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THE AP-DENJOY AND AP-HENSTOCK INTEGRALS

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Abstract. In this paper we define the ap-Denjoy integral and show that the ap-Denjoy integral is equivalent to the ap-Henstock integral and the integrals are equal.

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1. Introduction

For a measurable set E of real numbers we denote by $|E|$ its Lebesgue measure. Let E be a measurable set and let c be a real number. The *density* of E at c is defined by

$$d_cE = \lim_{h\to 0^+} \frac{|E\cap(c-h,c+h)|}{2h}$$

provided the limit exists. The point c is called a *point of density* of E if $d_cE = 1$ and a *point of dispersion* of E if $d_cE = 0$. The set E^d represents the set of all points $x \in E$ such that x is a point of density of E.

A function $F\colon [a,b] \to \mathbb{R}$ is said to be *approximately differentiable* at $c \in [a,b]$ if there exists a measurable set $E \subseteq [a,b]$ such that $c \in E^d$ and $\lim\limits_{\substack{x\to c\\ x\in E}} \frac{F(x)-F(c)}{x-c}$ exists. The approximate derivative of F at c is denoted by $F'_{\rm ap}(c)$.

An *approximate neighborhood* (or ap-nbd) of $x \in [a,b]$ is a measurable set $S_x \subseteq [a,b]$ containing x as a point of density. For every $x \in E \subseteq [a,b]$, choose an ap-nbd $S_x \subseteq [a,b]$ of x. Then we say that $S = \{S_x\colon x \in E\}$ is a *choice* on E. A tagged interval $(x,[c,d])$ is said to be *subordinate* to the choice $S = \{S_x\}$ if $c, d \in S_x$. Let $\mathscr{P} = \{(x_i,[c_i,d_i])\colon 1 \leqslant i \leqslant n\}$ be a finite collection of non-overlapping tagged intervals. If $(x_i,[c_i,d_i])$ is subordinate to a choice S for each i, then we say that $\mathscr{P}$ is subordinate to S. If $\mathscr{P}$ is subordinate to S and $[a,b] = \bigcup_{i=1}^{n} [c_i, d_i]$, then we say that $\mathscr{P}$ is a tagged partition of $[a,b]$ that is subordinate to S.

2. The ap-Denjoy integral

We introduce the notion of the approximate Lusin function. This function is used to define the ap-Denjoy integral.

For a function $F\colon [a,b] \to \mathbb{R}$, F can be treated as a function of intervals by defining $F([c,d]) = F(d) - F(c)$.

Definition 2.1. Let $F\colon [a,b] \to \mathbb{R}$ be a function. The function F is an *approximate Lusin function* (or F is an AL function) on $[a,b]$ if for every measurable set $E \subseteq [a,b]$ of measure zero and for every $\varepsilon > 0$ there exists a choice S on E such that $|(\mathscr{P}) \sum F(I)| < \varepsilon$ for every finite collection $\mathscr{P}$ of non-overlapping tagged intervals that is subordinate to S.

Recall that $F\colon [a,b] \to \mathbb{R}$ is AC_s on a measurable set $E \subseteq [a,b]$ if for each $\varepsilon > 0$ there exist a positive number δ and a choice S on E such that $|(\mathscr{P}) \sum F(I)| < \varepsilon$ for every finite collection $\mathscr{P}$ of non-overlapping tagged intervals that is subordinate to S and satisfies $(\mathscr{P}) \sum |I| < \delta$. The function F is ACG_s on E if E can be expressed as a countable union of measurable sets on each of which F is AC_s.

