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EMBEDDING c_0 IN $\operatorname{bvca}(\Sigma, X)$

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Abstract. If (Ω, Σ) is a measurable space and X a Banach space, we provide sufficient conditions on Σ and X in order to guarantee that $\operatorname{bvca}(\Sigma, X)$, the Banach space of all X-valued countably additive measures of bounded variation equipped with the variation norm, contains a copy of c_0 if and only if X does.

Keywords: countably additive vector measure of bounded variation, Pettis integrable function space, copy of c_0, copy of ℓ_∞

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1. Preliminaries

Throughout this paper (Ω, Σ) will be a measurable space and, unless otherwise stated, X will be a Banach space over the field $\mathbb{K}$ of real or complex numbers. Our notation is standard [2]–[4]. So, $\operatorname{ca}(\Sigma, X)$ will denote the Banach space over $\mathbb{K}$ of all X-valued countably additive measures F defined on Σ, endowed with the semi-variation norm $\|F\|$, and $\operatorname{bvca}(\Sigma, X)$ will stand for the Banach space of all X-valued countably additive measures F of bounded variation defined in Σ, equipped with the variation norm $|F|$. If $\mu \in \operatorname{ca}^+(\Sigma)$, we shall denote by $\mathcal{P}(\mu, X)$ the linear space of all classes of scalarly equivalent weakly μ-measurable X-valued Pettis integrable functions f defined on Ω, equipped with the norm

$$\|f\|_{\mathcal{P}(\mu,X)} = \sup\left\{\int_\Omega |x^* f(\omega)| \,\mathrm{d}\mu(\omega)\colon\ x^* \in X^*,\ \|x^*\| \leqslant 1\right\}.$$

A Banach space X is said to have the weak Radon-Nikodým property (WRNP) with respect to a finite measure space (Ω, Σ, μ) if every μ-continuous measure $F\colon$

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$\Sigma \to X$ of σ-finite variation has a Pettis μ-integrable derivative $f\colon \Omega \to X$, i.e. satisfying that $F(E) = (\mathrm{P}) \int_E f \, \mathrm{d}\mu$ for each $E \in \Sigma$ [9]. If X has the WRNP with respect to every finite measure space, we say that X has the WRNP.

It has been shown [5] that if each nonzero finite positive measure $\mu \in \mathrm{ca}(\Sigma)$ is purely atomic, then $\mathrm{ca}(\Sigma, X)$ contains a copy of c_0 or ℓ_∞ if and only if X contains, respectively, a copy of c_0 or ℓ_∞. With the same hypotheses, the same result holds for $\mathrm{bvca}(\Sigma, X)$ [6]. However, if the range space X is a dual Banach space, then $\mathrm{bvca}(\Sigma, X)$ contains a copy of c_0 or ℓ_∞ if and only if X does, without any condition on $\mathrm{ca}^+(\Sigma)$ [10]. If X is not a dual Banach space this last statement is no longer true [11]. In this paper we deal again with the problem of copies of c_0 in $\mathrm{bvca}(\Sigma, X)$ by proving Theorems 1.1 and 1.2 below. Both of them are based upon Lemma 2.3 and use a mild consequence of the lifting theorem which will be conveniently recalled. We state our main theorems.

Theorem 1.1. *Assume that B_{X^*} is weak* sequentially dense in $B_{X^{***}}$ and $\Sigma = 2^\Omega$. If X is norm-one complemented in X^{**}, then $\mathrm{bvca}(\Sigma, X)$ contains a copy of c_0 if and only if X does.*

Theorem 1.2. *Assume that X^* contains a norming sequence. If X has the WRNP with respect to each $\mu \in \mathrm{ca}^+(\Sigma)$, then the space $\mathrm{bvca}(\Sigma, X)$ contains a copy of c_0 if and only if X does.*

It should be noted that the conditions imposed on X in Theorem 1.1 imply that X has the WRNP with respect to each $\mu \in \mathrm{ca}^+(\Sigma)$. So, a natural question arises whether Theorem 1.1 remains true under this more general setting. A partial answer to this issue is given by Theorem 1.2, with the additional requirement that X^* contains a norming sequence. As a counterpart, the restrictions on the measurable space (Ω, Σ) stated in Theorem 1.1 are waived.

