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SOME INEQUALITIES INVOLVING UPPER BOUNDS FOR SOME MATRIX OPERATORS I

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Abstract. In this paper we consider the problem of finding upper bounds of certain matrix operators such as Hausdorff, Nörlund matrix, weighted mean and summability on sequence spaces $l_p(w)$ and Lorentz sequence spaces $d(w,p)$, which was recently considered in [9] and [10] and similarly to [14] by Josip Pecaric, Ivan Peric and Rajko Roki. Also, this study is an extension of some works by G. Bennett on l_p spaces, see [1] and [2].

Keywords: inequality, norm, summability matrix, Hausdorff matrix, Nörlund matrix, weighted mean matrix, weighted sequence space and Lorentz sequence space

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1. Introduction

We study the norm of some matrix operators on $l_p(w)$ and Lorentz sequence spaces $d(w,p)$, $p \geqslant 1$, which is considered in [1], [2], [3], [4] and [5] on l_p spaces and in [10] and [11] on $l_p(w)$ and $d(w,p)$ for some matrix operators such as Cesàro, Copson, Hilbert, Hausdorff, Nörlund, weighted mean and summability. The problem of finding a lower bound of such matrices on weighted sequence spaces considered by authors in a companion paper [13].

Let l_p be the normed linear space of all sequences $x = (x_n)$ with finite norm $\|x\|_p$, where

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

Suppose that $w = (w_n)$ is a sequence with non-negative entries. For $p \geqslant 1$, we define the weighted sequence space $l_p(w)$ as

$$l_p(w) = \left\{(x_n)\colon \sum_{n=1}^{\infty} w_n |x_n|^p < \infty\right\},$$

with the norm $\|\cdot\|_{p,w}$, defined as:

$$\|x\|_{p,w}=\left(\sum_{n=1}^{\infty}w_n|x_n|^p\right)^{1/p}.$$

Also, if $w=(w_n)$ is a decreasing sequence of non-negative numbers such that $\lim\limits_{n\to\infty}w_n=0$ and $\sum\limits_{n=1}^{\infty}w_n=\infty$, then the Lorentz sequence space $d(w,p)$ is defined as

$$d(w,p)=\left\{(x_n)\colon\ \sum_{n=1}^{\infty}w_n x_n^{*^p}<\infty\right\},$$

where (x_n^*) is the decreasing rearrangement of $(|x_n|)$. In fact $d(w,p)$ is the space of null sequences x for which x^* is in $l_p(w)$, with the norm $\|x\|_{d(w,p)}=\|x^*\|_{p,w}$.

We write $\|A\|_{p,w}$ for the norm of A as an operator on $l_p(w)$, and $\|A\|_p$ for the norm of A as an operator on l_p, and $\|A\|_{d(w,p)}$ for the norm of A as an operator on $d(w,p)$.

Our objective in Section 2 is to give a generalization of some results obtained by Bennett [1], [2] and Jameson and Lashkaripour [10] for Hausdorff matrix operators on the weighted sequence space. In Section 3 we try to solve the problem of finding the norm of summability operators on the Lorentz sequence space $d(w,1)$, while in Section 4 we consider the same problem on the weighted sequence space $l_p(w)$. Summability operators on l_p were considered in [1], [2], [3], [4]. Finally, in Section 5, we get an estimate for a certain matrix operator on the Lorentz sequence space $d(w,p)$.

2. Hausdorff matrix operator on $l_p(w)$ and $d(w,p)$

In this section, we consider the Hausdorff matrix operator $H(\mu)=(h_{j,k})$ such that

$$h_{j,k}=\begin{cases}\dbinom{j-1}{k-1}\Delta^{j-k}a_k & \text{if}\quad 1\leqslant k\leqslant j,\\ 0 & \text{if}\quad k>j,\end{cases}$$

where Δ is the difference operator; that is,

$$\Delta a_k=a_k-a_{k+1}$$

and (a_k) is a sequence of real numbers, normalized so that $a_1=1$.

If

$$a_k=\int_0^1\theta^k\,\mathrm{d}\mu(\theta)\qquad (k=1,2,\ldots),$$

where μ is a probability measure on $[0,1]$, then for all $j,k=1,2,\ldots$, we have

$$h_{j,k}=\begin{cases}\binom{j-1}{k-1}\displaystyle\int_0^1\theta^{k-1}(1-\theta)^{j-k}\,\mathrm{d}\mu(\theta) & \text{if } 1\leqslant k\leqslant j,\\ 0 & \text{if } k>j.\end{cases}$$

The Hausdorff matrix is contained in famous classes of matrices. These classes are as follows:

i) The choice $\mathrm{d}\mu(\theta)=\alpha(1-\theta)^{\alpha-1}\,\mathrm{d}\theta$ gives the Cesàro matrix of order α.

ii) The choice $\mathrm{d}\mu(\theta)=$ point evaluation at $\theta=\alpha$ gives the Euler matrix of order α.

iii) The choice $\mathrm{d}\mu(\theta)=|\log\theta|^{\alpha-1}/\Gamma(\alpha)\,\mathrm{d}\theta$ gives the Hölder matrix of order α.

iv) The choice $\mathrm{d}\mu(\theta)=\alpha\theta^{\alpha-1}\,\mathrm{d}\theta$ gives the Gamma matrix of order α.

