Czechoslovak Mathematical Journal, 57 (132) (2007), 747–762

MATRIX REFINEMENT EQUATIONS: CONTINUITY AND SMOOTHNESS

Xing-Gang He and Chun-Tai Liu, Wuhan

(Received September 18, 2005)

Abstract. In this paper we give some criteria for the existence of compactly supported $C^{k+\alpha}$-solutions (k is an integer and $0 \leqslant \alpha < 1$) of matrix refinement equations. Several examples are presented to illustrate the general theory.

Keywords: matrix refinement equation, continuity, smoothness, iteration, multi-wavelet

MSC 2000: 39B42, 39B12, 42C40

1. Introduction

A functional equation is called a *matrix refinement equation* if it has the following form:

$$f(x) = \sum_{n=0}^{N} C_n f(2x - n), \tag{1.1}$$

where $f(x)$ is a real vector-valued function from $\mathbb{R}$ to $\mathbb{R}^d$, d is an integer and the coefficients C_n's are real $d \times d$ matrices. An $L^1[\mathbb{R}, \mathbb{R}^d]$ solution of (1.1) is termed a *refinable or scaling vector*. Applying Fourier transformation to (1.1) leads to

$$\hat{f}(\xi) = M(\xi/2)\hat{f}(\xi/2), \tag{1.2}$$

where $\hat{f}$ is defined componentwise, i.e., $\hat{f}(\xi) = (\widehat{f_1}(\xi), \ldots, \widehat{f_d}(\xi))^T$ with

$$\widehat{f_j}(\xi) = \int_{-\infty}^{+\infty} f_j(x) \exp(-\mathrm{i}\xi x)\,\mathrm{d}x, \qquad j = 1, 2, \ldots, d$$

and

$$M(\xi) = \frac{1}{2} \sum_{n=0}^{N} C_n \exp(-\mathrm{i}n\xi).$$

The matrix $M(0) = \frac{1}{2} \sum\limits_{n=0}^{N} C_n$ will be used frequently.

The matrix refinement equation (1.1) plays an important role in constructing multi-wavelets by using multiresolution analysis. The basic question on (1.1) is how to establish the existence of continuous and smooth solutions of (1.1) with compact support in terms of its coefficients. There are three major approaches to this question: the Fourier method (the frequency domain approach) ([2], [3]), the iteration method (the time domain approach) ([6], [7]) and the subdivision method [1]. In this paper we use the second to obtain several criteria.

Let T_0 and T_1 be

$$T_0 = [C_{2i-j-1}]_{1\leqslant i,j\leqslant N} = \begin{bmatrix} C_0 & 0 & \dots & 0 \\ C_2 & C_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_{N-1} \end{bmatrix}$$

and

$$T_1 = [C_{2i-j}]_{1\leqslant i,j\leqslant N} = \begin{bmatrix} C_1 & C_0 & \dots & 0 \\ C_3 & C_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_N \end{bmatrix}$$

respectively. We will show that, if (1.1) has a compactly supported continuous solution $\varphi(x)$, then $\varphi(x)$ must be Hölder continuous and $\hat{\varphi}(0) \neq 0$ (Theorem 2.4 and Lemma 2.2). The following theorem is a characterization for a continuous solution of (1.1).

Theorem 1.1. *The matrix refinement equation* (1.1) *has a nonzero compactly supportede Hölder continuous solution with exponent* $\alpha = |\ln \lambda| / \ln 2$ *if and only if there exists a 2-eigenvector* v *of the matrix* $(T_0 + T_1)$ *such that*

$$\max_{d_i=0 \text{ or } 1} \|T_{d_1} \dots T_{d_m} \tilde{v}\| \leqslant c\lambda^m, \qquad m = 1, 2, \dots, \tag{1.3}$$

where $\tilde{v} = T_0 v - v$ *and* $0 < \lambda < 1$.

In general, we concern ourselves mainly with the sufficient conditions of Theorem 1.1, but it is not easy to check them because (1.3) contains infinitely many inequalities. Instead of it, we have the following practical criterion, which is a corollary of Theorem 2.4.

Proposition 1.2. *Let H be a common invariant subspace of T_0 and T_1 which contains $\tilde{v}$ defined in Theorem* 1.1. *Suppose there exists an integer m such that*

$$\max_{d_i=0 \text{ or } 1} \|T_{d_1} \dots T_{d_m}|_H\| < 1.$$

Then the equation (1.1) *has a Hölder continuous solution with compact support.*

In order to study smooth solutions of (1.1), we assume that all the eigenvalues of $M(0)$ except for 1 are inside the unit disk, that is, the absolute values of these eigenvalues are less than 1, and 1 is a simple eigenvalue of $M(0)$. If $M(0)$ satisfies these assumptions, we say that $M(0)$ satisfies *condition* $E(1)$. There are two reasons for using the condition $E(1)$ like Shen [14]: (1) it guarantees that (1.1) has at least one nonzero compactly supported solution in $L^1[\mathbb{R}, \mathbb{R}^d]$; (2) it is necessary if we assume that the sequence $\{f(x-n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence. We use $C^{k+\alpha}[I]$ (k is an integer and $0 \leqslant \alpha < 1$) to denote the set of $f(x)$ which belongs to $C^k[I]$ and satisfy

$$\|f^{(k)}(x) - f^{(k)}(y)\| \leqslant c|x-y|^\alpha, \qquad \forall x, y \in I,$$

where I is an interval.