2. Supporting lemmas

If (Ω, Σ, μ) is a complete finite measure space, following [2] we denote by $\mathcal{L}_{w^*}(\mu, X^*)$ the linear space of all μ-essentially bounded functions $\varphi\colon \Omega \to X^*$ which are weak* measurable, and represent by $\mathrm{bvca}_\mu(\Sigma, X^*)$ the linear subspace of $\mathrm{bvca}(\Sigma, X^*)$ of all those measures F for which there is $a > 0$ (which depends of F) such that $\|F(E)\| \leqslant a\mu(E)$ for each $E \in \Sigma$. We state the lifting theorem in the way we shall need in the proof of our main theorems (Lemma 2.1), as well as an averaging result due to Bourgain (Lemma 2.2) that we shall use to prove Lemma 2.3 below.

Lemma 2.1 ([2, Theorem 1.5.2]). *There is a linear injective map* $T\colon \operatorname{bvca}_\mu(\Sigma, X^*) \to \mathcal{L}_{w^*}(\mu, X^*)$ *such that for each* $F \in \operatorname{bvca}_\mu(\Sigma, X^*)$ *the function* $f = T(F)$ *satisfies the following two conditions:*

1. *For each* $E \in \Sigma$ *and* $x \in X$ *one has*
$$F(E)x = \int_E f(\omega)x \, \mathrm{d}\mu(\omega).$$
2. *The function* $\omega \to \|f(\omega)\|$ *is measurable, belongs to* $L_1(\mu)$ *and for each* $E \in \Sigma$ *satisfies*
$$|F|(E) = \int_E \|f(\omega)\| \, \mathrm{d}\mu(\omega).$$

Lemma 2.2 ([1], [2, Lemma 2.1.2]). *Let* X *be a seminormed space and let* $\{r_n\}_{n=0}^{\infty}$ *denote the sequence of Rademacher functions in* $[0,1]$. *If* $\{x_n\}$ *is a sequence in* X *such that* $\inf\limits_{n\in\mathbb{N}} \|x_n\| > 0$ *and*
$$\sup_{n\in\mathbb{N}} \int_0^1 \left\| \sum_{i=1}^n r_i(t)x_i \right\| \mathrm{d}t < \infty,$$
then there exists a subsequence $\{z_n\}$ *of* $\{x_n\}$ *which is a* c_0*-sequence in* X, *i.e., there are* $K_1 > 0$ *and* $K_2 > 0$ *such that*
$$K_1 \sup_{1\leqslant i\leqslant n} |a_i| \leqslant \left\| \sum_{i=1}^n a_i x_i \right\| \leqslant K_2 \sup_{1\leqslant i\leqslant n} |a_i|$$
for all $a_1, \ldots, a_n \in \mathbb{K}$ *and* $n \in \mathbb{N}$.

Lemma 2.3. *Let* (Ω, Σ, μ) *be a complete finite measure space and* X *a Banach space. If there is a sequence* $\{g_n\}$ *of functions* $g_n\colon \Omega \to X$ *such that*

(i) $\left\| \sum\limits_{i=1}^n \varepsilon_i g_i(\cdot) \right\| \in L_1(\mu)$ *for each finite set of signs* $\{\varepsilon_1, \ldots, \varepsilon_n\}$,

(ii) $\inf\limits_{n\in\mathbb{N}} \int_\Omega \|g_n(\omega)\| \, \mathrm{d}\mu(\omega) > 0$ *and*

(iii) *there exists* $K > 0$ *with*
$$\sup_{n\in\mathbb{N}} \int_\Omega \left\| \sum_{i=1}^n g_i(\cdot) \right\| \mathrm{d}\mu(\omega) < K,$$

then X *contains a copy of* c_0.