The Cesàro, Hölder and Gamma matrices have non-negative entries whenever $\alpha>0$, also the Euler matrix is non-negative when $0\leqslant\alpha\leqslant 1$. So that, if we obtain the norm of the Hausdorff matrix, then it is also an upper bound for the above matrices.

Note that, if T is an operator with non-negative entries on $l_p(w)$ (or $d(w,p)$), then we can get the norm of T by non-negative sequences, since $\|Tx\|_{p,w}\leqslant\|T|x|\|_{p,w}$ (or $\|Tx\|_{d(w,p)}\leqslant\|T|x|\|_{d(w,p)}$).

It is a much more delicate problem to find conditions under which the norm is determined by decreasing sequences x. The following statements give us some conditions adequate for the operators considered below, ensuring that $\|T\|_{d(w,p)}$ is determined by decreasing, non-negative sequences.

Proposition 2.1 ([11], Proposition 1.4.1). *Let $p\geqslant 1$ and let $T=(t_{i,j})$ be an operator with non-negative entries. If for all subsets M,N of natural numbers having m,n elements respectively, we have*

$$\sum_{i\in M}\sum_{j\in N}t_{i,j}\leqslant\sum_{i=1}^{m}\sum_{j=1}^{n}t_{i,j}, \tag{1}$$

then $\|T(u)\|_{d(w,p)}\leqslant\|T(u^*)\|_{d(w,p)}$ for all non-negative elements u of $d(w,p)$. Hence decreasing, non-negative elements are sufficient for $\|T\|_{d(w,p)}$ to be determined.

Proposition 2.2 ([9], Lemma 1). *Let $p\geqslant 1$ and let $T=(t_{i,j})$ be an operator with non-negative entries. Also, let T map $d(w,p)$ into itself. If we set $Tu=v$ for $u\in d(w,p)$ where $v_i=\sum\limits_{j=1}^{\infty}t_{i,j}u_j$, then the following conditions are equivalent:*

(a) *$v_1\geqslant v_2\geqslant\ldots\geqslant 0$ when $u_1\geqslant u_2\geqslant\ldots\geqslant 0$.*

(b) *$r_{i,n}=\sum\limits_{j=1}^{n}t_{i,j}$ decreases with i for each n.*

The following theorem is needed for the main result. Let μ be a Borel probability measure on $[0,1]$ with $\mu(0)=\mu(0+)=0$.

Theorem 2.1 ([6], Theorem 216)**.** *Let (x_n) be a non-negative sequence and $p>1$. Then*

$$\sum_{m=1}^{\infty}\Bigl(\sum_{n=1}^{m}\binom{m-1}{n-1}\Delta^{m-n}a_n x_n\Bigr)^p<\Bigl(\int_0^1\theta^{-1/p}\,\mathrm{d}\mu(\theta)\Bigr)^p\sum_{n=1}^{\infty}x_n^p,$$

unless $x_n=0$ for all n or the transformation reduces to the identity.

Theorem 2.2. *Let $H(\mu)$ be the Hausdorff matrix operator and $p>1$. Let (w_n) be a non-negative decreasing sequence such that $\sum\limits_{n=1}^{\infty}w_n/n=\infty$. Then the Hausdorff matrix operator maps $l_p(w)$ into itself, and*

$$\|H\|_{p,w}=\int_0^1\theta^{-1/p}\,\mathrm{d}\mu(\theta).$$

P r o o f. Let x be a non-negative sequence. Then since (w_n) is decreasing, applying Theorem 2.1 we have

$$\begin{aligned}\|Hx\|_{p,w}^p&=\sum_{j=1}^{\infty}w_j\Bigl(\sum_{k=1}^{j}\binom{j-1}{k-1}\Bigl(\int_0^1\theta^{k-1}(1-\theta)^{j-k}\,\mathrm{d}\mu(\theta)\Bigr)x_k\Bigr)^p\\&\leqslant\sum_{j=1}^{\infty}\Bigl(\sum_{k=1}^{j}\binom{j-1}{k-1}\Bigl(\int_0^1\theta^{k-1}(1-\theta)^{j-k}\,\mathrm{d}\mu(\theta)\Bigr)w_k^{1/p}x_k\Bigr)^p\\&\leqslant\Bigl(\int_0^1\theta^{-1/p}\,\mathrm{d}\mu(\theta)\Bigr)^p\sum_{j=1}^{\infty}w_jx_j^p=\Bigl(\int_0^1\theta^{-1/p}\,\mathrm{d}\mu(\theta)\Bigr)^p\|x\|_{p,w}^p.\end{aligned}$$

Hence

$$\|Hx\|_{p,w}\leqslant\Bigl(\int_0^1\theta^{-1/p}\,\mathrm{d}\mu(\theta)\Bigr)\|x\|_{p,w},$$

and so

$$\|H\|_{p,w}\leqslant\int_0^1\theta^{-1/p}\,\mathrm{d}\mu(\theta).$$

It remains to prove that the value $\int_0^1\theta^{-1/p}\,\mathrm{d}\mu(\theta)$ is the best possible. To show this, we follow an argument of Hardy ([5], page 47) with some slight modifications. For any $\varepsilon\in(0,1)$, choose α and N such that

$$\begin{gathered}\Bigl(1+\frac{1}{\alpha}\Bigr)^{-2/p}>1-\varepsilon,\\\int_{\alpha/n}^1\theta^{-1/p}\,\mathrm{d}\mu(\theta)>(1-\varepsilon)\int_0^1\theta^{-1/p}\,\mathrm{d}\mu(\theta)\qquad(n\geqslant N).\end{gathered}$$

