A NEW CHARACTERIZATION OF ANDERSON'S INEQUALITY IN C_1-CLASSES

S. Mecheri, Riyadh

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Abstract. Let $\mathscr{H}$ be a separable infinite dimensional complex Hilbert space, and let $\mathscr{L}(H)$ denote the algebra of all bounded linear operators on $\mathscr{H}$ into itself. Let $A = (A_1, A_2, \ldots, A_n)$, $B = (B_1, B_2, \ldots, B_n)$ be n-tuples of operators in $\mathscr{L}(H)$; we define the elementary operators $\Delta_{A,B}\colon \mathscr{L}(H) \mapsto \mathscr{L}(H)$ by

$$\Delta_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i - X.$$

In this paper, we characterize the class of pairs of operators $A, B \in \mathscr{L}(H)$ satisfying Putnam-Fuglede's property, i.e, the class of pairs of operators $A, B \in \mathscr{L}(H)$ such that $\sum\limits_{i=1}^{n} B_i T A_i = T$ implies $\sum\limits_{i=1}^{n} A_i^* T B_i^* = T$ for all $T \in \mathscr{C}_1(H)$ (trace class operators). The main result is the equivalence between this property and the fact that the ultraweak closure of the range of the elementary operator $\Delta_{A,B}$ is closed under taking adjoints. This leads us to give a new characterization of the orthogonality (in the sense of Birkhoff) of the range of an elementary operator and its kernel in C_1 classes.

Keywords: C_1-class, generalized p-symmetric operator, Anderson Inequality

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1. Introduction

Let $\mathscr{H}$ be a separable infinite dimensional complex Hilbert space, and let $\mathscr{L}(H)$ denote the algebra of all bounded linear operators on $\mathscr{H}$ into itself. Given $A, B \in \mathscr{L}(H)$, we define the generalized derivation $\delta_{A,B}\colon \mathscr{L}(H) \mapsto \mathscr{L}(H)$ by $\delta_{A,B}(X) = AX - XB$. Note $\delta_{A,A} = \delta_A$. Let $A = (A_1, A_2, \ldots, A_n)$, $B =$

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$(B_1, B_2, \ldots, B_n)$ be n-tuples of operators in $\mathscr{L}(H)$, and define the elementary operator $\Delta_{A,B}\colon \mathscr{L}(H) \mapsto \mathscr{L}(H)$ by $\Delta_{A,B}(X) = \sum\limits_{i=1}^{n} A_i X B_i - X$. In [2] J. Anderson, J. Bunce, J. A. Deddens and J. P. Williams show that, if A is D-symmetric, (i.e. $\overline{R(\delta_A)} = \overline{R(\delta_{A^*})}$, where $\overline{R(\delta_A)}$ is the closure of the range, $R(\delta_A)$, of δ_A in the norm topology), then $AT = TA$ implies $A^*T = TA^*$ for every $T \in \mathscr{C}_1(H)$ (trace class operators).

S. Bouali and J. Charles in [3] gave some properties of P-symmetric operators, the class of operators A such that $AT = TA$ implies $A^*T = TA^*$ for every $T \in \mathscr{C}_1(H)$. In order to generalize these results we initiated in [4] the study of a more general class of P-symmetric operators, namely the class of pairs of operators $A, B \in \mathscr{L}(H)$ such that $BT = TA$ implies $A^*T = TB^*$ for all $T \in \mathscr{C}_1(H)$. We call such operators generalized P-symmetric operators. In this paper we characterize the class of pairs of operators $A, B \in \mathscr{L}(H)$ such that $\sum\limits_{i=1}^{n} B_i T A_i = T$ implies $\sum\limits_{i=1}^{n} A_i^* T B_i^* = T$ for all $T \in \mathscr{C}_1(H)$. This leads us to present a new characterization of the orthogonality (in the sense of Birkhoff) of the range of an elementary operator and its kernel in C_1 classes.

2. Preliminaries

The ideal $C_1(H)$ of $\mathscr{L}(H)$ admits a trace function $\operatorname{tr}(T)$, given by $\operatorname{tr}(T) = \sum\limits_{n}(Te_n, e_n)$ for any complete orthonormal system (e_n) in H. As a Banach space $C_1(H)$ can be identified with the dual of the ideal K of compact operators by means of the linear isometry $T \mapsto f_T$, where $f_T(X) = \operatorname{tr}(XT)$. Moreover $\mathscr{L}(H)$ is the dual of $C_1(H)$. The ultraweakly continuous linear functionals on $\mathscr{L}(H)$ are those of the form f_T for $T \in C_1(H)$ and the weakly continuous ones are those of the form f_T with T of finite rank.