Proof. Let us remark that no measurability requirement is imposed on the functions g_n. To start with, let us see that condition (iii) implies that the sequence

$\{\|g_n(\cdot)\|\}$ is uniformly integrable. This statement is almost contained in the proof of [2, Theorem 2.1.1], so we just sketch the argument. Assume on the contrary that $\{\|g_n(\cdot)\|\}$ is not uniformly integrable. Then the scalar measures $\int_{(\cdot)} \|g_n(\omega)\| \,\mathrm{d}\mu(\omega)$ are not uniformly countably additive, so there are $\varepsilon > 0$, a sequence $\{A_n\} \subseteq \Sigma$ of pairwise disjoint sets and a subsequence of $\{g_n\}$, that we shall denote in the same way, such that $\int_{A_n} \|g_n(\omega)\| \,\mathrm{d}\mu(\omega) > 2\varepsilon$ for each $n \in \mathbb{N}$. Defining $\mu_n\colon \Sigma \to \mathbb{R}$ by $\mu_n(E) = \int_E \|g_n(\omega)\| \,\mathrm{d}\mu(\omega)$ for each $n \in \mathbb{N}$, Rosenthal's disjointification lemma [3, Chapter 7] determines a strictly increasing sequence $\{n_i\} \subseteq \mathbb{N}$ with $\mu_{n_i}\Big(\bigcup\limits_{j\in\mathbb{N},\, j\neq i} A_{n_j}\Big) < \varepsilon$ for each $i \in \mathbb{N}$. Hence $\sum\limits_{j=1,\, j\neq i}^{\infty} \int_{A_{n_j}} \|g_{n_i}(\omega)\| \,\mathrm{d}\mu(\omega) < \varepsilon$ for each $i \in \mathbb{N}$. Since (iii) ensures $\int_\Omega \Big\|\sum\limits_{i=1}^{m} g_{n_i}(\omega)\Big\| \,\mathrm{d}\mu(\omega) \leqslant K$ for each $m \in \mathbb{N}$, we have

$$\begin{aligned}
K &\geqslant \int_{\bigcup\limits_{i=1}^{m} A_{n_i}} \Big\|\sum_{j=1}^{m} g_{n_j}(\omega)\Big\| \,\mathrm{d}\mu(\omega) = \sum_{i=1}^{m} \int_{A_{n_i}} \Big\|\sum_{j=1}^{m} g_{n_j}(\omega)\Big\| \,\mathrm{d}\mu(\omega) \\
&\geqslant \sum_{i=1}^{m} \int_{A_{n_i}} \|g_{n_i}(\omega)\| \,\mathrm{d}\mu(\omega) - \sum_{i=1}^{m} \sum_{j=1,\, j\neq i}^{m} \int_{A_{n_i}} \|g_{n_j}(\omega)\| \,\mathrm{d}\mu(\omega) \geqslant m\varepsilon
\end{aligned}$$

for each $m \in \mathbb{N}$, a contradiction.

Setting $A_1 = \{\omega \in \Omega\colon \overline{\lim}_{n\to\infty} \|g_n(\omega)\| > 0\}$, we claim that $\mu(A_1) > 0$. Otherwise $\lim\limits_{n\to\infty} \|g_n(\omega)\| = 0$ for almost all $\omega \in \Omega$ and since the sequence $\{\|g_n(\cdot)\|\}$ is uniformly integrable, it follows from Vitali's lemma [8, Exercise 13.38] that

$$\lim_{n\to\infty} \int_\Omega \|g_n(\omega)\| \,\mathrm{d}\mu(\omega) = 0,$$

contradicting condition (ii).