For ε and N mentioned above, there exists δ such that $0<\delta<1/p$ and

$$\varepsilon\sum_{n=1}^{\infty} w_n n^{-1-p\delta} > \sum_{n=1}^{N-1} w_n n^{-1-p\delta},$$

(because, if $\varepsilon\sum\limits_{n=1}^{\infty} w_n n^{-1-p\delta} \leqslant \sum\limits_{n=1}^{N-1} w_n n^{-1-p\delta}$ for any $\delta>0$ by letting δ tend to 0, we deduce that $\sum\limits_{n=1}^{\infty} w_n/n$ is convergent, which contradicts the assumption $\sum\limits_{n=1}^{\infty} w_n/n = \infty$). Taking

$$s=\frac{1}{p}+\delta, \qquad x_n = n^{-s},$$

we obtain

$$\sum_{n=N}^{\infty} w_n x_n^p > (1-\varepsilon)\sum_{n=1}^{\infty} w_n x_n^p.$$

Since $(x_n)\in l_p$ and $0<w_n\leqslant w_1$, we deduce that $(x_n)\in l_p(w)$. If we set

$$e_n(\theta)=\sum_{m=1}^{n}\binom{n-1}{m-1}\theta^{m-1}(1-\theta)^{n-m}x_m,$$

then

$$x_n=\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{e}^{-nt}t^{s-1}\,\mathrm{d}t, \qquad e_n(\theta)=\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{e}^{-t}t^{s-1}(1-\theta+\theta\mathrm{e}^{-t})^{n-1}\,\mathrm{d}t.$$

For $t>0$ and $0<\theta<1$ we have $1-\theta+\theta\mathrm{e}^{-t}>\mathrm{e}^{-\theta t}$. Hence

$$e_n(\theta)\geqslant\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathrm{e}^{-(1-\theta+n\theta)t}\,\mathrm{d}t=(1-\theta+n\theta)^{-s}.$$

If $\alpha/n<\theta<1$, then

$$(1-\theta+n\theta)^{-s}>n^{-s}\theta^{-s}(1+\tfrac{1}{\alpha})^{-s}>\theta^{-1/p}(1+\tfrac{1}{\alpha})^{-2/p}x_n>(1-\varepsilon)\theta^{-1/p}x_n,$$

therefore

$$e_n(\theta)\geqslant(1-\varepsilon)\theta^{-1/p}x_n.$$

For $n\geqslant N$ we have

$$\begin{aligned}(Hx)_n&=\int_0^1 e_n(\theta)\,\mathrm{d}\mu(\theta)\geqslant\int_{\alpha/n}^1 e_n(\theta)\,\mathrm{d}\mu(\theta)\\&\geqslant(1-\varepsilon)x_n\int_{\alpha/n}^1\theta^{-1/p}\,\mathrm{d}\mu(\theta)\geqslant(1-\varepsilon)^2x_n\int_0^1\theta^{-1/p}\,\mathrm{d}\mu(\theta)\end{aligned}$$

and

$$\sum_{n=1}^{\infty} w_n (Hx)_n^p \geqslant \sum_{n=N}^{\infty} w_n (Hx)_n^p \geqslant (1-\varepsilon)^{2p} \Big(\int_0^1 \theta^{-1/p} \,\mathrm{d}\mu(\theta) \Big)^p \sum_{n=N}^{\infty} w_n x_n^p$$
$$\geqslant (1-\varepsilon)^{2p+1} \Big(\int_0^1 \theta^{-1/p} \,\mathrm{d}\mu(\theta) \Big)^p \sum_{n=1}^{\infty} w_n x_n^p.$$

Therefore

$$\|H\|_{p,w} \geqslant (1-\varepsilon)^{2+1/p} \int_0^1 \theta^{-1/p} \,\mathrm{d}\mu(\theta).$$

Since ε is arbitrary, letting $\varepsilon \longrightarrow 0$ we have

$$\|H\|_{p,w} \geqslant \Big(\int_0^1 \theta^{-1/p} \,\mathrm{d}\mu(\theta) \Big),$$

and this completes the proof of the statement.

Corollary 2.1. *Suppose that $p > 1$ and $p^* = p/(p-1)$. If (w_n) is a non-negative decreasing sequence and $\sum\limits_{n=1}^{\infty} w_n/n$ is divergent, then Cesàro, Hölder, Gamma and Euler operators map $l_p(w)$ into itself. Also, we have:*

$$\|C(\alpha)\|_{p,w} = \frac{\Gamma(\alpha+1)\Gamma(1/p^*)}{\Gamma(\alpha+1/p^*)} \qquad (\alpha > 0);$$
$$\|H(\alpha)\|_{p,w} = \frac{1}{\Gamma(\alpha)} \int_0^1 \theta^{-1/p} |\log \theta|^{\alpha-1} \,\mathrm{d}\theta \qquad (\alpha > 0);$$
$$\|G(\alpha)\|_{p,w} = \frac{\alpha p}{\alpha p - 1} \qquad (\alpha p > 1);$$
$$\|E(\alpha)\|_{p,w} = \alpha^{-1/p} \qquad (0 < \alpha < 1).$$

P r o o f. It is elementary. □

Corollary 2.2 ([10], Proposition 5.1)**.** *If u, w are non-negative sequences, w is decreasing and $\sum\limits_{n=1}^{\infty} w_n/n$ is divergent, then*

$$\sum_{n=1}^{\infty} w_n \Big(\frac{1}{n} \sum_{i=1}^{n} u_i \Big)^p \leqslant p^{*p} \Big(\sum_{n=1}^{\infty} w_n u_n^p \Big).$$

The value of p^{*p} is the best possible.