Definition 1. Let E be a complex Banach space. We say that $b \in E$ is orthogonal to $a \in E$ if for all complex λ there holds

$$\|a + \lambda b\| \geqslant \|a\|. \tag{1.1}$$

This definition has a natural geometric interpretation. Namely, $b \perp a$ if and only if the complex line $\{a + \lambda b\colon \lambda \in \mathbb{C}\}$ is disjoint with the open ball $K(0, \|a\|)$, i.e., iff this complex line is a tangent one. Note that if b is orthogonal to a, then a need not be orthogonal to b. If E is a Hilbert space, then from (1.1) follows $\langle a, b\rangle = 0$, i.e, orthogonality in the usual sense.

Definition 2. Let $A = (A_1, A_2, \ldots, A_n)$, $B = (B_1, B_2, \ldots, B_n)$ be n-tuples of operators in $\mathscr{L}(H)$. The pair (A, B) is called generalized D-symmetric pair of operators if $\overline{R(\Delta_{A,B})} = \overline{R(\Delta_{B^*,A^*})}$. The set of such pairs is denoted by $\mathrm{GS}(H)$. Here $\overline{R(\Delta_{A,B})}$ is the closure of the range $R(\Delta_{A,B})$ of $\Delta_{A,B}$ in the norm topology.

Definition 3. Let $A = (A_1, A_2, \ldots, A_n)$, $B = (B_1, B_2, \ldots, B_n)$ be n-tuples of operators in $\mathscr{L}(H)$. The pair (A, B) of operators such that $\sum\limits_{i=1}^{n} B_i T A_i = T$ implies $\sum\limits_{i=1}^{n} A_i^* T B_i^* = T$ for all $T \in \mathscr{C}_1(H)$ is called generalized P-symmetric pair of operators. The set of such pairs is denoted by $\mathrm{GF}_0\,(H)$.

2. Main Results

Theorem 4. *Let $A = (A_1, A_2 \ldots, A_n)$, $B = (B_1, B_2 \ldots, B_n)$ be n-tuples of operators in $\mathscr{L}(H)$. Then*

$$(A, B) \in \mathrm{GF}_0\, H \;\Leftrightarrow\; \overline{R(\Delta_{A,B})}^{w*} \text{ is closed under taking adjoints.}$$

Proof. The w^*-topology is generated by all f_T with $T \in C_1$ and so $\overline{R(\Delta_{A,B})}^{w*}$ is the intersection

$$\bigcap \left\{ \ker f_T \colon\; f_T\left(\sum_{i=1}^{n} A_i X B_i - X\right) = 0, \quad \text{for all}\;\; X \in \mathscr{L}(H) \right\}.$$

Since

$$\begin{aligned} f_T\left(\sum_{i=1}^{n} A_i X B_i - X\right) &= \operatorname{tr}\left(T\left(\sum_{i=1}^{n} A_i X B_i - X\right)\right) \\ &= \operatorname{tr}\left(\left(\sum_{i=1}^{n} B_i T A_i - T\right) X\right) \end{aligned}$$

this intersection is

$$\bigcap \{\ker f_T \colon\; T \in \ker \Delta_{B,A} \cap \mathscr{C}_1(\mathscr{H})\}.$$

If $(A, B) \in \mathrm{GF}_0(\mathscr{H})$, then

$$\ker \Delta_{B,A} \cap \mathscr{C}_1(\mathscr{H}) = \ker \Delta_{A^*,B^*} \cap \mathscr{C}_1(\mathscr{H})$$

and so the weak*-closure of

$$R(\Delta_{B^*,A^*}) = (R(\Delta_{A,B}))^*.$$

Conversely, if $\overline{R(\Delta_{A,B})}^{w*}$ is self-adjoint the set of $T \in \mathscr{C}_1(\mathscr{H})$ for which f_T vanishes on $R(\Delta_{A,B})$ must be self-adjoint ($Y \in R(\Delta_{A,B})$ implies $0 = f_T(Y^*) = \operatorname{tr}(TY^*) = \overline{\operatorname{tr}(T^*Y)}$). Hence

$$\ker \Delta_{B,A} \cap \mathscr{C}_1(H) = \ker \Delta_{A^*,B^*} \cap \mathscr{C}_1(H),$$

and $(A, B) \in \mathrm{GF}_0(H)$. □

Theorem 5 ([5]). *Let $A = (A_1, A_2, \ldots, A_n)$, $B = (B_1, B_2, \ldots, B_n)$ be n-tuples of operators in $\mathscr{L}(H)$, and let $T \in \mathscr{C}_1$ have the polar decomposition $T = U|T|$. Then the following statements are equivalent:*

1. $\|T + \Delta_{A,B}(X)\|_1 \geqslant \|T\|_1$, *for all $T \in \ker \Delta_{A,B}|_{\mathscr{C}_1}$ and for all $X \in \mathscr{C}_1$.*
2. $(A, B) \in \mathrm{GF}_0(H)$.