Denote by Δ the product space $\{-1,1\}^{\mathbb{N}}$, by Γ the σ-algebra of subsets of Δ generated by the n-cylinders of Δ, $n = 1, 2, \ldots$, and by ν the probability measure $\bigotimes\limits_{i=1}^{\infty} \nu_i$ on Γ, where $\nu_i\colon 2^{\{-1,1\}} \to [0,1]$ satisfies $\nu_i(\emptyset) = 0$, $\nu_i(\{-1\}) = \nu_i(\{1\}) = \frac{1}{2}$ and $\nu_i(\{-1,1\}) = 1$ for each $i \in \mathbb{N}$. Now consider the non-negative μ-measurable map $\varphi_n\colon \Omega \to \mathbb{R}$ defined by

$$\varphi_n(\omega) = \int_\Delta \Big\|\sum_{i=1}^{n} \varepsilon_i g_i(\omega)\Big\| \,\mathrm{d}\nu(\varepsilon)$$

for $n = 1, 2, \ldots$. Hence (iii) and Fubini's theorem yield

$$\sup_{n\in\mathbb{N}} \int_\Omega \varphi_n(\omega) \,\mathrm{d}\mu(\omega) \leqslant K.$$

Taking into account that

$$\int_{\Delta}\left\|\sum_{i=1}^{n}\varepsilon_i g_i(\omega)\right\| \mathrm{d}\nu(\varepsilon) = \frac{1}{2^n}\sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)\in\{-1,1\}^n}\left\|\sum_{i=1}^{n}\varepsilon_i g_i(\omega)\right\|$$
$$= \int_0^1\left\|\sum_{i=1}^{n} r_i(t)g_i(\omega)\right\| \mathrm{d}t$$

we have for each $\omega\in\Omega$

$$\varphi_n(\omega) = \int_0^1\left\|\sum_{i=1}^{n} r_i(t)g_i(\omega)\right\| \mathrm{d}t.$$

So it follows from [2, Proposition 2.1.2] that $\{\varphi_n\}$ is a monotone increasing sequence of non negative functions. Consequently, the monotone convergence theorem provides a μ-null set $A_2\in\Sigma$ such that $\sup\limits_{n\in\mathbb{N}}\varphi_n(\omega)<\infty$ for each $\omega\in\Omega\setminus A_2$. Considering the set $A := A_1\cap(\Omega\setminus A_2)$, it is obvious that $\mu(A)>0$, hence $A\neq\emptyset$. Moreover, $\overline{\lim}_{n\to\infty}\|g_n(\omega)\|>0$ and

$$\sup_{n\in\mathbb{N}}\int_0^1\left\|\sum_{i=1}^{n} r_i(t)g_i(\omega)\right\| \mathrm{d}t<\infty$$

for each $\omega\in A$. Choose $\omega_0\in A$ and a strictly increasing sequence of positive integers $\{n_i\}$ such that $\inf\limits_{i\in\mathbb{N}}\|g_{n_i}(\omega_0)\|>0$. Then, setting $y_i := g_{n_i}(\omega_0)$ for each $i\in\mathbb{N}$ and using the properties of the measure space, we conclude that

$$\sup_{n\in\mathbb{N}}\int_0^1\left\|\sum_{i=1}^{n} r_i(t)y_i\right\| \mathrm{d}t<\infty.$$

Since X is a Banach space, according to Lemma 2.2 there is a subsequence of $\{y_n\}$ which is a basic sequence in X equivalent to the unit vector basis of c_0. □

3. Proof of Theorem 1.1

Let us suppose that $\{F_n\}$ is a normalized basic sequence in $\mathrm{bvca}(\Sigma,X)$ equivalent to the unit vector basis of c_0 and define $\mu\in\mathrm{ca}^+(\Sigma)$ by $\mu=\sum\limits_{n=1}^{\infty}2^{-n}|F_n|$. We will assume without loss of generality that the measure space (Ω,Σ,μ) is complete. Clearly $\|F_n(E)\|\leqslant 2^n\mu(E)$ for each $E\in\Sigma$ and $n\in\mathbb{N}$ and, if we consider each F_n as a map from Σ into X^{**}, then $\mathrm{span}(\{F_n\colon n\in\mathbb{N}\})\subseteq\mathrm{bvca}_\mu(\Sigma,X^{**})$.