Theorem 1.3. *Assume that the matrix $M(0)$ satisfies the condition $E(1)$. Then the matrix refinement equation* (1.1) *has a nonzero compactly supported solution in $C^{k+\alpha}$, where $0 \leqslant \alpha = |\ln\lambda|/\ln 2 < 1$, if and only if there is a 2^{1-k}-eigenvector w of the matrix (T_0+T_1) satisfying $(E_d, \dots, E_d)w = 0$ and*

$$\max_{d_i=0 \text{ or } 1} \|T_{d_1} \dots T_{d_m}\tilde{w}\| \leqslant c\Big(\frac{\lambda}{2^k}\Big)^m \tag{1.4}$$

for all $m \geqslant 1$, where $\tilde{w} = 2^k T_0 w - w$, $0 < \lambda < 1$ and E_d is the $d\times d$ unit matrix.

We remark that the corresponding result to Proposition 1.2 holds in the smooth case. The conditions (1.3) and (1.4) in Theorem 1.1 and 1.3 are analogs of Daubechies and Lagarias [6] and [7], Micchelli and Prautzsch [12]. However, our conditions are simpler and they apply to the vector-valued case. Moreover, to obtain results similar to Theorems 1.1 and 1.3 in the real-valued case, [6], [7], [11] demand that the coefficients of a refinement equation satisfy the 'sum rules', which is equivalent to insisting that $M(\xi)$ has a factor $(\frac{1}{2}(1+\xi))^k$ for some $k \geqslant 1$. It is known that no good analogs of 'sum rules' or factors $(\frac{1}{2}(1+\xi))^k$ exist in the vector-valued case [2], which causes more difficulty in treating the same problems. In order to get over them, Cohen, Daubechies and Plonk [2] assume some more complicated conditions on the coefficients, whereas we use the same method to deal with both the real and vector cases simultaneously. The initial idea of this paper comes from [10].

2. Continuous solutions of the matrix refinement equation (1.1)

In this section we study compactly supported continuous solutions of (1.1). If such a solution $f(x)$ exists, it is easy to verify that its support is contained in the interval $[0, N]$. We can decompose f into N pieces and form a multi-vector function as follows. Let

$$f_i(x) = f(x+i)\chi_{[0,1)}, \qquad i = 0, 1, \dots, N-1,$$

where $\chi_{[0,1)}$ is the characteristic function of $[0,1)$, and define a multi-vector function $F(x)$ by

$$F(x) = (f_0^T(x), f_1^T(x), \dots, f_{N-1}^T(x))^T,$$

where v^T is the transpose of a vector v. Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^d$ and $\|\cdot\|_\infty = \sup\limits_{0 \leqslant x < 1} \|\cdot\|$. The multi-vector function $F(x)$ is called the *unfold* of the vector function $f(x)$ and $f(x)$ is the *fold* of $F(x)$.

For a refinable vector $f(x)$, it's easy to check that its unfold $F(x)$ satisfies

$$F(x) = \begin{cases} T_0 F(2x) & \text{if } 0 \leqslant x < 1/2, \\ T_1 F(2x-1) & \text{if } 1/2 \leqslant x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Now we define an operator T on the multi-vector function $F(x)$ by

$$TF(x) = T_0F(2x) + T_1F(2x-1) = \begin{cases} T_0 F(2x) & \text{if } 0 \leqslant x < 1/2, \\ T_1 F(2x-1) & \text{if } 1/2 \leqslant x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Comparing (2.1) with (2.2), it's easy to show that the fold of the fixed point of the operator T is a compactly supported solution of (1.1), and the converse is true, too.

For any $x \in [0,1)$, x can be written uniquely as

$$x = \sum_{j=1}^{\infty} d_j(x) 2^{-j}, \qquad \text{where} \quad d_j = 0 \text{ or } 1 \quad \text{for all} \quad j,$$

if we assume that the above expression is a finite sum for all rational numbers which have two expressions. Let τ be the shift operator on [0,1) defined by

$$\tau x = \sum_{j=2}^{\infty} d_j(x) 2^{-j+1}$$

or equivalently by

$$\tau x = \begin{cases} 2x & \text{if } 0 \leqslant x < 1/2, \\ 2x-1 & \text{if } 1/2 \leqslant x < 1. \end{cases}$$

Then the operator T defined by (2.2) can be written as

$$TF(x) = T_{d_1(x)}F(\tau x)$$

for all $x \in [0,1)$. Then

$$T^m F(x) = T_{d_1(x)} \dots T_{d_m(x)} F(\tau^m(x)), \qquad x \in [0,1). \tag{2.3}$$

Proposition 2.1. *If the matrix $M(0)$ has eigenvalue 1, then the matrix (T_0+T_1) has eigenvalue 2. Conversely, if (T_0+T_1) has a 2-eigenvector v and $(E_d,\dots,E_d)v \neq 0$, then $M(0)$ has eigenvalue 1.*

Proof. Since

$$\begin{pmatrix} E_d & E_d & \dots & E_d \\ 0 & E_d & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & E_d \end{pmatrix} (T_0+T_1) \begin{pmatrix} E_d & -E_d & \dots & -E_d \\ 0 & E_d & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & E_d \end{pmatrix} = \begin{pmatrix} 2M(0) & 0 & \dots & 0 \\ & & * & \\ & & & \end{pmatrix},$$

it follows that (T_0+T_1) has eigenvalue 2. Conversely, by the hypothesis, we have

$$\begin{aligned} 2(E_d,\dots,E_d)v &= (E_d,\dots,E_d)(T_0+T_1)v \\ &= 2(M(0),\dots,M(0))v = 2M(0)(E_d,\dots,E_d)v. \end{aligned}$$

Then $M(0)$ has eigenvalue 1 and $(E_d,\dots,E_d)v$ is a corresponding eigenvector. □

Proof of the sufficiency part of Theorem 1.1.