Now by using Theorem 4 and Theorem 5 it is easy to prove the following theorem which gives us an other characterization of the orthogonality of the range of $\Delta_{A,B}$ and its kernel.

Theorem 6. *Let $A = (A_1, A_2 \ldots, A_n)$, $B = (B_1, B_2 \ldots, B_n)$ be n-tuples of operators in $\mathscr{L}(H)$, and let T have the polar decomposition $T = U|T|$. Then the following statements are equivalent:*

(i) $\overline{R(\Delta_{A,B})}^{w*}$ *is closed under taking adjoints*
(ii) $(A, B) \in \mathrm{GF}_0\, H$
(iii) $\|T + \Delta_{A,B}(X)\|_1 \geqslant \|T\|_1$, *for all $T \in \ker \Delta_{A,B}|_{\mathscr{C}_1}$ and for all $X \in \mathscr{C}_1$.*

Let $A = (A_1, A_2 \ldots, A_n)$, $B = (B_1, B_2 \ldots, B_n)$ be n-tuples of operators in $\mathscr{L}(H)$. We define the elementary operator

$$\Delta'_{A,B} \colon \mathscr{L}(H) \mapsto \mathscr{L}(H)$$

by

$$\Delta'_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i.$$

By the same arguments as in the proof of Theorem 4 we can prove the following theorem.

Theorem 7. *Let $A = (A_1, A_2 \ldots, A_n)$, $B = (B_1, B_2 \ldots, B_n)$ be n-tuples of operators in $\mathscr{L}(H)$. Then $(A, B) \in \mathrm{GF}_1(\mathscr{H}) \Leftrightarrow \overline{R(\Delta'_{A,B})}^{w*}$ is closed under taking adjoints. Here $\mathrm{GF}_1(\mathscr{H})$ is the set of pairs of operators such that $\sum_{i=1}^{n} B_i T A_i = 0$ implies $\sum_{i=1}^{n} A_i^* T B_i^* = 0$ for all $T \in \mathscr{C}_1(\mathscr{H})$.*

Theorem 8 ([5]). *Let $A, B \in \mathscr{L}(H)$ and let $T \in \mathscr{C}_1$ have the polar decomposition $T = U|T|$. Then the follwing statements are equivalent:*

1. $\|T + \delta_{A,B}(X)\|_1 \geqslant \|T\|_1$, *for all* $T \in \ker \delta_{A,B}|_{\mathscr{C}_1}$ *and for all* $X \in \mathscr{C}_1$, *where* $\delta_{A,B}$ *is the generalized derivation defined on* $\mathscr{L}(H)$ *by* $\delta_{A,B}(X) = AX - XB$.
2. $(A, B) \in \mathrm{GF}_1(\mathscr{H})$.

Remark 9. Theorem 6 remains hold if we consider instead of $\Delta_{A,B}$ the generalized derivation $\delta_{A,B}$. This leads us to pose the following open problem.

Are the following statements equivalent:

(i) $\overline{R(\Delta'_{A,B})}^{w*}$ is closed under taking adjoints

(ii) $(A, B) \in \mathrm{GF}_1(\mathscr{H})$

(iii) $\|T + \Delta'_{A,B}(X)\|_1 \geqslant \|T\|_1$, for all $T \in \ker \Delta'_{A,B}|_{C_1}$ and for all $X \in C_1$, where $\Delta'_{A,B}$ is the elementary operators defined on $\mathscr{L}(H)$ by $\Delta'_{A,B}(X) = AXB - CXD$ and $\mathrm{GF}_1(\mathscr{H})$ is the set of pairs of operators satisfying $\ker \Delta'_{A,B} \subseteq \ker \Delta'^*_{A,B}$. Here $\Delta'^*_{A,B}$ is defined by

$$\Delta'^*_{A,B}(X) = A^*XB^* - C^*XD^*.$$

The generalization of the above results to the elementary operators $\sum\limits_{i=1}^{n} A_iXB_i$ for $n > 2$ is not possible. In [7] Shulman stated that there exists a normally represented elementary operator of the form $\sum\limits_{i=1}^{n} A_iXB_i$ with $n > 2$ such that $\operatorname{asc} E > 1$, i.e. the range and kernel have no trivial intersection.

In [8, p. 276], J.P. Williams showed that if $A \in B(H)$, then

$$R(\delta_A)^\circ \simeq R(\delta_A)^\circ \cap K^\circ(H) \oplus \ker(\delta_A) \cap C_1,$$

where $R(\delta_A)$, $K(H)$, $\ker(\delta_A)$ and C_1 denote, respectively, the range of δ_A, the ideal of compact operators, the kernel of δ_A and the trace class operators. The following theorems generalize this result.