According to Lemma 2.1 there is a linear injective map T from $\operatorname{bvca}_\mu(\Sigma, X^{**})$ into $\mathcal{L}_{w^*}(\mu, X^{**})$ satisfying conditions 1 and 2 therein. Therefore, if $f := T(F)$ with $F \in \operatorname{bvca}_\mu(\Sigma, X^{**})$, then

(a) $x^*F(E) = \int_E f(\omega)x^* \,\mathrm{d}\mu(\omega)$ for each $x^* \in X^*$ and $E \in \Sigma$, and

(b) $|F|(E) = \int_E \|f(\omega)\| \,\mathrm{d}\mu(\omega)$ for each $E \in \Sigma$.

Since B_{X^*} is weak* sequentially dense in $B_{X^{***}}$, given $u \in B_{X^{***}}$ there is a sequence $\{x_n^*\}$ in B_{X^*} that converges to u under the weak* topology of $B_{X^{***}}$. Then, choosing a fixed $F \in \operatorname{bvca}_\mu(\Sigma, X)$ and setting $f := T(F)$, the latter implies that $x_n^* f(\omega) \to uf(\omega)$ for each $\omega \in \Omega$. So, given that $|f(\omega)x_n^*| \leqslant \|f(\omega)\|$ for μ-almost all $\omega \in \Omega$ and $\|f(\cdot)\| \in L_1(\mu)$, we have $uf \in L_1(\mu)$ and

$$\int_E x_n^* f(\omega) \,\mathrm{d}\mu(\omega) \to \int_E uf(\omega) \,\mathrm{d}\mu(\omega) \tag{3.1}$$

for each $E \in \Sigma$ by virtue of the dominated convergence theorem. This shows that the map $f\colon \Omega \to X^{**}$ is Dunford integrable in Ω, so that $(\mathrm{D}) \int_E f \,\mathrm{d}\mu \in X^{****}$. Since (3.1) and condition (a) above imply that

$$uF(E) = \int_E uf(\omega) \,\mathrm{d}\mu(\omega)$$

for each $u \in X^{***}$ and $E \in \Sigma$, it follows that $(\mathrm{D}) \int_E f \,\mathrm{d}\mu \in X^{**}$ and, consequently, $F(E) = (\mathrm{P}) \int_E f \,\mathrm{d}\mu$ for each $E \in \Sigma$, i.e. $f \in \mathcal{P}(\mu, X^{**})$.

As a consequence of the fact that $uf \in L_1(\mu)$ for each $u \in X^{***}$, if S is a norm-one linear projection operator from X^{**} onto X and $x^* \in X^*$, the fact that $S^*x^* \in X^{***}$ guarantees that the integral

$$\int_E x^*(Sf) \,\mathrm{d}\mu = \int_E (S^*x^*)f \,\mathrm{d}\mu$$

is well defined, i.e. $x^*(Sf) \in L_1(\mu)$ for each $x^* \in X^*$. So, keeping in mind that $F(E) \in X$, one has

$$\Big\langle x^*, (\mathrm{D})\int_E Sf \,\mathrm{d}\mu \Big\rangle = \int_E x^*(Sf) \,\mathrm{d}\mu = \langle S^*x^*, F(E)\rangle = \langle x^*, S(F(E))\rangle = x^*F(E)$$

for each $x^* \in X^*$. This establishes that $Sf \in \mathcal{P}(\mu, X)$ and

$$F(E) = (\mathrm{P})\int_E (S \circ f)(\omega) \,\mathrm{d}\mu(\omega) \tag{3.2}$$

for each $E \in \Sigma$. Using the fact that the mapping $\omega \to \|Sf(\omega)\|$ is μ-measurable (since each set in Ω is μ-measurable), we conclude that (3.2) implies

$$\|F(E)\| \leqslant \sup_{\|x^*\| \leqslant 1} \int_E |x^*(S \circ f)(\omega)| \,\mathrm{d}\mu(\omega) \leqslant \int_E \|(Sf)(\omega)\| \,\mathrm{d}\mu(\omega),$$

which leads to the inequality

$$|F| \leqslant \int_\Omega \|Sf(\omega)\| \,\mathrm{d}\mu(\omega). \tag{3.3}$$

On the other hand, since $\|S\| = 1$, we have

$$\|Sf(\omega)\| \leqslant \|f(\omega)\|$$

for each $\omega \in \Omega$. Thus, according to property (b) one has

$$\int_\Omega \|Sf(\omega)\| \,\mathrm{d}\mu(\omega) \leqslant \int_\Omega \|f(\omega)\| \,\mathrm{d}\mu(\omega) = |F|. \tag{3.4}$$