P r o o f. Apply Corollary 2.1 for Cesàro operator with $\alpha = 1$. □

Remark 2.1. By taking $w_n = 1$ for all n, we deduce that Hausdorff, Cesàro, Holder, Gamma and Euler operators map l_p into itself.

We now state the extension of the Hardy inequality to the weighted sequence space. The following lemma is needed for the main result.

Lemma 2.3. *Suppose that a_n, b_n are non-negative numbers such that $\sum\limits_{n=1}^{\infty} a_n$ is divergent and $\lim\limits_{n\to\infty} b_n = 0$. Then*

$$\frac{\sum\limits_{n=1}^{m} a_n b_n}{\sum\limits_{n=1}^{m} a_n} \to 0 \qquad \text{as } m \to \infty.$$

Proof. It is elementary. $\square$

Theorem 2.3. *Suppose that $p > 1$, $w = (w_n)$ is a decreasing sequence with non-negative entries and $\sum\limits_{n=1}^{\infty} w_n/n$ is divergent. Let $N \geqslant 0$ and let $C_N = (c_{n,k}^N)$ be the matrix with*

$$c_{n,k}^N = \begin{cases} \dfrac{1}{n+N} & \text{for } n \geqslant k, \\ 0 & \text{for } n < k. \end{cases}$$

Then $\|C_N\|_{p,w} = p^*$.

Proof. C_0 is the Cesàro matrix of order $\alpha = 1$ and $0 \leqslant c_{n,k}^N \leqslant c_{n,k}^0$ for all $n, k \geqslant 1$. Since $w = (w_n)$ is a decreasing sequence, by Corollary 2.2 we have

$$\|C_N\|_{p,w} \leqslant \|C_0\|_{p,w} = p^*.$$

Fix m such that $m \geqslant N$, and let

$$x_n = \begin{cases} (n+m)^{-1/p} & \text{for } 1 \leqslant n \leqslant m, \\ 0 & \text{for } n > m. \end{cases}$$

Then $\sum\limits_{n=1}^{\infty} w_n x_n^p = \sum\limits_{n=1}^{m} w_n/(n+m)$. Also, for $n \leqslant m$,

$$X_n \geqslant \int_1^n (t+m)^{-1/p}\,\mathrm{d}t = p^*((n+m)^{1/p^*} - (m+1)^{1/p^*}),$$

so that

$$y_n = \frac{X_n}{n+N} \geqslant \frac{p^*}{(n+m)^{1/p}}\Big(1 - \Big(\frac{m+1}{n+m}\Big)^{1/p^*}\Big).$$

Since $(1-t)^p \geqslant 1-pt$ for $0<t<1$, we have

$$y_n^p \geqslant \frac{(p^*)^p}{n+m}\Big(1-p\Big(\frac{m+1}{n+m}\Big)^{1/p^*}\Big),$$

and hence

$$\sum_{n=1}^{m} w_n y_n^p \geqslant (p^*)^p \sum_{n=1}^{m} \frac{w_n}{n+m} - p(p^*)^p(m+1)^{1/p^*} \sum_{n=1}^{m} \frac{w_n}{(n+m)^{1+1/p^*}}.$$

Since (w_n) is a decreasing sequence, $w_n \geqslant w_{n+m}$ and

$$\sum_{n=1}^{\infty} \frac{w_n}{n+m} \geqslant \sum_{n=1}^{\infty} \frac{w_{n+m}}{n+m} = \sum_{n=m+1}^{\infty} \frac{w_n}{n} = \infty.$$

Therefore $\sum\limits_{n=1}^{\infty} w_n/(n+m)$ is divergent, so that setting $a_n = w_n/(n+m)$, $b_n = 1/(n+m)^{1/p^*}$ and applying Lemma 2.1 we obtain the statement.

3. Summability operator on $d(w,1)$

In this part we consider the upper bound problem for summability matrix operators. These are lower triangular matrices with entries of the form

(i) $d_{j,k} \geqslant 0$;

(ii) $d_{j,k} = 0$ if $k > j$;

(iii) $\sum\limits_{k=1}^{j} d_{j,k} = 1$.

It is natural to ask what can be said about the norm of an arbitrary summability matrix on $d(w,1)$. We give an interesting answer to this question in the following statement.

Theorem 3.1. *Suppose $D = (d_{i,j})$ is a summability matrix operator satisfying condition* (1) *of Proposition* 2.1. *If*

$$\sup \frac{S_n}{W_n} < \infty,$$

where $S_n = s_1 + \ldots + s_n$ and $s_n = \sum\limits_{k=n}^{\infty} w_k d_{k,n}$ and $W_n = w_1 + \ldots + w_n$, then D is a bounded operator from $d(w,1)$ into itself, and

$$\|D\|_{d(w,1)} = \sup_n \frac{S_n}{W_n}.$$

P r o o f. By Proposition 2.1, it is sufficient to consider decreasing, non-negative sequences. Let x be in $d(w,1)$ such that $x_1 \geqslant x_2 \geqslant \ldots \geqslant 0$. Then

$$\|Dx\|_{d(w,1)} = \sum_{n=1}^{\infty} w_n \left(\sum_{k=1}^{n} d_{n,k} x_k \right) = \sum_{n=1}^{\infty} s_n x_n = \sum_{n=1}^{\infty} S_n (x_n - x_{n+1}).$$