(1) Let $F_0(x) = v$ for $x \in [0,1)$ and $F_k(x) = TF_{k-1}(x)$ for all $k \geqslant 1$. Then

$$\begin{aligned} \|F_{m+1}(x) - F_m(x)\|_\infty &= \|T_{d_1(x)} \dots T_{d_{m+1}(x)}v - T_{d_1(x)} \dots T_{d_m(x)}v\|_\infty \\ &= \|T_{d_1(x)} \dots T_{d_m(x)}\tilde{v}\|_\infty \\ &= \max_{d_i = 0 \text{ or } 1} \|T_{d_1} \dots T_{d_m}\tilde{v}\| \leqslant c\lambda^m \end{aligned} \tag{2.4}$$

for $x \in [0,1)$ and any positive integer m. Hence

$$\|F_m(x)\|_\infty \leqslant \|F_0(x)\|_\infty + \frac{c}{1-\lambda} = \|v\| + \frac{c}{1-\lambda}. \tag{2.5}$$

The inequalities (2.4) and (2.5) imply that the vector function sequence $\{F_m(x)\}$ converges uniformly to a vector function $F(x)$ in $[0,1)$.

(2) We claim that $\int_0^1 F_m(x)\,\mathrm{d}x = v$ for each m. This follows from (2.2) and the following induction argument:

$$\begin{aligned}\int_0^1 F_{m+1}(x)\,\mathrm{d}x &= \int_0^{\frac{1}{2}} T_0 F_m(2x)\,\mathrm{d}x + \int_{\frac{1}{2}}^1 T_1 F_m(2x-1)\,\mathrm{d}x \\ &= \frac{1}{2}(T_0+T_1)\int_0^1 F_m(x)\,\mathrm{d}x = v \neq 0.\end{aligned}$$

Hence the vector function $F(x)$ is nonzero.

(3) For any integer $j \geqslant 0$ and $x \in [0,1)$, we have

$$\begin{aligned}(2.6)\qquad & \|F_{m+j}(x) - F_m(x)\|_\infty \\ &\leqslant \|F_{m+j}(x) - F_{m+j-1}(x)\|_\infty + \ldots + \|F_{m+1}(x) - F_m(x)\|_\infty \\ &\leqslant c\lambda^m + \ldots + c\lambda^{m+j-1} \\ &\leqslant \frac{c}{1-\lambda}\lambda^m := C_1\lambda^m.\end{aligned}$$

As j tends to infinity, then (2.6) implies that

$$(2.7)\qquad \sup_{0\leqslant x<N} \|f(x) - f_m(x)\| \leqslant \sup_{0\leqslant x^*<1} \|F(x^*) - F_m(x^*)\| \leqslant C_1\lambda^m,$$

where $f(x)$ and $f_m(x)$ are the folds of $F(x)$ and $F_m(x)$, respectively.

(4) For any $m \geqslant 1$ and $x, y \in [0,N)$ with $2^{-(m+1)} \leqslant y - x < 2^{-m}$ there exists an odd integer $n \in \mathbb{N}$ such that one of the following two inequalities holds:

$$(n-1)2^{-m} \leqslant x \leqslant y \leqslant n2^{-m},$$

or

$$(n-1)2^{-m} < x \leqslant n2^{-m} < y < (n+1)2^{-m}.$$

We only discuss the second case, the first is similar. Note that there exists a $k \in \mathbb{N}$ such that $n2^{-m} - k \in (0,1)$. This implies that

$$n2^{-m} - k = n'2^{-m} = \frac{d_1}{2} + \frac{d_2}{2^2} + \ldots + \frac{d_{m-1}}{2^{m-1}} + \frac{1}{2^m}.$$

Since $y < (n+1)2^{-m}$, we have

$$y - k = y' = \frac{d_1}{2} + \frac{d_2}{2^2} + \ldots + \frac{d_{m-1}}{2^{m-1}} + \frac{1}{2^m} + \ldots.$$

Similarly we have, if $x \neq n2^{-m}$,

$$x - k = x' = \frac{d_1}{2} + \frac{d_2}{2^2} + \ldots + \frac{d_{m-1}}{2^{m-1}} + \frac{0}{2^m} + \ldots + \frac{1}{2^{m+q}} + \ldots$$

for some $q \geqslant 1$. It is clear that

$$\begin{aligned} \|f_m(y) - f_m(n2^{-m})\| &\leqslant \|F_m(y') - F_m(n'2^{-m})\| \\ &= \|T_{d_1} \ldots T_{d_{m-1}} T_1 v - T_{d_1} \ldots T_{d_{m-1}} T_1 v\| \\ &= 0. \end{aligned}$$

Similarly,

$$\begin{aligned} \|f_m(x) - f_m(n2^{-m})\| &\leqslant \|F_m(x') - F_m(n'2^{-m})\| \\ &= \|T_{d_1} \ldots T_{d_{m-1}} T_0 v - T_{d_1} \ldots T_{d_{m-1}} T_1 v\| \\ &= \|2T_{d_1} \ldots T_{d_{m-1}} T_0 \tilde{v}\| \\ &\leqslant 2c\lambda^{m-1}. \end{aligned}$$

Hence,

$$\begin{aligned} \|f(x) - f(y)\| &\leqslant \|f(x) - f_m(x)\| + \|f_m(x) - f_m(n2^{-m})\| \\ &\quad + \|f_m(n2^{-m}) - f_m(y)\| + \|f_m(y) - f(y)\| \\ &\leqslant 2(c\lambda^{-1} + C_1)\lambda^m \leqslant C_2|y - x|^\alpha, \end{aligned} \tag{2.8}$$

where $C_2 = 2(c\lambda^{-1} + C_1)2^\alpha$.