On the basis of (3.3) and (3.4) we conclude that

$$|F| = \int_\Omega \|Sf(\omega)\| \,\mathrm{d}\mu(\omega). \tag{3.5}$$

Summarizing, if $F \in \mathrm{bvca}_\mu(\Sigma, X)$ and $f = T(F)$, we have shown that
(i) $Sf \in \mathcal{P}(\mu, X)$,
(ii) $F(E) = (\mathrm{P}) \int_E Sf \,\mathrm{d}\mu$ for each $E \subseteq \Omega$ and
(iii) $|F| = \int_\Omega \|Sf(\omega)\| \,\mathrm{d}\mu(\omega)$.
Let us write $g_n := ST(F_n)$ for each positive integer n.

Given that the series $\sum\limits_{n=1}^{\infty} F_n$ is weakly unconditionally Cauchy there is $K > 0$ such that $\Big|\sum\limits_{i=1}^{n} F_i\Big| < K$ for all $n \in \mathbb{N}$. So, according to (3.5), we have

$$\int_\Omega \Big\|\sum_{i=1}^n \varepsilon_i g_i(\omega)\Big\| \,\mathrm{d}\mu(\omega) = \int_\Omega \Big\|ST\Big(\sum_{i=1}^n \varepsilon_i F_i\Big)(\omega)\Big\| \,\mathrm{d}\mu(\omega) = \Big|\sum_{i=1}^n \varepsilon_i F_i\Big| < K$$

for each $n \in \mathbb{N}$. This verifies conditions (i) and (iii) of Lemma 2.3. As moreover

$$\int_\Omega \|g_n(\omega)\| \,\mathrm{d}\mu(\omega) = |F_n| = 1$$

for each $n \in \mathbb{N}$, an application of Lemma 2.3 concludes the proof. □

Remark 3.1. Note that if the dual ball of X is weak* dense in $B_{X^{***}}$ and, in addition, X^* is separable, then X cannot contain any copy of c_0. Otherwise, since X is separable, Sobczyk's theorem [3, Chapter 7] ensures that c_0 embeds complementably in X and, consequently, X^{**} does not contain a copy of ℓ_∞. This is a contradiction since, according to Odell-Rosenthal's theorem [3, Chapter 8], $B_{X^{***}}$ is weak* sequentially compact.

Corollary 3.1. *Assume that $\Sigma = 2^{\Omega}$ and let X be a Banach space such that X^* is separable and contains no copy of ℓ_1. If X is norm-one complemented in X^{**}, then* $\operatorname{bvca}(\Sigma, X)$ *does not contain a copy of* c_0.

Proof. Another application of Odell-Rosenthal's theorem guarantees that the dual ball of X is weak* dense in $B_{X^{***}}$. Hence, the previous remark prevents X from containing a copy of c_0 and Theorem 1.1 applies. $\square$

4. Proof of Theorem 1.2

Let $\{F_n\}$ be a normalized basic sequence in $\operatorname{bvca}(\Sigma, X)$ equivalent to the unit vector basis of c_0 and define $\mu \in \operatorname{ca}^+(\Sigma)$ by $\mu = \sum\limits_{n=1}^{\infty} 2^{-n}|F_n|$. As in the proof of the previous theorem, $\|F_n(E)\| \leqslant 2^n \mu(E)$ for each $E \in \Sigma$ and $n \in \mathbb{N}$ and $\operatorname{span}(\{F_n\}) \subseteq \operatorname{bvca}_{\mu}(\Sigma, X^{**})$ when each F_n is considered as a map from Σ into X^{**}. By Lemma 2.1 there is a linear injective map T from $\operatorname{bvca}_{\mu}(\Sigma, X^{**})$ into $\mathcal{L}_{w^*}(\mu, X^{**})$ such that

(i) $x^*F(E) = \int_E TF(\omega)x^* \,\mathrm{d}\mu(\omega)$ for each $x^* \in X^*$ and $E \in \Sigma$, and

(ii) $|F|(E) = \int_E \|Tf(\omega)\| \,\mathrm{d}\mu(\omega)$ for each $E \in \Sigma$.