Also, we have

$$\|x\|_{d(w,1)} = \sum_{n=1}^{\infty} W_n (x_n - x_{n+1}).$$

Let $M = \sup_n S_n / W_n$. Then

$$\|Dx\|_{d(w,1)} \leqslant M \sum_{n=1}^{\infty} w_n x_n.$$

Hence

$$\|D\|_{d(w,1)} \leqslant M.$$

To show that the constant M is the best possible, we take $x_1 = x_2 = \ldots = x_n = 1$ and $x_k = 0$ for all $k \geqslant n+1$. Then

$$\|x\|_{d(w,1)} = W_n, \qquad \|Dx\|_{d(w,1)} = S_n.$$

Therefore

$$\|D\|_{d(w,1)} = M.$$

We now state some consequences of the above theorem.

Let (d_n) be a non-negative sequence with $d_1 > 0$, and $D_n = d_1 + \ldots + d_n$. The Nörlund matrix $N_d = (d_{n,k})$ is defined as follows:

$$d_{n,k} = \begin{cases} \dfrac{d_{n-k+1}}{D_n}, & 1 \leqslant k \leqslant n, \\ 0 & k > n, \end{cases}$$

Further, the weighted mean matrix $D_d = (d_{n,k})$ is defined by

$$d_{n,k} = \begin{cases} \dfrac{d_k}{D_n}, & 1 \leqslant k \leqslant n, \\ 0 & k > n. \end{cases}$$

We note that the Hausdorff matrix, Nörlund mean matrix and weighted mean matrix are summability matrices so that we have the following statement.

Corollary 3.1. *Suppose $D = (d_{i,j})$ is a Hausdorff (Nörlund mean or weighted mean) matrix operator satisfying condition* (1). *If*

$$\sup_n \frac{S_n}{W_n} < \infty,$$

then D is a bounded operator from $d(w,1)$ into itself, and

$$\|D\|_{d(w,1)} = \sup_n \frac{S_n}{W_n}.$$

Proposition 3.1. *Suppose d_n is a non-negative, increasing sequence and for all $n < i$ we have*

$$\frac{1}{D_i}\sum_{k=1}^{n} d_{i-k+1} \geqslant \frac{1}{D_{i+1}}\sum_{k=1}^{n} d_{i-k+2}.$$

If

$$\sup \frac{S_n}{W_n} < \infty,$$

then N_d is a bounded operator from $d(w,1)$ into itself, and

$$\|N_d\|_{d(w,1)} = \sup_n \frac{S_n}{W_n}.$$

Proof. The Nörlund mean operator, N_d, satisfies condition (1). So, applying Corollary 3.1 we have the statement. □

Proposition 3.2. *Suppose d_n is a non-negative, decreasing sequence. If*

$$\sup_n \frac{S_n}{W_n} < \infty,$$

then D_d is a bounded operator from $d(w,1)$ into itself, and

$$\|D_d\|_{d(w,1)} = \sup_n \frac{S_n}{W_n}.$$

Proof. Since d_n is a non-negative, decreasing sequence, the weighted mean matrix operator D_d satisfies condition (1). If we apply Corollary 3.1, then we have the statement. □

As we mentioned in the previous section, the Hausdorff matrix is contained in the class of the famous Cesàro and Gamma matrices. Also, for $\alpha > 0$, the Cesàro matrix $C(\alpha)$ and the Gamma matrix $G(\alpha)$ are the Nörlund matrix N_d and the Weighted mean matrix D_d, respectively, with

$$d_n = \binom{n+\alpha-2}{n-1}.$$

If $\alpha = 1$, then $G(1) = C(1)$. Hence for $w_n = 1/n^p$, where $0 < p \leqslant 1$, by ([12], Theorem 6) we have

$$\|G(1)\|_{d(w,1)} = \|C(1)\|_{d(w,1)} = \zeta(1+p),$$

where ζ is Riemann's zeta function.

In the next statement we give the norm of $C(2)$ on $d(w,1)$. It is enough to consider the sequence (s_n/w_n) instead of (S_n/W_n), because of the well-known fact listed in the following lemma.

Lemma 3.1. *If $m \leqslant s_n/w_n \leqslant M$ for all n, then $m \leqslant S_n/W_n \leqslant M$ for all n.*

Proof. It is elementary. □

Proposition 3.3. *If $w_n = 1/n$, then $C(2)$ is a bounded operator from $d(w,1)$ into itself, and*

$$\|C(2)\|_{d(w,1)} = 2.$$

Proof. We note that $s_n/w_n \leqslant s_1/w_1$ for all n. Therefore, applying Lemma 3.1, we deduce that $S_n/W_n \leqslant S_1/W_1 = s_1$, and by Corollary 3.1 we have

$$\|C(2)\|_{d(w,1)} = 2.$$

Since

$$s_1 = \sum_{k=1}^{\infty} \frac{1}{\frac{1}{2}k(k+1)} = 2,$$

we have for all n

$$\frac{s_n}{w_n} = n \sum_{k=n}^{\infty} \frac{1}{\frac{1}{2}k(k+1)} \frac{k-n+1}{k} \leqslant 2n \sum_{k=n}^{\infty} \frac{1}{k(k+1)} = 2n\frac{1}{n} = 2 = s_1.$$

This completes the proof of the proposition.