(5) For any $x \neq y \in [0, N)$, if $|x - y| \leqslant 1/2$, then there exists m such that $2^{-(m+1)} \leqslant |x - y| < 2^{-m}$, and so (2.8) holds. If $|x - y| > 1/2$, we assume that $x < y$. Let $x_i = i/4$, $i = 0, 1, \ldots, 4N$. Then there exist i and l such that $x_{i-1} < x \leqslant x_i$ and $x_{i+l} \leqslant y < x_{i+l+1}$. Consequently,

$$\begin{aligned} \|f(x) - f(y)\| &\leqslant \|f(x) - f(x_i)\| + \|f(x_i) - f(x_{i+1})\| + \ldots + \|f(x_{i+l}) - f(y)\| \\ &\leqslant C_2|x - x_i|^\alpha + C_2 l 4^{-\alpha} + C_2|x_{i+l} - y|^\alpha \\ &\leqslant 12NC_2|y - x|^\alpha, \end{aligned}$$

where we have used the simple inequality $a^\alpha + 4^{-\alpha} + b^\alpha \leqslant 3(a + 4^{-1} + b)^\alpha$ for nonnegative real numbers a and b. $\square$

To prove the necessary condition of Theorem 1.1, we need the following lemma.

Lemma 2.2. *Assume that the vector function $f(x)$ is a nonzero compactly supported continuous solution of* (1.1). *Then*

$$v_0 = \Big(\int_0^1 f^T(x)\,\mathrm{d}x, \dots, \int_{N-1}^N f^T(x)\,\mathrm{d}x\Big)^T \neq 0$$

and v_0 is a 2-eigenvector of (T_0+T_1). Moreover, let $F_0(x) = v_0$ for $x \in [0,1)$ and $F_k(x) = TF_{k-1}(x)$ for $k \geqslant 1$. Then

$$F_k(x) = T^k F_0(x) = 2^k \int_{D_k(x)}^{D_k(x)+1/2^k} F(t)\,\mathrm{d}t \tag{2.9}$$

where $D_k(x) = \frac{1}{2}d_1(x) + \ldots + \frac{1}{2^k}d_k(x)$ if $x = \sum\limits_{j=1}^{\infty} \frac{1}{2^j}d_j(x)$, and the vector function $F(x)$ is the unfold of the vector function $f(x)$.

Proof. Since $f(x)$ is a compactly supported continuous solution of (1.1), we have $TF(x) = F(x)$. By (2.3) we have $F(x) = T_{d_1(x)}\dots T_{d_k(x)}F(\tau^k x)$. Integrating this over $[D_k(x), D_k(x)+1/2^k]$ and by (2.3) again, we obtain

$$2^k \int_{D_k(x)}^{D_k(x)+1/2^k} F(t)\,\mathrm{d}t = T_{d_1(x)}\dots T_{d_k(x)}F_0(x) = T^k F_0(x) = F_k(x),$$

which converges to $F(x)$ as $k \to \infty$. Then $v_0 = F_0(x) \neq 0$. From $TF(x) = F(x)$, we have $(T_0+T_1)v_0 = 2v_0$ by integrating both sides of (2.2) over [0,1], that is, v_0 is a 2-eigenvector of (T_0+T_1). □

Proof of the necessity part of Theorem 1.1. Let v_0 be the vector defined in Lemma 2.2. Let $F_0(x) = v_0$ for $x \in [0,1)$ and $F_k(x) = TF_{k-1}(x)$ for $k \geqslant 1$. By Lemma 2.2 and the integral mean value theorem we have

$$\begin{aligned}
\max_{d_i=0 \text{ or } 1} \|T_{d_1}\dots T_{d_m}\tilde{v}\| &= \|T_{d_1(x)}\dots T_{d_m(x)}T_0 v_0 - T_{d_1(x)}\dots T_{d_m(x)}v_0\|_\infty \\
&= \|F_{m+1}(x) - F_m(x)\|_\infty \\
&= \Big\|2^{m+1}\int_{D_m(x)}^{D_m(x)+1/2^{m+1}} F(x)\,\mathrm{d}x - 2^m \int_{D_m(x)}^{D_m(x)+1/2^m} F(x)\,\mathrm{d}x\Big\|_\infty \\
&\leqslant c\Big(\frac{1}{2^m}\Big)^\alpha = c\lambda^m.
\end{aligned}$$

□

For any 2-eigenvector v of (T_0+T_1), let $\tilde{v} = T_0 v - v$ and let $H(\tilde{v})$ be the subspace in $\mathbb{R}^{dN}$ defined by

$$H(\tilde{v}) = \operatorname{span}\{\tilde{v}, T_{d_1}\dots T_{d_n}\tilde{v} : d_j = 0 \text{ or } 1, 1 \leqslant j \leqslant n, n = 1,2,\dots\}. \tag{2.10}$$

Proposition 2.3. *$H(\tilde{v})$ is the smallest common invariant subspace of T_0 and T_1 which contains $\tilde{v}$.*

Proof. It is trivial by the definition of $H(\tilde{v})$. □

Theorem 2.4. *The following statements are equivalent:*

(a) *The matrix refinement equation* (1.1) *has a nonzero compactly supported continuous solution.*

(b) *There exists a 2-eigenvector v of the matrix $(T_0 + T_1)$ satisfying*

$$\lim_{n\to\infty} \max_{d_i=0 \text{ or } 1} \|T_{d_1}\dots T_{d_n}\tilde{v}\| = 0. \tag{2.11}$$

(c) *There exists a 2-eigenvector v of the matrix $(T_0 + T_1)$ such that there exists an integer $m \geqslant 1$ satisfying*

$$\alpha_m = \max_{d_i=0 \text{ or } 1} \|T_{d_1}\dots T_{d_m}|_{H(\tilde{v})}\|^{\frac{1}{m}} < 1. \tag{2.12}$$

Proof. (a) $\Rightarrow$ (b). Let $F_0(x) = v_0 = (\int_0^1 f^T(x)\,\mathrm{d}x, \dots, \int_{N-1}^N f^T(x)\,\mathrm{d}x)^T$ for $x \in [0,1)$ and $F_k(x) = TF_{k-1}(x)$ for all $k \geqslant 1$. By Lemma 2.2,

$$F_k(x) = 2^k \int_{D_k(x)}^{D_k(x)+1/2^k} F(t)\,\mathrm{d}t$$

converges to $F(x)$ uniformly on $[0,1)$. Hence

$$\sup_{x\in[0,1)} \|F_{m+1}(x) - F_m(x)\| \to 0$$

as $m \to \infty$, and (b) follows immediately by (2.9).