Set $h_n := TF_n$ for each $n \in \mathbb{N}$.

Since X has the WRNP with respect to (Ω, Σ, μ), there exists a sequence $\{f_n\}$ in $\mathcal{P}(\mu, X)$ such that $F_n(E) = (\mathrm{P}) \int_E f_n(\omega) \,\mathrm{d}\mu(\omega)$ for each $E \in \Sigma$ and $n \in \mathbb{N}$. Given n fixed numbers $\varepsilon_i \in \{-1, 0, 1\}$, $1 \leqslant i \leqslant n$, it follows from

$$\int_E \Big\| \sum_{i=1}^n \varepsilon_i h_i \Big\| \,\mathrm{d}\mu(\omega) = \int_E \Big\| T\Big(\sum_{i=1}^n \varepsilon_i F_i\Big) \Big\| \,\mathrm{d}\mu(\omega) = \Big| \sum_{i=1}^n \varepsilon_i F_i \Big|(E)$$

that

$$\int_E \Big| x^*\Big(\sum_{i=1}^n \varepsilon_i f_i\Big) \Big| \,\mathrm{d}\mu \leqslant \Big\| \sum_{i=1}^n \varepsilon_i F_i \Big\|(E) \leqslant \Big| \sum_{i=1}^n \varepsilon_i F_i \Big|(E) = \int_E \Big\| \sum_{i=1}^n \varepsilon_i h_i(\omega) \Big\| \,\mathrm{d}\mu(\omega)$$

for each $x^* \in B_{X^*}$ and $E \in \Sigma$. Consequently, for each $x^* \in B_{X^*}$ there is a null set $N(\varepsilon_1, \ldots, \varepsilon_n, x^*) \in \Sigma$ such that $\Big| x^*\Big(\sum\limits_{i=1}^n \varepsilon_i f_i(\omega)\Big) \Big| \leqslant \Big\| \sum\limits_{i=1}^n \varepsilon_i h_i(\omega) \Big\|$ for all $\omega \in \Omega \setminus N(\varepsilon_1, \ldots, \varepsilon_n, x^*)$. If $\{x_m^*\}$ denotes a norming sequence in (the unit sphere of) X^*, setting $N(\varepsilon_1, \ldots, \varepsilon_n) := \bigcup\limits_{m=1}^{\infty} N(\varepsilon_1, \ldots, \varepsilon_n, x_m^*)$ we have $\Big| x_m^*\Big(\sum\limits_{i=1}^n \varepsilon_i f_i(\omega)\Big) \Big| \leqslant \Big\| \sum\limits_{i=1}^n \varepsilon_i h_i(\omega) \Big\|$ for each $\omega \in \Omega \setminus N(\varepsilon_1, \ldots, \varepsilon_n)$ and all $m \in \mathbb{N}$. This implies that $\Big\| \sum\limits_{i=1}^n \varepsilon_i f_i(\omega) \Big\| \leqslant \Big\| \sum\limits_{i=1}^n \varepsilon_i h_i(\omega) \Big\|$ for each $\omega \in \Omega \setminus N(\varepsilon_1, \ldots, \varepsilon_n)$. Therefore, setting

