformulate some of them, to fit them into specific notation which comes from the theory of automata. In what follows, $X$ is always a nonempty alphabet, $X^*$ denotes the free monoid over $X$, and $e$ denotes its identity. An *algebra of type* $X$, or an $X$*-algebra*, is a system $A = (A, X)$ where $A$ is a nonempty set and every symbol $x \in X$ is realized as a unary operation $x^A\colon A \to A$. For any $a \in A$ and $x \in X$, we write $ax^A$ for $x^A(a)$. For any word $w = x_1x_2\ldots x_n \in X^*$, $w^A\colon A \to A$ is defined as the composition of the mappings $x_1^A, x_2^A, \ldots, x_n^A$, that is to say, $aw^A = ax_1^Ax_2^A\ldots x_n^A$ for any $a \in A$. In particular, $e^A$ is the identity mapping of $A$. If $A$ is known from the context, we write simply $aw$ instead of $aw^A$.

We define *terms* of type $X$ over a set $V$ of *variables* as expressions of the form $gu$, where $g \in V$ and $u \in X^*$, and we denote by $T_X(V)$ the set of all such terms. The *term* $X$*-algebra* $T_X(V) = (T_X(V), X)$ is defined so that $(gu)x = g(ux)$ for all $gu \in T_X(V)$ and $x \in X$ (see § 1.6 of [8]). An *identity* of type $X$ over $V$ is an expression $gu \approx hv$,

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where $gu, hv \in T_X(V)$. An $X$-algebra $A$ is said to *satisfy* an identity $gu \approx hv$ if $(g\alpha)u^A = (h\alpha)v^A$ for all valuations $\alpha\colon V \to A$ of the variables. Identities of the form $gu \approx gv$ are called *regular*, whereas identities of the form $gu \approx hv$, with $g \neq h$, are *irregular*. A variety of $X$-algebras is *regular* if it is determined by a set of regular identities, otherwise it is *irregular*. A class of $X$-algebras is said to be a *generalized variety* if it is closed under subalgebras, homomorphic images, finite direct products and arbitrary direct powers, or equivalently, if it is a directed union of varieties (see [1], [2], [3]).

In the paper we consider only algebras of a fixed type $X$, and for brevity we simply say '*algebra*' instead of '*$X$-algebra*'. The least subalgebra of an algebra $A$, if it exists, is called the *kernel* of $A$, and if $A$ has a least nontrivial subalgebra, it is called the *core* of $A$. The *monogenic subalgebra* of $A$ generated by $a \in A$ is denoted by $\langle a \rangle$. It is obvious that $\langle a \rangle = \{aw\colon\ w \in X^*\}$. An element $a \in A$ is called a *trap* if $ax = a$, for every $x \in X$. An algebra is called *discrete* if all of its elements are traps. For a set $H$, $\Delta_H$ and $\nabla_H$ denote respectively the *diagonal* and the *universal* relation on $H$. The *Rees congruence* $\varrho_B$ on an algebra $A$ modulo a subalgebra $B$ of $A$ is defined by $\varrho_B = \nabla_B \cup \Delta_A$. The corresponding *Rees quotient* $A/\varrho_B$ is denoted by $A/B$, and $A$ is said to be an *extension* of $B$ by an algebra $C$ if $A/B \cong C$. If this is the case, $C$ evidently has a trap which corresponds to the image of $B$ under the canonical epimorphism of $A$ onto $A/B$. In other words, we may regard $C$ as the result of contracting the subalgebra $B$ of $A$ into one element, a trap of' $C$. A *trap-extension* of an algebra is obtained by adjoining to it a trap, that is to say, $A$ is a trap-extension of $B$ if it is an extension of $B$ by a two-element discrete algebra. If $B$ is a subalgebra of $A$, a congruence $\theta$ on $A$ is called a *$B$-congruence* if $\theta \cap \nabla_B = \Delta_B$, and if $\Delta_A$ is the only $B$-congruence on $A$, we say that $A$ is a *dense extension* of $B$. In particular, every algebra is a dense extension of itself.