Lemma 2.2. *If $F\colon [a,b] \to \mathbb{R}$ is ACG_s on $[a,b]$, then F is an AL function on $[a,b]$.*

P r o o f. Suppose that $E \subseteq [a,b]$ is a measurable set of measure zero. Let $E = \bigcup_{n=1}^{\infty} E_n$, where $\{E_n\}$ is a sequence of disjoint measurable sets and F is AC_s on each E_n. Let $\varepsilon > 0$. For each positive integer n there exist a choice $S^n = \{S_x^n \colon x \in E_n\}$ on E_n and a positive number δ_n such that $|(\mathscr{P}) \sum F(I)| < \varepsilon/2^n$ whenever $\mathscr{P}$ is subordinate to S^n and $(\mathscr{P}) \sum |I| < \delta_n$. For each positive integer n, choose an open set O_n such that $E_n \subseteq O_n$ and $|O_n| < \delta_n$. Let $S_x = S_x^n \cap (x - \varrho(x, O_n^c), x + \varrho(x, O_n^c))$ for each $x \in E_n$, where $\varrho(x, O_n^c)$ is the distance from x to $O_n^c = [a,b] - O_n$. Then $S = \{S_x \colon x \in E\}$ is a choice on E. Suppose that $\mathscr{P}$ is subordinate to S. Let $\mathscr{P}_n$ be a subset of $\mathscr{P}$ that has tags in E_n and note that $(\mathscr{P}_n) \sum |I| < |O_n| < \delta_n$. Hence, we have

$$\Bigl|(\mathscr{P}) \sum F(I)\Bigr| \leqslant \sum_{n=1}^{\infty} \Bigl|(\mathscr{P}_n) \sum F(I)\Bigr| < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

□

Definition 2.3. A function $f\colon [a,b] \to \mathbb{R}$ is *ap-Denjoy integrable* on $[a,b]$ if there exists an AL function F on $[a,b]$ such that F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{\mathrm{ap}} = f$ almost everywhere on $[a,b]$. The function f is ap-Denjoy integrable on a measurable set $E \subseteq [a,b]$ if $f\chi_E$ is ap-Denjoy integrable on $[a,b]$.

If we add the condition $F(a) = 0$, then the function F is unique. We will denote this function $F(x)$ by $(\mathrm{AD})\int_a^x f$.

It is easy to show that if $f\colon [a,b] \to \mathbb{R}$ is ap-Denjoy integrable on $[a,b]$, then f is ap-Denjoy integrable on every subinterval of $[a,b]$. This gives rise to an interval function F such that $F(I) = (\mathrm{AD})\int_I f$ for every subinterval $I \subseteq [a,b]$. The function F is called the primitive of f.

Recall that a function $F\colon [a,b] \to \mathbb{R}$ is AC_* on a measurable set $E \subseteq [a,b]$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $(\mathscr{P})\sum \omega(F,I) < \varepsilon$ for every finite collection $\mathscr{P}$ of non-overlapping intervals that have endpoints in E and satisfy $(\mathscr{P})\sum |I| < \delta$, where $\omega(F,I) = \sup\{|F(y) - F(x)|\colon\ x, y \in I\}$. A function F is ACG_* on E if $F|_E$ is continuous on E, $E = \bigcup\limits_{n=1}^{\infty} E_n$ and F is AC_* on each E_n. It is easy to show that if F is ACG_* on $[a,b]$, then F is ACG_s on $[a,b]$. A function $f\colon [a,b] \to \mathbb{R}$ is *Denjoy integrable* on $[a,b]$ if there exists an ACG_* function $F\colon [a,b] \to \mathbb{R}$ such that $F' = f$ almost everywhere on $[a,b]$.

The following theorem shows that the ap-Denjoy integral is an extension of the Denjoy integral.

Theorem 2.4. *If $f\colon [a,b] \to \mathbb{R}$ is Denjoy integrable on $[a,b]$, then f is ap-Denjoy integrable on $[a,b]$.*

P r o o f. Suppose that $f\colon [a,b] \to \mathbb{R}$ is Denjoy integrable on $[a,b]$. Then there exists an ACG_* function $F\colon [a,b] \to \mathbb{R}$ such that $F' = f$ almost everywhere on $[a,b]$. Since F is ACG_s on $[a,b]$, by Lemma 2.2 F is an AL function on $[a,b]$ and $F'_{\mathrm{ap}} = F' = f$ almost everywhere on $[a,b]$. Hence, f is ap-Denjoy integrable on $[a,b]$. $\square$

There exists a function that is ap-Denjoy integrable on $[a,b]$, but not Denjoy integrable on $[a,b]$.