Note that the weakly continuous linear form (resp. the ultra-weakly continuous linear form) on $B(H)$, Φ_T, where $T \in F(H)$ (resp. $T \in C_1$), is defined by

$$\Phi_T(X) = \operatorname{tr}(XT) = \operatorname{tr}(TX)$$

for all $X \in \mathscr{L}(H)$ (see [6, p. 23]).

Let $\mathscr{S}$ be a subspace of $B(H)$. Let

$$\mathscr{S}^0 = \{f \in B(H)' \colon\ f(x) = 0,\ \text{ if } x \in \mathscr{S}\}.$$

Let $\mathscr{B}$ be a Banach space and let $\mathscr{S}$ be a subspace of $\mathscr{B}$. Denote

$$\mathscr{S}^\circ = \{f \in \mathscr{B}' \colon\ f(x) = 0, \text{ if } x \in \mathscr{S}\}.$$

Lemma 10. *Let $\mathscr{S}_1$, $\mathscr{S}_2$ be two sub-vectorspaces of $\mathscr{B}$. Then $\mathscr{S}_1^\circ \subset \mathscr{S}_2^\circ$ if and only if $\mathscr{S}_2 \subset \overline{\mathscr{S}_1}$.*

The following theorem is proved in [6]. For the convenience of the reader we will prove it.

Theorem 11. *Let E, F be Banach spaces and $S \in B(E,F)$ a bounded operator. Then*

$$R(S^{**})^\circ = (R(S^{**})^\circ \cap F^\circ) \oplus \ker(S^*). \tag{3.1}$$

P r o o f. One has $F^{***} = F^\circ \oplus F^*$ (here we have identified F^* with its isometric image in F^{***} and F° is really $(i(F))^\circ$ where $i(F)$ is the image of F in F^{**} under the canonical isometric embedding i), since $f \in F^{***}$ has the unique decomposition $f = f_0 + f_1$, where $f_1 = f|_F \in F^*$ and $f_0 = f - f_1 \in F^\circ$. Suppose that $f \in R(S^{**})^\circ$. Decompose f as above: $f = f_0 + f_1 \in F^\circ \oplus F^*$. Recall that $\ker(S^*) = R(S)^\circ$ (considered in F^*). For $u \in E$ one has

$$0 = f(Su) = f_0(Su) + f_1(Su) = f_1(Su),$$

since $Su \in F$ and $f_0 \in F^\circ$. Thus $f_1 \in \ker(S^*)$.

Recall that $F^* = (F^{**}, w^*)^*$ (the w^*-continuous functionals on F^{**}). Since E is w^*-dense in E^{**} (Goldstine's theorem) and $f_1 \in F^*$ is w^*-continuous on F^{**}, it follows from $f_1|_{SE} = 0$ that $f_1|_{S^{**}E^{**}} = 0$, that is, $f_1 \in R(S^{**})^\circ$. Thus $f_0 = f - f_1 \in (R(S^{**})^\circ$, so that $f = f_0 + f_1$ is the desired decomposition.

Conversely, if $f = f_0 + f_1 \in (R(S^{**})^\circ \cap F^\circ) \oplus \ker(S^*)$, then one uses the w^*-continuity of f_1 as above to deduce that $f_1 \in R(S^{**})^\circ$. It follows that $f \in R(S^{**})^\circ$. $\square$

The following theorem generalizes the result of J. P. Williams [8].

Theorem 12. *Let $A = (A_1, A_2, \dots, A_n)$ and $B = (B_1, B_2, \dots, B_n)$ be n-tuples in $B(H)$, then*

$$R(E_{A,B})^\circ = R(E_{A,B})^\circ \cap K^\circ(H) \oplus \ker(E_{B,A}) \cap C_1. \tag{3.2}$$

P r o o f. It suffices to take in (3.1) $E = F = K(H)$ and

$$S = E_{A,B}\colon\ K(H) \to K(H),$$

where $S^* = E_{B,A}\colon\ C_1 \to C_1$ using trace duality. $\square$

Note that $\overline{R(\Delta_{A,B})}^{w*}$ is self-adjoint if and only if

$$R(\Delta_{A,B})^{\circ} \cap \mathscr{L}(H)'^{w*}$$

is also self-adjoint. By using Theorem 11 we obtain in particular

$$R(\Delta_{A,B})^{\circ} \cap \mathscr{L}(H)'^{w*} \simeq \{A\}' \cap \mathscr{C}_1(\mathscr{H}).$$

Thus $(A, B) \in \mathrm{GF}_1(\mathscr{H})$ if and only if $\overline{R(\Delta_{A,B})}^{w*}$ is self-adjoint.

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Author's address: Salah Mecheri, Department of Mathematics, King Saud University, College of Science, P.O. Box 2455, Riyadh 11451, Saudi Arabia, e-mail: `mecherisalah@hotmail.com`.