An algebra $A$ is *connected* if for all $a, b \in A$ there exist $u, v \in X^*$ such that $au = bv$, and it is *strongly connected* if for all $a, b \in A$ there exists $u \in X^*$ such that $au = b$. Obviously, $A$ is strongly connected if and only if $\langle a \rangle = A$, for every $a \in A$. A connected algebra can have at most one trap, and if it has a trap, it is called *trap-connected.* Furthermore, a nontrivial algebra $A$ is *strongly trap-connected* if it has a trap $a_0$ and $\langle a \rangle = A$, for every $a \in A \setminus \{a_0\}$. Every strongly trap-connected algebra is trap-connected, but the converse does not hold. An algebra $A$ is *directable* if there exists a word $u \in X^*$ such that $au = bu$, for every pair of elements $a, b \in A$. Any directable algebra is connected, so it can have at most one trap, and if it has a trap, it is called *trap-directable.* Let $\boldsymbol{Dir}$, $\boldsymbol{TDir}$ and $\boldsymbol{D}$ denote respectively the classes of all directable, trap-directable and discrete algebras. It is known that $\boldsymbol{D}$ is a variety and $\boldsymbol{Dir}$ and $\boldsymbol{TDir}$ are generalized varieties (see [2], [5], [10]). Moreover, a variety $\boldsymbol{K}$ of unary algebras is regular if and only if it contains $\boldsymbol{D}$, and it is irregular if and only if it is contained in $\boldsymbol{Dir}$ (cf. [2], [5], [10]).

An algebra $A$ is the *direct sum* of algebras $A_\alpha$, $\alpha \in Y$, if $A = \bigcup_{\alpha\in Y} A_\alpha$ and $A_\alpha \cap A_\beta = \emptyset$, for all $\alpha, \beta \in Y$ such that $\alpha \neq \beta$. If $A$ can not be decomposed into a direct sum of two or more algebras, then it is *direct sum indecomposable*. For more information on direct sum decompositions we refer to [7]. Let $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ be two classes of algebras. The *subdirect product* $\boldsymbol{K}_1 \otimes \boldsymbol{K}_2$ of $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ is the class of all subdirect products of an algebra from $\boldsymbol{K}_1$ and an algebra from $\boldsymbol{K}_2$, and the *Mal'cev product* $\boldsymbol{K}_1 \circ \boldsymbol{K}_2$ is the class of all algebras $A$ which have a congruence $\varrho$ such that $A/\varrho \in \boldsymbol{K}_2$ and every $\varrho$-class which is a subalgebra of $A$ belongs to $\boldsymbol{K}_1$. In particular, $\boldsymbol{K} \circ \boldsymbol{D}$ is the class of all direct sums of algebras from $\boldsymbol{K}$.

J. Płonka in [11], [12] studied the regularization operator $\mathscr{R}\colon \boldsymbol{K} \mapsto \mathscr{R}(\boldsymbol{K})$ on the lattice of varieties of unary algebras and proved, among other things, that

$$\mathscr{R}(\boldsymbol{K}) = \boldsymbol{K} \vee \boldsymbol{D} = \boldsymbol{K} \circ \boldsymbol{D}$$

(for some related results we refer to [2]). He also noted in [12] that one could expect that $\mathscr{R}(\boldsymbol{K}) = \boldsymbol{K} \otimes \boldsymbol{D}$, but is not the case. In terminology from the theory of automata, in the example which confirms this note he assumed $\boldsymbol{K}$ to be the variety of *reset* or 1-*definite* algebras, and $A$ to be a trap-extension of a two-element reset algebra, and showed that $A$ belongs to $\mathscr{R}(\boldsymbol{K})$, but does not belong to $\boldsymbol{K} \otimes \boldsymbol{D}$.