(b) $\Rightarrow$ (c). Note that α_m^m has an equivalent form

$$\max_{d_i=0 \text{ or } 1} \|T_{d_1}\dots T_{d_m}u\| < 1$$

for all $u \in H(\tilde{v})$ and $\|u\| \leqslant 1$. The subspace $H(\tilde{v})$ is finite dimensional and has a finite basis consisting of $T_{d_1'}\dots T_{d_l'}\tilde{v}$'s. Let $u = T_{d_1'}\dots T_{d_l'}\tilde{v}$ be one of the elements of the basis. Then we have

$$\max_{d_j=0 \text{ or } 1} \|T_{d_1}\dots T_{d_m}u\| \leqslant \max_{d_j,d_i'=0 \text{ or } 1} \|T_{d_1}\dots T_{d_m}T_{d_1'}\dots T_{d_l'}\tilde{v}\| \to 0 \tag{2.13}$$

as $m \to \infty$, hence (2.13) holds for all elements of the basis uniformly. So the convergence is uniform for all $\|u\| \leqslant 1$. Hence (c) follows by taking m sufficiently large.

(c) $\Rightarrow$ (a). For any $u \in H(\tilde{v})$ we have

$$\max_{d_j=0\text{ or }1} \|T_{d_1} \dots T_{d_m} u\| \leqslant \alpha_m^m \|u\|.$$

Let $n = qm + r$ with $q, r \in \mathbb{N}$ and $0 \leqslant r < m$. Then

$$\max_{d_j=0\text{ or }1} \|T_{d_1} \dots T_{d_n} \tilde{v}\| \leqslant \alpha_m^{qm} \alpha_r^r \leqslant \max(1, \alpha_1, \dots, \alpha_{m-1}^{m-1}) \alpha_m^{-m+1} \alpha_m^n = c\alpha_m^n.$$

By Theorem 1.1, (a) follows. □

Using the notation from Theorem 2.4, we remark that if a solution of (1.1) exists, then $v \notin H(\tilde{v})$ and the dimension of $H(\tilde{v})$ is not more than $dN - 1$. In fact, if $v \in H(\tilde{v})$, then

$$\|v\| = \frac{1}{2^m} \|(T_0 + T_1)^m v\| \leqslant \max_{d_j=0\text{ or }1} \|T_{d_1} \dots T_{d_m} v\| \leqslant c\lambda^m \to 0$$

as $m \to \infty$. This contradicts $v \neq 0$.

Corollary 2.5. *If the matrix refinement equation* (1.1) *has a compactly continuous solution, then the solution is Hölder continuous.*

3. Smooth solutions of the matrix refinement equation (1.1)

In this section we assume that the matrix $M(0)$ satisfies the condition $E(1)$, which is necessary for constructing multi-wavelet by the multiresolution analysis. Now we consider the matrix refinement equation

$$\varphi(x) = 2^k \sum_{n=0}^{N} C_n \varphi(2x - n), \tag{3.1}$$

where k is a positive integer. It's easy to verify that $\operatorname{supp} \varphi(x) \subseteq [0, N]$ if a solution $\varphi(x)$ of (3.1) has compact support.

Similarly to Section 2, we define an operator A on a vector function $\Phi(x)$ by

$$\begin{aligned} A\Phi(x) &= 2^k T_0 \Phi(2x) + 2^k T_1 \Phi(2x-1) \\ &= \begin{cases} 2^k T_0 \Phi(2x) & \text{if } 0 \leqslant x < \frac{1}{2}, \\ 2^k T_1 \Phi(2x-1) & \text{if } \frac{1}{2} \leqslant x < 1, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{3.2}$$

where T_0 and T_1 are the same matrices as defined in Section 1. Let $\Phi_0(x)$ be a vector function and $\Phi_m(x) = A\Phi_{m-1}(x)$ for all $m \geqslant 1$. We have

$$\Phi_m(x) = 2^{mk} T_{d_1(x)} \dots T_{d_m(x)} \Phi_0(\tau^m x) \tag{3.3}$$

for $x \in [0,1)$, where $x = \sum\limits_{i=1}^{\infty} \frac{1}{2^i} d_i(x)$, $d_i = 0$ or 1 for all $i \geqslant 1$.

Lemma 3.1. *Assume that φ is a nonzero compactly supported continuous solution of* (3.1). *Then*

$$w_0 = \left(\int_0^1 \varphi^T(x), \dots, \int_{N-1}^N \varphi^T(x) \right)^T \neq 0$$

and w_0 is a right 2^{1-k}-eigenvector of the matrix $(T_0 + T_1)$. Moreover, let $\Phi_0(x) = w_0$ for $0 \leqslant x < 1$, $\Phi_m(x) = A\Phi_{m-1}(x)$, $m = 1, 2, \dots$. We have

$$\Phi_m(x) = 2^{mk} T_{d_1(x)} \dots T_{d_m(x)} w_0 = 2^m \int_{D_m(x)}^{D_m(x)+1/2^m} \Phi(t)\,\mathrm{d}t$$

where $D_m(x) = \frac{1}{2} d_1(x) + \dots + \frac{1}{2^m} d_m(x)$ if $x = \sum\limits_{m=1}^{\infty} \frac{1}{2^m} d_m(x)$, and $\Phi(x)$ is the unfold of $\varphi(x)$.