Example 2.5. Let $\{(a_n, b_n)\}$ be a sequence of disjoint open intervals in (a,b) with the following properties:

(1) $b_1 < b$ and $b_{n+1} < b_n$ for all n;

(2) $\{a_n\}$ converges to a;

(3) a is a point of dispersion of $O = \bigcup\limits_{n=1}^{\infty} (a_n, b_n)$.

Define $F\colon [a,b] \to \mathbb{R}$ by $F(x) = 0$ for all $x \in [a,b] - O$ and

$$F(x) = \sin^2\Bigl(\frac{x - a_n}{b_n - a_n}\pi\Bigr)$$

for $x \in (a_n, b_n)$. Then it is easy to show that the function F is differentiable on $(a, b]$ and approximately differentiable at a, but F is not continuous at a. Hence $F' = F_{\mathrm{ap}}$ almost everywhere on $[a, b]$, but F'_{ap} is not Denjoy integrable on $[a, b]$, since F is not continuous on $[a, b]$.

To show that F'_{ap} is ap-Denjoy integrable on $[a, b]$, it is sufficient to show that F is an AL function on $[a, b]$. Let E be a measurable set in $[a, b]$ of measure zero and let $\varepsilon > 0$. For each positive integer n, choose an open set O_n such that $E \cap [a_n, b_n] \subseteq O_n$ and $|O_n| < (b_n - a_n)\varepsilon/\pi 2^n$.

For each $x \in E$, define

$$S_x = \begin{cases} [a, b] - \bigcup\limits_{n=1}^{\infty} (a_n, b_n) & \text{if } x = a; \\ (b_{n+1}, a_n) & \text{if } b_{n+1} < x < a_n, \ \ n = 1, 2, 3, \ldots; \\ (x - \varrho(x, O_n^c), x + \varrho(x, O_n^c)) & \text{if } a_n \leqslant x \leqslant b_n, \ \ n = 1, 2, 3, \ldots. \end{cases}$$

Then $S = \{S_x \colon x \in E\}$ is a choice on E. Let $\mathscr{P} = \{(x, [a, b])\}$ be a finite collection of non-overlapping tagged intervals that is subordinate to S. Then we have

$$\begin{aligned} (\mathscr{P}) \sum |F([c, d])| &= \sum_{n=1}^{\infty} \sum_{x \in (b_{n+1}, a_n)} |F([c, d])| + \sum_{n=1}^{\infty} \sum_{x \in [a_n, b_n]} |F([c, d])| \\ &\leqslant \sum_{n=1}^{\infty} \sum_{x \in [a_n, b_n]} \frac{\pi(d - c)}{b_n - a_n} \leqslant \sum_{n=1}^{\infty} \frac{\pi}{b_n - a_n} |O_n| < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon. \end{aligned}$$

Hence, F is an AL function on $[a, b]$.

Theorem 2.6. *Let $f \colon [a, b] \to \mathbb{R}$ be ap-Denjoy integrable on $[a, b]$ and let $F(x) = (\mathrm{AD})\int_a^x f$ for each $x \in [a, b]$. Then*

(a) *the function F is approximately differentiable almost everywhere on $[a, b]$ and $F'_{\mathrm{ap}} = f$ almost everywhere on $[a, b]$; and*

(b) *the functions F and f are measurable.*

P r o o f. (a) follows from the definition of the ap-Denjoy integral. Since F is approximately continuous almost everywhere on $[a, b]$, F is measurable by [4, Theorem 14.7]. It follows from [4, Theorem 14.12] that f is measurable. $\square$

Theorem 2.7. *Let $F \colon [a, b] \to \mathbb{R}$ be an AL function on $[a, b]$. If F is approximately differentiable almost everywhere on $[a, b]$, then F'_{ap} is ap-Denjoy integrable on $[a, b]$ and $(\mathrm{AD})\int_a^x F'_{\mathrm{ap}} = F(x) - F(a)$ for each $x \in [a, b]$.*