Let $D = (d_{n,k})$ be a summability matrix operator defined as before, and let its transpose be D^t which is defined as

$$(D^t x)_n = \sum_{k=n}^{\infty} d_{k,n} x_k.$$

D^t is a quasi-summability matrix.

Note: If D is a Summability matrix satisfying condition (1), then D^t is so.

Theorem 3.2. *Suppose D is a summability matrix operator on $d(w,1)$ satisfying condition* (1)*. If*

$$M = \sup_n \frac{R_n}{W_n} < \infty,$$

where $R_n = r_1 + \ldots + r_n$, $r_n = \sum\limits_{k=1}^{n} w_k d_{n,k}$ and $W_n = w_1 + \ldots + w_n$, then D^t is a bounded operator from $d(w,1)$ into $d(w,1)$ and we have

$$\|D^t\|_{d(w,1)} = M.$$

Proof. Applying Proposition 2.1 and the above note, it is sufficient to consider decreasing, non-negative sequences. Let x be in $d(w,1)$ such that $x_1 \geqslant x_2 \geqslant \ldots \geqslant 0$. Then

$$\|D^t x\|_{d(w,1)} = \sum_{n=1}^{\infty} w_n \left(\sum_{k=n}^{\infty} d_{k,n} x_k \right) = \sum_{n=1}^{\infty} r_n x_n = \sum_{n=1}^{\infty} R_n (x_n - x_{n+1}).$$

Hence

$$\|D^t x\|_{d(w,1)} \leqslant M \|x\|_{d(w,1)}.$$

To show that this constant is the best possible, we take $x_1 = x_2 = \ldots = x_n = 1$ and $x_k = 0$ for all $k \geqslant n+1$. Then

$$\|x\|_{d(w,1)} = W_n, \qquad \|D^t x\|_{d(w,1)} = R_n.$$

Therefore

$$\|D^t\|_{d(w,1)} = M.$$

Using the above notation, we have the following statement. □

Corollary 3.2. *Suppose $D = (d_{i,j})$ is a Hausdorff (Nörlund mean or weighted mean) matrix operator satisfying condition* (1)*. If*

$$M = \sup_n \frac{R_n}{W_n} < \infty,$$

then D^t is a bounded operator from $d(w,1)$ into itself, and we have

$$\|D^t\|_{d(w,1)} = M.$$

If $\alpha = 1$, then $G^t(1) = C^t(1)$. Hence for $w_n = 1/n^p$, where $0 < p \leqslant 1$, applying ([12], Theorem 9) we deduce that

$$G^t(1)\|_{d(w,1)} = \|C^t(1)\|_{d(w,1)} = \frac{1}{1-p}.$$

4. Summability matrix operator on $l_p(w)$

In this section we consider the upper bound problem for summability matrix operators. It is natural to ask what can be said about the norm of an arbitrary summability matrix on $l_p(w)$ (or $d(w,p)$).

First, we compare the norm of the quasi-summability matrix with that of the Copson matrix. Then we give an estimate for the quasi-matrix, where the Copson matrix is the transpose of the Cesàro matrix.

Let $p, q \geqslant 1$. We write $\|A\|_{p,q,w}$ for the norm of A as an operator from $l_p(w)$ into $l_q(w)$.

Lemma 4.1. *Let $p \geqslant 1$ and let u, v and w be non-negative sequences. If v, w are decreasing and*

$$\sum_{i=1}^{n} v_i \leqslant \sum_{i=1}^{n} u_i \qquad (n = 1, 2, \ldots),$$

then

$$\sum_{i=1}^{\infty} w_i v_i^p \leqslant \sum_{i=1}^{\infty} w_i u_i^p.$$

P r o o f. It is elementary. □

Lemma 4.2. *Suppose p, $q \geqslant 1$ and $A = (a_{i,j})$, $D = (d_{i,j})$ are matrices with non-negative entries. Let (w_n) be a decreasing sequence. If the columns of D are decreasing, i.e.*

$$\text{(I)} \qquad d_{k,j} \geqslant d_{k+1,j} \qquad (j, k = 1, 2, \ldots),$$

and also

$$\text{(II)} \qquad \sum_{i=1}^{k} a_{i,j} \geqslant \sum_{i=1}^{k} d_{i,j} \qquad (j, k = 1, 2, \ldots),$$

then

$$\|A\|_{p,q,w} \geqslant \|D\|_{p,q,w}.$$

Proof. Let x be a sequence of non-negative entries. We define u and v by

$$u_k = \sum_{i=1}^{\infty} d_{k,i} x_i, \qquad v_k = \sum_{i=1}^{\infty} a_{k,i} x_i, \qquad (k = 1, 2, \ldots).$$