Proof. By $A\Phi(x) = \Phi(x)$ and (3.2) we have

$$\Phi(x) = A^m \Phi(x) = 2^{mk} T_{d_1(x)} \dots T_{d_m(x)} \Phi(\tau^m x).$$

Integrating the above equation over $[D_m(x), D_m(x) + \frac{1}{2^m}]$, we obtain

$$A^m \Phi_0(x) = 2^m \int_{D_m(x)}^{D_m(x)+1/2^m} \Phi(t)\,\mathrm{d}t.$$

Since $\Phi(t)$ is continuous on $[0,1)$, it's easy to show that

$$\lim_{m \to \infty} A^m \Phi_0(x) = \Phi(x) \not\equiv 0.$$

Hence $w_0 = \Phi_0(x) \neq 0$ for $x \in [0,1)$. The fact that w_0 is a right 2^{1-k}-eigenvector of $(T_0 + T_1)$ follows by integrating (3.2) over [0,1]. $\square$

Proof of Theorem 1.3. First we prove sufficiency. Let $\Phi_0(x) = w$ for $x \in [0,1)$ and $\Phi_m(x) = A\Phi_{m-1}(x)$ for $m \geqslant 1$. Then

$$\begin{aligned}\|\Phi_{m+1}(x) - \Phi_m(x)\|_\infty &= \|2^{mk}T_{d_1(x)}\dots T_{d_m(x)}2^kT_{d_{m+1}}w - 2^{mk}T_{d_1(x)}\dots T_{d_m(x)}w\|_\infty \\ &\leqslant 2^{mk}\max_{d_j=0\text{ or }1}\|T_{d_1}\dots T_{d_m}\tilde{w}\| \\ &\leqslant c\lambda^m.\end{aligned}$$

Similarly to the proof of Theorem 1.1, we know that the vector function sequence $\{\Phi_n(x)\}$ converges uniformly to $\Phi(x)$ and $\Phi(x) \not\equiv 0$ on $[0,1)$. Let $\varphi(x)$ be the fold of $\Phi(x)$. Then $\varphi(x)$ is a continuous solution of equation (3.1) and $\operatorname{supp}\varphi(x) \subseteq [0,N]$. Taking Fourier transforms of (3.1) gives

$$\hat{\varphi}(\xi) = 2^{kn}\prod_{i=1}^{n} M\Big(\frac{\xi}{2^i}\Big)\hat{\varphi}\Big(\frac{\xi}{2^n}\Big).$$

The results of Colella and Heil [5] show that $\prod\limits_{i=1}^{n} M(\frac{1}{2^i}\xi)$ converges to $\prod\limits_{i=1}^{\infty} M(\frac{1}{2^i}\xi) \neq 0$. Hence $\hat{\varphi}(0) = 0$, i.e.

$$\int_{-\infty}^{\infty}\varphi(x)\,\mathrm{d}x = 0. \tag{3.4}$$

Let $f_1(x) = \int_{-\infty}^{x}\varphi(t)\,\mathrm{d}t$. It's clear that

$$f_1(x) = 2^{k-1}\sum_{n=0}^{N} C_n f_1(2x-n)$$

and $f_1(x)$ has compact support contained in $[0,N]$.

Repeating the above procedure finite times, it's easy to see that

$$f(x) = \int_{-\infty}^{x}\int_{-\infty}^{x_1}\dots\int_{-\infty}^{x_{k-1}}\varphi(t)\,\mathrm{d}t$$

is a solution of (1.1) satisfying $f \in C^{k+\alpha}$ and $\operatorname{supp} f(x) \subseteq [0,N]$.

Now suppose that there is a nonzero compactly supported solution $f(x)$ of the equation (1.1) in $C^{k+\alpha}$. If we take k derivatives on both sides of (1.1), we see that (3.1) has a Hölder continuous solution $\varphi = f^{(k)}$ with compact support in $[0,N]$ and Hölder exponent $\alpha = |\ln\lambda|/\ln 2$. Let $w_0 = (\int_0^1 \varphi^T(x), \dots, \int_{N-1}^{N}\varphi^T(x))^T$. From the proof of sufficiency, we have $(E_d,\dots,E_d)w_0 = \int_0^N \varphi(x)\,\mathrm{d}x = 0$. Lemma 3.1 implies

$$2^{mk}T_{d_1(x)}\dots T_{d_m(x)}w_0 = 2^m\int_{D_m(x)}^{D_m(x)+1/2^m}\Phi(t)\,\mathrm{d}t$$

for $x \in [D_m(x), D_m(x)+\frac{1}{2^m})$. By the integral mean value theorem, we get

$$\begin{aligned}\|T_{d_1(x)}\dots T_{d_m(x)}\tilde{w_0}\| &= \frac{1}{2^{mk}}\|2^{mk}T_{d_1(x)}\dots T_{d_m(x)}2^kT_0w_0 - 2^{mk}T_{d_1(x)}\dots T_{d_m(x)}w_0\| \\ &= \frac{1}{2^{mk}}\left\|2^{m+1}\int_{D_m(x)}^{D_m(x)+1/2^{m+1}}\Phi(x)\,\mathrm{d}x - 2^m\int_{D_m(x)}^{D_m(x)+1/2^m}\Phi(x)\,\mathrm{d}x\right\| \\ &\leqslant \frac{1}{2^{mk}}c\Big(\frac{1}{2^m}\Big)^\alpha = c\Big(\frac{\lambda}{2^k}\Big)^m.\end{aligned}$$

The inequality (1.4) follows by taking maximum on the left hand side. □

Similarly to Theorem 2.4, the following theorem is obvious.

Theorem 3.2. *The following statements are equivalent:*

(a) *The matrix refinement equation of* (1.1) *has a nonzero compactly supported solution in* C^k.

(b) *There exists a* 2^{1-k}*-eigenvector* w *of the matrix* (T_0+T_1) *satisfying* $(E_d,\dots E_d)$ $w=0$ *and*

$$\max_{d_i=0\text{ or }1}\|T_{d_1}\dots T_{d_n}\tilde{w}\| \longrightarrow 0, \quad \text{as } n\longrightarrow\infty,$$

where $\tilde{w}=2^kT_0w-w$.