P r o o f. Suppose that F is an AL function on $[a, b]$ and F is approximately differentiable almost everywhere on $[a, b]$. It follows from the definition that F'_{ap} is ap-Denjoy integrable on $[a,b]$. For a constant C, $F+C$ is also an AL function on $[a,b]$, approximately differentiable almost everywhere on $[a,b]$ and $(F+C)'_{\rm ap} = F'_{\rm ap}$ almost everywhere on $[a,b]$. Hence, we have

$$F(x) + C = (\mathrm{AD})\int_a^x F'_{\rm ap} \quad \text{for each } x \in [a,b].$$

Since $F(a) + C = 0$, $C = -F(a)$ and

$$(\mathrm{AD})\int_a^x F'_{\rm ap} = F(x) - F(a) \quad \text{for each } x \in [a,b].$$

□

We can easily show that if f is ap-Denjoy integrable on each of intervals $[a,c]$ and $[c,b]$, then f is ap-Denjoy integrable on $[a,b]$ and

$$(\mathrm{AD})\int_a^b f = (\mathrm{AD})\int_a^c f + (\mathrm{AD})\int_c^b f.$$

Theorem 2.8. *Suppose that $f\colon [a,b] \to \mathbb{R}$ is ap-Denjoy integrable on each subinterval $[c,d] \subseteq (a,b)$. If $(\mathrm{AD})\int_c^d f$ converges to a finite limit as $c \to a^+$ and $d \to b^-$, then f is ap-Denjoy integrable on $[a,b]$ and $(\mathrm{AD})\int_a^b f = \lim\limits_{\substack{c\to a^+\\ d\to b^-}} (\mathrm{AD})\int_c^d f$.*

Proof. Choose a point $p \in (a,b)$ and fix it. First, we will prove that if f is ap-Denjoy integrable on $[p,d]$ for each $d \in (p,b)$ and $(\mathrm{AD})\int_p^d f$ converges to a finite limit as $d \to b^-$, then f is ap-Denjoy integrable on $[p,b]$ and $(\mathrm{AD})\int_p^b f = \lim\limits_{d\to b^-} (\mathrm{AD})\int_p^d f$.

Let $L = \lim\limits_{d\to b^-} (\mathrm{AD})\int_p^d f$, let $a_0 = p$ and let $\{a_k\}$ be an increasing sequence in (p,b) that converges to b. Define a function $F\colon [p,b] \to \mathbb{R}$ by

$$F(x) = F_i(x) \quad \text{if } x \in [a_{i-1}, a_i] \quad \text{for each } i = 1,2,3,\ldots$$

and $F(b) = L$, where F_i is the primitive of f on $[a_{i-1}, a_i]$ and $F_i(a_{i-1}) = 0$ for each i. Since each F_i is an AL function on $[a_{i-1}, a_i]$ such that F_i is approximately differentiable almost everywhere on $[a_{i-1}, a_i]$ and $(F_i)'_{\rm ap} = f$ almost everywhere on $[a_{i-1}, a_i]$, the function F is an AL function on $[p,b]$ such that F is approximately differentiable almost everywhere on $[p,b]$ and $F'_{\rm ap} = f$ almost everywhere on $[p,b]$. Hence, f is ap-Denjoy integrable on $[p,b]$ and

$$(\mathrm{AD})\int_p^b f = F(b) = L = \lim_{d\to b^-} (\mathrm{AD})\int_p^d f.$$

Similarly, we can prove that if f is ap-Denjoy integrable on $[c,p]$ for each $c \in (a,p)$ and $(\mathrm{AD})\int_c^p f$ converges to a finite limit as $c \to a^+$, then f is ap-Denjoy integrable on $[a,p]$ and $(\mathrm{AD})\int_a^p f = \lim\limits_{c\to a^+} (\mathrm{AD})\int_c^p f$.