In this paper we show that a considerably large class of varieties of unary algebras fulfills the Płonka's expectation. Namely, for an irregular variety $\boldsymbol{K}$ of unary algebras[1] we prove that $\mathscr{R}(\boldsymbol{K}) = \boldsymbol{K} \otimes \boldsymbol{D}$ if and only if $\boldsymbol{K} \subseteq \boldsymbol{TDir}$. For that purpose we use a lot of specific notions which come from the theory of automata (cf. [2], [4], [5], [7], [10]), and a general characterization of subdirectly irreducible unary algebras from [4]. This is the following result:

**Theorem 1.** *A nontrivial algebra $A$ is subdirectly irreducible if and only if it is a dense extension of a nontrivial subdirectly irreducible subalgebra $B$ by a trap-connected algebra and this $B$ satisfies one of the following conditions:*

(C0) *$B$ is the core of $A$ and strongly connected;*

(C1) *$B$ is the core of $A$ and strongly trap-connected, or $B$ is a trap-extension of the core of $A$ and the core is strongly connected;*

(C2) *$B$ is the core of $A$ and a two-element discrete algebra.*

*Moreover, for each $k = 0, 1, 2$, $B$ satisfies the condition* (Ck) *if and only if $A$ has exactly $k$ traps.*

[1] Contrary to Płonka, who studied algebras having both unary and nullary fundamental operations, here we consider only algebras all of whose operations are unary.

We also need the following lemma.

**Lemma 1.** *Let $A'$ be a trap-extension of an algebra $A$. Then $A'$ is a dense extension of $A$ if and only if $A$ does not have a trap.*

P r o o f. Let $a \in A' \setminus A$ be the trap adjoined to $A$. We shall prove that $A'$ is not a dense extension of $A$ if and only if $A$ has a trap.

Suppose that $A'$ is not a dense extension of $A$, i.e., there exists an $A$-congruence $\theta$ on $A'$ different than $\Delta_{A'}$. Then there exists $(b, c) \in \theta$ such that $b \neq c$, and since $\theta$ is an $A$-congruence, then one of $b$ and $c$, say $c$, must be equal to $a$. For every $x \in X$ we have that $(ax, bx) \in \theta$, i.e., $(a, bx) \in \theta$, which together with $(b, a) \in \theta$ yields

$$(b, bx) \in \theta \cap \nabla_A = \Delta_A.$$

Therefore, $b = bx$, and we have obtained that $A$ has a trap $b$.

Conversely, suppose that $A$ has a trap $b$. Then $C = \{a, b\}$ is a subalgebra of $A'$ and the Rees congruence on $A'$ modulo $C$ is an $A$-congruence on $A'$ different than $\Delta_{A'}$, so $A'$ can not be a dense extension of $A$. □

Recall that every irregular variety of unary algebras is contained in the generalized variety $\boldsymbol{Dir}$ of all directable automata (Corollary 5.1 of [2]).

**Theorem 2.** *Let $\boldsymbol{K}$ be an irregular variety of algebras. Then the following conditions are equivalent:*

(i) $\boldsymbol{K} \subseteq \boldsymbol{TDir}$;

(ii) *$\boldsymbol{K}$ does not contain a nontrivial strongly connected algebra;*

(iii) *$\boldsymbol{K}$ does not contain a nontrivial subdirectly irreducible strongly connected algebra.*

P r o o f. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious, and it remains to prove the implication (iii) $\Rightarrow$ (i).

Suppose that (iii) holds. Let $A \in \boldsymbol{K}$ be any nontrivial subdirectly irreducible algebra. Then $A$ has a nontrivial subdirectly irreducible subalgebra $B$ which satisfies one of the conditions (C0), (C1) and (C2) of Theorem 1. We can immediately exclude the case (C2), since $A$ is directable and can not have two different traps, whereas the case (C0) is excluded by our hypothesis (iii), because $B \in \boldsymbol{K}$. Therefore, $B$ must satisfy (C1), and we conclude that $A$ has a trap. Having in mind that $A$ is directable, the existence of a trap in $A$ implies $A \in \boldsymbol{TDir}$. Hence, we have proved that every subdirectly irreducible algebra from $\boldsymbol{K}$ belongs to $\boldsymbol{TDir}$.