(c) *There exists a* 2^{1-k}*-eigenvector* w *of* (T_0+T_1) *with* $(E_d,\dots E_d)w=0$ *such that there exists an integer* $m\geqslant 1$ *satisfying*

$$\alpha_m = \max_{d_i=0\text{ or }1}\|T_{d_1}\dots T_{d_m}|_{H(\tilde{w})}\|^{\frac{1}{m}} < \frac{1}{2^k}.$$

4. Examples

Example 4.1. Consider the refinement equation

$$f(x)=\tfrac{3}{4}f(2x)-\tfrac{1}{2}f(2x-1)+\tfrac{3}{2}f(2x-2)-\tfrac{1}{2}f(2x-3)+\tfrac{3}{4}f(2x-4).$$

It has a compactly supported solution which is continuous but not continuously differentiable.

P r o o f. By definitions in Section 1 we obtain that

$$T_0=\begin{pmatrix}3/4&0&0&0\\3/2&-1/2&3/4&0\\3/4&-1/2&3/2&-1/2\\0&0&3/4&-1/2\end{pmatrix},\qquad T_1=\begin{pmatrix}-1/2&3/4&0&0\\-1/2&3/2&-1/2&3/4\\0&3/4&-1/2&3/2\\0&0&0&3/4\end{pmatrix}.$$

Then there is a 2-eigenvector $v = (1, \frac{7}{3}, \frac{7}{3}, 1)^T$ of $T_0 + T_1$ and $\tilde{v} = T_0 v - v = \frac{1}{4}(-1,-1,1,1)^T$. Let $e_1 = 2^{-1/2}(1,0,-1,0)^T$ and $e_2 = 2^{-1/2}(0,1,0,-1)^T$. We have

$$H(\tilde{v}) = \operatorname{span}\{e_1, e_2\}$$

and $\{e_1, e_2\}$ is an orthonormal basis of the linear subspace $H(\tilde{v})$. Note that

$$T_0(e_1,e_2) = (e_1,e_2)\begin{pmatrix} 3/4 & 0 \\ 3/4 & -1/2 \end{pmatrix}, \quad T_1(e_1,e_2) = (e_1,e_2)\begin{pmatrix} -1/2 & 3/4 \\ 0 & 3/4 \end{pmatrix}.$$

Therefore

$$\begin{aligned} \|T_0T_0|_{H(\tilde{v})}\| &= \|T_1T_1|_{H(\tilde{v})}\| = (1/256(53+1513^{1/2}))^{1/2} \doteq 0.599144, \\ \|T_1T_0|_{H(\tilde{v})}\| &= \|T_0T_1|_{H(\tilde{v})}\| = (9/256(9+65^{1/2}))^{1/2} \doteq 0.774497. \end{aligned}$$

By Theorem 2.4, it follows that the refinement equation has a continuous solution with compact support.

Moreover, since $w = (-1,-1,1,1)^T$ is a unique 1-eigenvector of (T_0+T_1) up to a scalar multiple, then $\tilde{w} = 2T_0w - w = \frac{1}{2}(-1,1,1,-1)$ and $H(\tilde{w}) = H(\tilde{v})$ by the definitions in Section 3. It is clear that

$$\max_{d_i=0 \text{ or } 1} \|T_{d_1}\dots T_{d_m}|_{H(\tilde{w})}\|^{1/m} \geqslant \|T_0\dots T_0|_{H(\tilde{w})}\|^{1/m} \geqslant \tfrac{3}{4} > \tfrac{1}{2}$$

for all positive integers m. By Theorem 3.2 we conclude that the solution is not continuously differentiable. □

We remark that the coefficients of the refinement equation of Example 4.1 do not satisfy the 'sum rule' conditions.

Example 4.2. Consider the refinement equation

$$f(x) = \begin{pmatrix} 3/4 & 1/2 \\ 1/4 & 1/4 \end{pmatrix} f(2x) + \begin{pmatrix} 1/2 & 0 \\ 0 & 3/4 \end{pmatrix} f(2x-1) + \begin{pmatrix} 3/4 & -1/2 \\ -1/4 & 0 \end{pmatrix} f(2x-2).$$

It has a continuous but not continuously differentiable solution with compact support.

P r o o f. Since

$$T_0 = \begin{pmatrix} 3/4 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 \\ 3/4 & -1/2 & 1/2 & 0 \\ -1/4 & 0 & 0 & 3/4 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 1/2 & 0 & 3/4 & 1/2 \\ 0 & 3/4 & 1/4 & 1/4 \\ 0 & 0 & 3/4 & -1/2 \\ 0 & 0 & -1/4 & 0 \end{pmatrix},$$

it is easy to obtain that one of the 2-eigenvectors of $T_0 + T_1$ is

$$v = (-0.65653, -0.26261, -0.65653, 0.26261)^T$$

and

$$\tilde{v} = T_0 v - v = (0.032827, 0.032827, -0.032827, 0.09848)^T.$$

Let

$$\begin{aligned}
e_1 &= (-0.42329, -0.3415, 0.41397, -0.72996)^T, \\
e_2 &= (-0.16734, -0.27631, 0.70811, 0.62788)^T, \\
e_3 &= (-0.88816, 0.15176, -0.36223, 0.2386)^T.
\end{aligned}$$

Then e_1, e_2 and e_3 form an orthonormal basis of $H(\tilde{v})$. Note that

$$T_0\,[e_1, e_2, e_3] = [e_1, e_2, e_3]\,A, \qquad T_1\,[e_1, e_2, e_3] = [e_1, e_2, e_3]\,B,$$

where

$$A = \begin{pmatrix} 0.61928 & -0.072999 & -0.36214 \\ -0.1001 & 0.65638 & -0.25226 \\ 0.27739 & 0.20685 & 0.92634 \end{pmatrix},$$
$$B = \begin{pmatrix} 0.58226 & -0.14645 & -0.0037973 \\ 0.55045 & -0.11982 & -0.14309 \\ -0.083846 & -0.77786 & 0.70556 \end{pmatrix}.$$