If $(\mathrm{AD})\int_c^d f$ converges to a finite limit as $c \to a^+$ and $d \to b^-$, then for any $p \in (a,b)$ the integral $(\mathrm{AD})\int_c^p f$ converges to a finite limit as $c \to a^+$ and $(\mathrm{AD})\int_p^d f$ converges to a finite limit as $d \to b^-$. By the proof of the previous parts, f is ap-Denjoy integrable on $[a,p] \cup [p,b] = [a,b]$ and

$$\begin{aligned}(\mathrm{AD})\int_a^b f &= (\mathrm{AD})\int_a^p f + (\mathrm{AD})\int_p^b f \\ &= \lim_{c\to a^+} (\mathrm{AD})\int_c^p f + \lim_{d\to b^-} (\mathrm{AD})\int_p^d f = \lim_{\substack{c\to a^+\\ d\to b^-}} (\mathrm{AD})\int_c^d f.\end{aligned}$$

□

Recall that a function $f\colon [a,b] \to \mathbb{R}$ is *ap-Henstock integrable* on $[a,b]$ if there exists a real number A with the following property: for each $\varepsilon > 0$ there exists a choice S on $[a,b]$ such that $|(\mathscr{P})\sum f(x)|I| - A| < \varepsilon$ whenever $\mathscr{P} = \{(x,I)\colon x \in [a,b]\}$ is a tagged partition of $[a,b]$ that is subordinate to S. The real number A is called the ap-Henstock integral of f on $[a,b]$ and is denoted by $(\mathrm{AH})\int_a^b f$. If f is ap-Henstock integrable on $[a,b]$, then f is also ap-Henstock integrable on any subinterval I of $[a,b]$. Hence, an interval function F can be defined by $F(I) = (\mathrm{AH})\int_I f$. The function F is called the primitive of f.

The following theorem shows that the ap-Denjoy integral is equivalent to the ap-Henstock integral and the integrals are equal to each other.

Theorem 2.9. *The function $f\colon [a,b] \to \mathbb{R}$ is ap-Denjoy integrable on $[a,b]$ if and only if f is ap-Henstock integrable on $[a,b]$ and the integrals are equal to each other.*

Proof. If f is ap-Henstock integrable on $[a,b]$ with the primitive F, then F is ACG_s on $[a,b]$ and $F'_{\mathrm{ap}} = f$ almost everywhere on $[a,b]$ by [4, Theorem 16.18]. By Lemma 2.2, f is ap-Denjoy integrable on $[a,b]$.

Suppose that f is ap-Denjoy integrable on $[a,b]$ with the primitive F. Then F is an AL function on $[a,b]$ such that F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{\mathrm{ap}} = f$ almost everywhere on $[a,b]$. Let

$$E = \{x \in [a,b]\colon F'_{\mathrm{ap}}(x) \neq f(x)\}.$$

Then $|E| = 0$. Let $D = [a,b] - E$ and let $\varepsilon > 0$.

For each $x \in D$ there exists a measurable set $D_x \subseteq [a,b]$ such that $x \in D_x^d$ and

$$F'_{\mathrm{ap}}(x) = \lim_{\substack{y \to x \\ y \in D_x}} \frac{F(y) - F(x)}{y - x}.$$

Hence, there exists $\delta_x > 0$ such that for every $y \in D_x \cap (x - \delta_x, x + \delta_x) = S_x$

$$|F(y) - F(x) - F'_{\mathrm{ap}}(x)(y - x)| \leqslant \varepsilon |y - x|.$$

If $(x, [u,v])$ is a tagged interval that is subordinate to $\{S_x\}$, then

$$\begin{aligned}
&|F(v) - F(u) - F'_{\mathrm{ap}}(x)(v - u)| \\
&\qquad \leqslant |F(v) - F(x) - F'_{\mathrm{ap}}(x)(v - x)| + |F(x) - F(u) - F'_{\mathrm{ap}}(x)(x - u)| \\
&\qquad < \varepsilon(v - x) + \varepsilon(x - u) = \varepsilon(v - u).
\end{aligned}$$

Hence, there exists a choice S' on D such that $|(\mathscr{P}) \sum f(x)|I| - (\mathscr{P}) \sum F(I)| < \varepsilon (\mathscr{P}) \sum |I|$ whenever $\mathscr{P}$ is a collection of tagged intervals that is subordinate to S'.