Further, consider an arbitrary algebra $A \in \boldsymbol{K}$. Then $A$ is a subdirect product of subdirectly irreducible algebras $A_i$, $i \in I$, and evidently, $A_i \in \boldsymbol{K}$, and hence

$A_i \in \boldsymbol{TDir}$, for every $i \in I$. Let $P$' be the direct product of the algebras $A_i$, $i \in I$. Since every $A_i$ has exactly one trap, $P$ also has exactly one trap. On the other hand, $P \in \boldsymbol{K} \subseteq \boldsymbol{Dir}$, so we conclude that $P \in \boldsymbol{TDir}$, and now $A \in \boldsymbol{TDir}$, as a subalgebra of $P$. Therefore, we have proved (i). This completes the proof of the theorem. $\square$

Note that a variety $\boldsymbol{K}$ is contained in $\boldsymbol{TDir}$ if and only if it satisfies a set of identities $\{gux \approx hu\colon\ x \in X\}$, for some $u \in X^*$ (see [10] or [5]).

Now we are ready to state and prove the main theorem of the paper.

**Theorem 3.** *Let $\boldsymbol{K}$ be an irregular variety of algebras. Then*

$$\boldsymbol{K} \vee \boldsymbol{D} = \boldsymbol{K} \otimes \boldsymbol{D} \Longleftrightarrow \boldsymbol{K} \subseteq \boldsymbol{TDir}.$$

Proof. Let $\boldsymbol{K} \vee \boldsymbol{D} = \boldsymbol{K} \otimes \boldsymbol{D}$. Suppose that $\boldsymbol{K} \nsubseteq \boldsymbol{TDir}$. Then by Theorem 2, there exists a nontrivial subdirectly irreducible strongly connected algebra $A \in \boldsymbol{K}$. Since $A$ is a nontrivial strongly connected algebra, it has no trap, thus $A \in \boldsymbol{K} \backslash \boldsymbol{TDir}$. According to Lemma 1, $A'$ is a dense extension of $A$. Now, by Theorem 5.1 of [4] it follows that $A'$ is subdirectly irreducible. On the other hand, $A'$ is a direct sum of $A$ and a one-element algebra, which both belong to $\boldsymbol{K}$, so our starting hypothesis yields

$$A' \in \boldsymbol{K} \circ \boldsymbol{D} = \boldsymbol{K} \vee \boldsymbol{D} = \boldsymbol{K} \otimes \boldsymbol{D}.$$

But, $A' \in \boldsymbol{K} \otimes \boldsymbol{D}$ and subdirect irreducibility of $A'$ imply that

$$A' \in \boldsymbol{D} \quad \text{or} \quad A' \in \boldsymbol{K},$$

which is not true, because $A'$ is neither discrete nor directable algebra. Therefore, we conclude that $\boldsymbol{K} \subseteq \boldsymbol{TDir}$.

Conversely, let $\boldsymbol{K} \subseteq \boldsymbol{TDir}$. Since every algebra from $\boldsymbol{K} \vee \boldsymbol{D}$ is a subdirect product of subdirectly irreducible algebras from $\boldsymbol{K} \vee \boldsymbol{D}$, it is enough to prove that every subdirectly irreducible algebra from $\boldsymbol{K} \vee \boldsymbol{D}$ belongs either to $\boldsymbol{K}$ or to $\boldsymbol{D}$.

Let $A \in \boldsymbol{K} \vee \boldsymbol{D} = \boldsymbol{K} \circ \boldsymbol{D}$ be an arbitrary subdirectly irreducible algebra. Then $A$ is a direct sum of algebras $A_\alpha$, $\alpha \in Y$, where $A_\alpha \in \boldsymbol{K} \subseteq \boldsymbol{TDir}$, for each $\alpha \in Y$. This means that every $A_\alpha$ has exactly one trap, and by Theorem 1, $|Y| \leqslant 2$. If $|Y| = 2$, then Theorem 1 says that $A$ has exactly two traps $a_1$ and $a_2$, and $B = \{a_1, a_2\}$ is the core of $A$. If $B \neq A$, then $A$ is connected and direct sum indecomposable, which contradicts the hypothesis $|Y| = 2$. Thus, we conclude that $A$ must be a two-element discrete algebra, and hence $A \in \boldsymbol{D}$. Finally, if $|Y| = 1$, then clearly $A \in \boldsymbol{K}$. This completes the proof of the theorem. $\square$

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