It is clear from (I) that u_k decreases with k, and by (II) we have

$$\sum_{k=1}^{n} u_k \leqslant \sum_{k=1}^{n} v_k \qquad (n = 1, 2, \ldots).$$

Hence applying Lemma 4.1 we deduce that

$$\sum_{k=1}^{\infty} w_k u_k^p \leqslant \sum_{k=1}^{\infty} w_k v_k^p.$$

Therefore $\|Dx\|_{q,w} \leqslant \|Ax\|_{q,w}$, and so

$$\|A\|_{p,q,w} \geqslant \|D\|_{p,q,w}.$$

□

Theorem 4.1. *Suppose $p, q \geqslant 1$ and $A = (a_{i,j})$ is a summability matrix. If $C = (c_{i,j})$ is the Cesàro matrix of order $\alpha = 1$ and the rows of A are decreasing, then*

$$\|A^t\|_{p,q,w} \geqslant \|C^t\|_{p,q,w}.$$

P r o o f. We apply Lemma 4.1 for A^t and C^t. It is clear that (I) holds for C^t. We show that

$$\sum_{i=1}^{k} a^t_{i,j} \geqslant \sum_{i=1}^{k} c^t_{i,j} \qquad (j, k = 1, 2, \ldots),$$

or

$$\sum_{i=1}^{k} a_{j,i} \geqslant \sum_{i=1}^{k} c_{j,i} \qquad (j, k = 1, 2, \ldots).$$

When $k \geqslant j$, it is easy to see that we have the above inequality. When $k < j$, we have

$$\sum_{i=1}^{k} a_{j,i} \geqslant \frac{k}{j} \qquad (j = 1, 2, \ldots)$$

because the j^{th} row of A is decreasing, therefore the average

$$\frac{1}{k} \sum_{i=1}^{k} a_{j,i}$$

decreases with k, and the j^{th} term of this average is precisely $1/j$.

We now state a consequence of Theorem 4.1.

Corollary 4.1. *Suppose $p \geqslant 1$ and $A = (a_{i,j})$ is a summability matrix with decreasing rows. If $0 \leqslant \beta < 1$ and w is defined either by $w_n = 1/n^{\beta}$ or by $W_n = \sum\limits_{k=1}^{n} w_k = n^{1-\beta}$, then*

$$\|A^t\|_{p,w} \geqslant \frac{p}{1-\beta}.$$

P r o o f. Let C be the Cesàro matrix of order $\alpha = 1$. By Theorem 4.2 of [10], we have

$$\|C^t\|_{p,w} = \frac{p}{1-\beta}.$$

This completes the proof of the statement. $\square$

In the following, we extend the so-called Maximal Theorem of Hardy and Littlewood to $l_p(w)$ spaces, and then we establish an upper bound for the summability matrix with increasing rows.

Theorem 4.2. *If $p > 1$ and x, w are non-negative sequences and w is decreasing, then*

$$\sum_{j=1}^{\infty} w_j \max_{1\leqslant i\leqslant j}\Big(\frac{1}{j-i+1}\sum_{k=i}^{j} x_k\Big)^p \leqslant (p^*)^p \sum_{k=1}^{\infty} w_k x_k^p.$$

P r o o f. If we set $a_k = w_k^{1/p} x_k$ in Theorem 8 of [7], then we have

$$\sum_{j=1}^{\infty} \max_{1\leqslant i\leqslant j}\Big(\frac{1}{j-i+1}\sum_{k=i}^{j} w_k^{1/p} x_k\Big)^p \leqslant (p^*)^p \sum_{k=1}^{\infty} w_k x_k^p.$$

Since w is decreasing, we deduce that

$$\sum_{j=1}^{\infty} w_j \max_{1\leqslant i\leqslant j}\Big(\frac{1}{j-i+1}\sum_{k=i}^{j} x_k\Big)^p \leqslant (p^*)^p \sum_{k=1}^{\infty} w_k x_k^p.$$

The next statement is an easy consequence of the previous theorem. □

Corollary 4.2 ([1], Corollary 1.15). *If $p > 1$ and x is a sequence of non-negative terms, then*

$$\sum_{j=1}^{\infty} \max_{1\leqslant i\leqslant j}\Big(\frac{1}{j-i+1}\sum_{k=i}^{j} x_k\Big)^p \leqslant (p^*)^p \sum_{k=1}^{\infty} x_k^p.$$

In the following statement, we give an upper bound for the summability matrix operator. Let D be a summability matrix with increasing rows, that is; for all j we have

$$d_{j,1} \leqslant d_{j,2} \leqslant \ldots \leqslant d_{j,j}.$$

Theorem 4.3. *Let $p > 1$, and let D be a summability matrix with increasing rows. Then*

$$\|D\|_{p,w} \leqslant p^*.$$

P r o o f. Let x be a non-negative sequence and let j be fixed. Setting

$$M = \max\Big\{\frac{x_j}{1}, \frac{x_j + x_{j-1}}{2}, \ldots, \frac{x_j + \ldots + x_1}{j}\Big\},$$

we have for all k with $1 \leqslant k \leqslant j$

$$x_j + \ldots + x_{j-k+1} \leqslant M(1 + \ldots + 1) \qquad (k \text{ terms}).$$

Since $0 \leqslant d_{j,1} \leqslant \ldots \leqslant d_{j,j}$, applying Lemma 4.1, we obtain

$$\sum_{k=1}^{j} d_{j,k} x_k \leqslant M \sum_{k=1}^{j} d_{j,k} = M.$$

Since

$$M = \max_{1 \leqslant i \leqslant j} \left(\frac{1}{j-i+1} \sum_{k=i}^{j} x_k \right) = \max_{1 \leqslant i \leqslant j} \left(\frac{1}{i} \sum_{k=j-i+1}^{j} x_k \right),$$

applying Theorem 4.2, we deduce that

$$\begin{aligned} \|Dx\|_{p,w}^p &= \sum_{j=1}^{\infty} w_j \left(\sum_{k=1}^{j} d_{j,k} x_k \right)^p \leqslant \sum_{j=1}^{\infty} w_j \max_{1 \leqslant i \leqslant j} \left(\frac{1}{i} \sum_{k=j-i+1}^{j} x_k \right)^p \\ &\leqslant (p^*)^p \sum_{k=1}^{\infty} w_k x_k^p = (p^*)^p \|x\|_{p,w}^p, \end{aligned}$$

and so $\|D\|_{p,w} \leqslant p^*$.