$N := \bigcup\limits_{n=1}^{\infty} \bigcup\limits_{(\varepsilon_1,\dots,\varepsilon_n)} N(\varepsilon_1,\dots,\varepsilon_n)$, we conclude that N is a μ-null set such that

$$\Big\|\sum_{i=1}^{n}\varepsilon_i f_i(\omega)\Big\| \leqslant \Big\|\sum_{i=1}^{n}\varepsilon_i h_i(\omega)\Big\|$$

for each $\omega \in \Omega\setminus N$ and each finite set $\{\varepsilon_1,\dots,\varepsilon_n\}$. Moreover, as each scalarly valued function $\omega \to \Big\|\sum\limits_{i=1}^{n}\varepsilon_i f_i(\omega)\Big\|$ is μ-measurable as a consequence of the fact that

$$\Big\|\sum_{i=1}^{n}\varepsilon_i f_i(\omega)\Big\| = \sup_{m\in\mathbb{N}}\Big|x_m^*\Big(\sum_{i=1}^{n}\varepsilon_i f_i(\omega)\Big)\Big|$$

for all $\omega \in \Omega$, it follows that $\Big\|\sum\limits_{i=1}^{n}\varepsilon_i f_i(\cdot)\Big\| \in L_1(\mu)$.

In particular, $\|f_n(\cdot)\| \in L_1(\mu)$ for all $n \in \mathbb{N}$ and, consequently,

$$\|F_n(E)\| = \sup_{\|x^*\|\leqslant 1}|x^*F(\omega)| \leqslant \sup_{\|x^*\|\leqslant 1}\int_E |x^*f_n(\omega)|\,\mathrm{d}\mu(\omega) \leqslant \int_E \|f_n(\omega)\|\,\mathrm{d}\mu(\omega)$$

for each $n \in \mathbb{N}$ and $E \in \Sigma$. This implies that

$$|F_n| \leqslant \int_\Omega \|f_n(\omega)\|\,\mathrm{d}\mu(\omega) \leqslant \int_\Omega \|h_n(\omega)\|\,\mathrm{d}\mu(\omega) = |F_n|,$$

that is, $\int_\Omega \|f_n(\omega)\|\,\mathrm{d}\mu(\omega) = 1$ for each $n \in \mathbb{N}$.

Since the series $\sum\limits_{n=1}^{\infty} F_n$ is weak unconditionally Cauchy there is $K > 0$ such that $\Big|\sum\limits_{i=1}^{n}\varepsilon_i F_i\Big| < K$ for all finite set of numbers $\varepsilon_i \in \{-1,0,1\}$. So, according to what we have established above, we have

$$\int_\Omega \Big\|\sum_{i=1}^{n}\varepsilon_i f_i(\omega)\Big\|\,\mathrm{d}\mu(\omega) \leqslant \int_\Omega \Big\|\sum_{i=1}^{n}\varepsilon_i h_i(\omega)\Big\|\,\mathrm{d}\mu(\omega) = \Big|\sum_{i=1}^{n}\varepsilon_i F_i\Big| < K$$

for each $\varepsilon_i \in \{-1,0,1\}$, $1 \leqslant i \leqslant n$ and $n \in \mathbb{N}$. Since $\int_\Omega \|f_n(\omega)\|\,\mathrm{d}\mu(\omega) = 1$ for each $n \in \mathbb{N}$, Lemma 2.3 leads to the conclusion. $\square$

Corollary 4.1. *Let X be a real Banach space of infinite dimension whose dual unit ball has countably many extreme points. If X has the WRNP with respect to each $\mu \in \mathrm{ca}^{+}(\Sigma)$, then the space* $\mathrm{bvca}(\Sigma, X)$ *contains a copy of c_0 if and only if X does.*

P r o o f. Since the set $K = \operatorname{Ext} B_{X^*}$ of the extreme points of the dual unit ball of X is a James boundary for B_{X^*} [7, Chapter 3], we have $\|x\| = \sup\{|x^*x| \colon x^* \in K\}$ and, consequently, $\operatorname{Ext} B_{X^*}$ is a norming set for X. Since $\operatorname{Ext} B_{X^*}$ is countable by hypothesis, the preceding theorem applies. $\square$

Remark 4.1. Let us mention that Theorem 1.2 clearly applies when X is separable, for X^* would contain a norming sequence. However, this case was already contained in [6, Remark], since separability implies that X possesses the Radon-Nikodým property with respect to each $\mu \in \mathrm{ca}^{+}(\Sigma)$.

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