We have

$$\begin{aligned}
\|T_0^4|_{H(\tilde{v})}\| &= (0.55976)^{1/2} = 0.74817 < 1, \\
\|T_0^3T_1|_{H(\tilde{v})}\| &= (0.83747)^{1/2} = 0.91513 < 1, \\
\|T_0^2T_1^2|_{H(\tilde{v})}\| &= (0.6987)^{1/2} = 0.83588 < 1, \\
\|T_0T_1^3|_{H(\tilde{v})}\| &= (0.76482)^{1/2} = 0.87454 < 1, \\
\|T_1^4|_{H(\tilde{v})}\| &= (0.58474)^{1/2} = 0.76468 < 1, \\
\|T_1^3T_0|_{H(\tilde{v})}\| &= (0.65839)^{1/2} = 0.81141 < 1, \\
\|T_1^2T_0^2|_{H(\tilde{v})}\| &= (0.87568)^{1/2} = 0.93578 < 1, \\
\|T_1T_0^3|_{H(\tilde{v})}\| &= (0.79495)^{1/2} = 0.89160 < 1, \\
\|T_0T_1T_0T_1|_{H(\tilde{v})}\| &= (0.97618)^{1/2} = 0.9880 < 1, \\
\|T_0T_1^2T_0|_{H(\tilde{v})}\| &= (0.82331)^{1/2} = 0.90736 < 1, \\
\|T_1T_0T_1T_0|_{H(\tilde{v})}\| &= (0.93476)^{1/2} = 0.96683 < 1, \\
\|T_1T_0^2T_1|_{H(\tilde{v})}\| &= (0.9385)^{1/2} = 0.9688 < 1, \\
\|T_0^2T_1T_0|_{H(\tilde{v})}\| &= (0.85229)^{1/2} = 0.92319 < 1,
\end{aligned}$$

$$\|T_0T_1T_0^2|_{H(\tilde{v})}\| = (0.90202)^{1/2} = 0.94975 < 1,$$
$$\|T_1^2T_0T_1|_{H(\tilde{v})}\| = (0.90416)^{1/2} = 0.95087 < 1,$$
$$\|T_1T_0T_1^2|_{H(\tilde{v})}\| = (0.87117)^{1/2} = 0.93336 < 1.$$

According to Theorem 2.4, the refinement equation has a continuous solution with compact support.

Moreover, since all the 1-eigenvectors of $T_0 + T_1$ are $\{\lambda w : \lambda \in \mathbb{R}\}$ where $w = (0.57735, 0.57735, -0.57735, 0.00000)^T$, we have $(E_2, E_2, E_2, E_2)w = 0.57735 \neq 0$. By Theorem 3.2 we conclude that the solution is not continuously differentiable. □

Acknowledgement. The authors wish to thank the referee for his (or her) invaluable comments and opinions.

References

[1] *A. S. Cavaretta, W. Dahmen and C. A. Micchelli*: Stationary subdivision. Memoirs of Amer. Math. Soc., Vol. 93, 1991.
[2] *A. Cohen, I. Daubechies and G. Plonka*: Regularity of refinable function vectors. J. Fourier Anal. Appl. *3* (1997), 295–324.
[3] *A. Cohen, K. Gröchenig and L. F. Villemoes*: Regularity of multivariate refinable functions. Constr. Approx. *15* (1999), 241–255.
[4] *D. Colella and C. Heil*: Characterizations of scaling functions: Continuous solutions. SIAM J. Matrix Anal. Appl. *15* (1994), 496–518.
[5] *C. Heil and D. Colella*: Matrix refinement equations: Existence and uniqueness. J. Fourier Anal. Appl. *2* (1996), 363–377.
[6] *I. Daubechies and J. Lagarias*: Two-scale difference equation I. SIAM J. Math. Anal. *22* (1991), 1388–1410.
[7] *I. Daubechies and J. Lagarias*: Two-scale difference equation II. Local regularity, infinite products of matrices, and fractals. SIAM J. Math. Anal. *22* (1991), 1388–1410.
[8] *T. A. Hogan*: A note on matrix refinement equation. SIAM J. Math. Anal. *4* (1998), 849–854.
[9] *R. Q. Jia, K. S. Lau and J. R. Wang*: L_p solutions of refinement equations. J. Fourier Anal. and Appli. *7* (2001), 143–167.
[10] *K. S. Lau and J. R. Wang*: Characterization of L^p-solutions for the two-scale dilation equations. SIAM J. Math. Anal. *26* (1995), 1018–1046.
[11] *P. Massopust, D. Ruch and P. Van Fleet*: On the support properties of scaling vectors. Appl. Comp. Harmonic Anal. *3* (1996), 229–238.
[12] *C. A. Micchelli and H. Prautzsch*: Uniform refinement of curves Linear Algebra and Its Applications.
[13] *G. Plonka and V. Strela*: From wavelets to multiwavelets. Mathematical methods for curves and surfaces, 2 (Lillehammer) (1997), 375–399.
[14] *Z. W. Shen*: Refinable function vectors. SIAM J. Math. Anal. *29* (1998), 235–250.
[15] *D. X. Zhou*: Existence of multiple refinable distributions. Michigan Math. J. *44* (1997), 317–329.

Authors' addresses: Xing-Gang He, Department of Mathematics, Central China Normal University, Wuhan, 430079, P. R. China, e-mail: `xingganghe@sina.com`; Chun-Tai Liu, Department of Mathematics, Central China Normal University, Wuhan, 430079, P. R. China, e-mail: `lct984@163.com`.