Let D_d and N_d be the weighted mean matrix and the Nörlund matrix respectively. We state some consequences of Theorem 4.3.

Corollary 4.3. *Let $p > 1$. If (d_n) is an increasing sequence, then*

$$\|D_d\|_{p,w} \leqslant p^*.$$

Corollary 4.4. *If $p > 1$. If (d_n) is a decreasing sequence, then*

$$\|N_d\|_{p,w} \leqslant p^*.$$

5. Matrix operator with $\sum_{i=1}^{\infty} |a_{i,j}| \leqslant 1$ for all j and $\sum_{j=1}^{\infty} |a_{i,j}| \leqslant 1$ for all i

In this section we consider some operators satisfying the above conditions. We apply some results of the majorization principle to show that such operators are bounded on the Lorentz sequence spaces $d(w,p)$. In the following, we state some lemmas which are needed throughout this section.

Lemma 5.1. *Let $A = (a_{i,j})$ be a matrix operator with entries of the form*

i) $\sum\limits_{i=1}^{\infty} |a_{i,j}| \leqslant 1$ *for all* j;

ii) $\sum\limits_{j=1}^{\infty} |a_{i,j}| \leqslant 1$ *for all* i.

Let $x = (x_i)$ be a null sequence and $y = Ax$. Then we have:

$$\sum_{i=1}^{n} y_i^* \leqslant \sum_{i=1}^{n} x_i^* \qquad (n = 1, 2, \ldots).$$

Proof. We may assume $|x_1| \geqslant |x_2| \ldots$. So for all j we have $x_j^* = |x_j|$. Let for all $r y_r^* = |y_{i_r}|$. Then

$$y_r^* = \left| \sum_{j=1}^{\infty} a_{i_r,j} x_j \right| \leqslant \sum_{j=1}^{\infty} |a_{i_r,j}| x_j^*.$$

Therefore

$$\sum_{r=1}^{n} y_r^* \leqslant \sum_{j=1}^{\infty} b_j x_j^*,$$

where $b_j = \sum\limits_{r=1}^{n} |a_{i_r,j}|$. Let $B_k = b_1 + \ldots + b_k$, then for all k we have: $B_k \leqslant k$. Also, for $k \geqslant n$,

$$B_k = \sum_{r=1}^{n} \sum_{j=1}^{k} |a_{i_r,j}| \leqslant n.$$

By the Abel summation, we have

$$\sum_{j=1}^{\infty} b_j x_j^* = \sum_{j=1}^{\infty} B_j (x_j^* - x_{j+1}^*) \leqslant \sum_{j=1}^{n} j (x_j^* - x_{j+1}^*) + n \sum_{j=n+1}^{\infty} (x_j^* - x_{j+1}^*) = \sum_{j=1}^{n} x_j^*.$$

This completes the proof of the statement. □

Lemma 5.2. *Let $1 \leqslant p \leqslant q$ and let $x = (x_i)$ be a sequence in $d(w,p)$. If $w_1 = 1$, then*

$$\|x\|_{d(w,q)} \leqslant \|x\|_{d(w,p)}.$$

Proof. Let the sequence x be such that $x_1 \geqslant x_2 \geqslant \ldots \geqslant 0$. Write $y_i = x_i^p$. Since $w_1 = 1$, by Proposition 1.3.2 of [11] we have

$$\sum_{i=1}^{\infty} w_i x_i^p = \sum_{i=1}^{\infty} w_i y_i \geqslant \left(\sum_{i=1}^{\infty} w_i y_i^{q/p} \right)^{p/q} = \left(\sum_{i=1}^{\infty} w_i x_i^q \right)^{p/q}.$$

This completes the proof of the proposition. □

Theorem 5.1. *Suppose* $A = (a_{i,j})$ *is a matrix operator with entries of the form*

i) $\sum\limits_{i=1}^{\infty} |a_{i,j}| \leqslant 1$ *for all* j;

ii) $\sum\limits_{j=1}^{\infty} |a_{i,j}| \leqslant 1$ *for all* i.

Let $1 \leqslant p \leqslant q$. *If* $w_1 = 1$, *then* A *is a bounded operator from* $d(w,p)$ *into* $d(w,q)$, *and we have*

$$\|A\|_{p,q,w} \leqslant 1.$$

Proof. Let x be in $d(w,p)$ and $y = Ax$. Since x convergents to zero, applying Lemma 5.1 we obtain

$$\sum_{i=1}^{n} y_i^* \leqslant \sum_{i=1}^{n} x_i^* \qquad (n = 1, 2, \ldots).$$

Applying Lemma 4.1 we deduce that

$$\|Ax\|_{d(w,q)}^q = \sum_{n=1}^{\infty} w_n (y_n^*)^q \leqslant \sum_{n=1}^{\infty} w_n (x_n^*)^q = \|x\|_{d(w,q)}^q.$$

Hence by Lemma 5.2, $\|Ax\|_{d(w,q)} \leqslant \|x\|_{d(w,p)}$, and so

$$\|A\|_{p,q,w} \leqslant 1.$$

□

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