By [4, Lemma 9.15] and the fact that F is an AL function on $[a,b]$, there exists a choice S'' on E such that $|(\mathscr{P}) \sum f(x)|I|| < \varepsilon$ and $|(\mathscr{P}) \sum F(I)| < \varepsilon$ whenever $\mathscr{P}$ is subordinate to S''. Let $S = S' \cup S''$. Then S is a choice on $[a,b]$.

Suppose that $\mathscr{P}$ is a tagged partition of $[a,b]$ that is subordinate to S. Let $\mathscr{P}_E$ be the subset of $\mathscr{P}$ that has tags in E and let $\mathscr{P}_D = \mathscr{P} - \mathscr{P}_E$. Then we have

$$\begin{aligned}
&\Big|(\mathscr{P}) \sum f(x)|I| - (\mathscr{P}) \sum F(I)\Big| \\
&\quad \leqslant \Big|(\mathscr{P}_D) \sum f(x)|I| - (\mathscr{P}_D) \sum F(I)\Big| + \Big|(\mathscr{P}_E) \sum f(x)|I|\Big| + \Big|(\mathscr{P}_E) \sum F(I)\Big| \\
&\quad < \varepsilon(b - a + 2).
\end{aligned}$$

Hence, f is ap-Henstock integrable on $[a,b]$ and $(\mathrm{AH})\int_a^b f = (\mathscr{P}) \sum F(I) = F(b) - F(a) = (\mathrm{AD})\int_a^b f$. $\square$

References

[1] *P. S. Bullen*: The Burkill approximately continuous integral. J. Austral. Math. Soc. (Ser. A) *35* (1983), 236–253.

[2] *T. S. Chew, K. Liao*: The descriptive definitions and properties of the AP-integral and their application to the problem of controlled convergence. Real Anal. Exch. *19* (1994), 81–97.

[3] *R. A. Gordon*: Some comments on the McShane and Henstock integrals. Real Anal. Exch. *23* (1997), 329–341.

[4] *R. A. Gordon*: The Integrals of Lebesgue, Denjoy, Perron and Henstock. Amer. Math. Soc., Providence, 1994.

[5] *J. Kurzweil*: On multiplication of Perron integrable functions. Czechoslovak Math. J. *23(98)* (1973), 542–566.
[6] *J. Kurzweil, J. Jarník*: Perron type integration on n-dimensional intervals as an extension of integration of step functions by strong equiconvergence. Czechoslovak Math. J. *46(121)* (1996), 1–20.
[7] *T. Y. Lee*: On a generalized dominated convergence theorem for the AP integral. Real Anal. Exch. *20* (1995), 77–88.
[8] *K. Liao*: On the descriptive definition of the Burkill approximately continuous integral. Real Anal. Exch. *18* (1993), 253–260.
[9] *Y. J. Lin*: On the equivalence of four convergence theorems for the AP-integral. Real Anal. Exch. *19* (1994), 155–164.
[10] *J. M. Park*: Bounded convergence theorem and integral operator for operator valued measures. Czechoslovak Math. J. *47(122)* (1997), 425–430.
[11] *J. M. Park*: The Denjoy extension of the Riemann and McShane integrals. Czechoslovak Math. J. *50(125)* (2000), 615–625.
[12] *J. M. Park, C. G. Park, J. B. Kim, D. H. Lee, and W. Y. Lee*: The s-Perron, sap-Perron and ap-McShane integrals. Czechoslovak Math. J. *54(129)* (2004), 545–557.
[13] *A. M. Russell*: Stieltjes type integrals. J. Austr. Math. Soc. (Ser. A) *20* (1975), 431–448.
[14] *A. M. Russell*: A Banach space of functions of generalized variation. Bull. Aust. Math. Soc. *15* (1976), 431–438.

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