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WEAK CHAIN-COMPLETENESS AND FIXED POINT PROPERTY FOR PSEUDO-ORDERED SETS

S. Parameshwara Bhatta, Karnataka

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Abstract. In this paper the notion of weak chain-completeness is introduced for pseudo-ordered sets as an extension of the notion of chain-completeness of posets (see [3]) and it is shown that every isotone map of a weakly chain-complete pseudo-ordered set into itself has a least fixed point.

Keywords: pseudo-ordered set, trellis, complete trellis, fixed point property, weak chain completeness

MSC 2000: 06B05

1. Introduction

A reflexive and antisymmetric binary relation $\sqsubseteq$ on a set P is called a *pseudo-order* on P. A *pseudo-ordered set* or a *psoset* $\langle P, \sqsubseteq\rangle$ consists of a nonempty set P and a pseudo-order $\sqsubseteq$ on P. For $a, b \in P$, we denote $a \sqsubset b$ to mean $a \sqsubseteq b$ and $a \neq b$. For a subset of P the notions of a lower bound, an upper bound, the greatest lower bound (or meet), the least upper bound (or join), the minimum (or the least) element and the maximum (or the greatest) element are defined analogous to the corresponding notions in a poset. As in the case of posets (see [1]) for the empty set $\emptyset$, $\bigvee\emptyset$ exists in P if and only if $\bigwedge P$ exists or equivalently P has the minimum element 0 and $\bigvee\emptyset = \bigwedge P = 0$. A psoset $\langle P, \sqsubseteq\rangle$ is said to be a *trellis* if every pair of elements of P has a g.l.b. and a l.u.b. or equivalently, a trellis can be regarded as an algebra $\langle P, \wedge, \vee\rangle$ satisfying certain axioms. A psoset $\langle P, \sqsubseteq\rangle$ is said to be a *complete trellis* if every subset of P has a g.l.b and a l.u.b. An extensive investigation of the notions of psoset, a trellis and a complete trellis can be found in H. L. Skala [4] and H. Skala [5].

Let $\langle P, \sqsubseteq \rangle$ be psoset. A map $f\colon P \to P$ is said to be *isotone* if $a \sqsubseteq b$ implies $f(a) \sqsubseteq f(b)$. An element $a \in P$ is said to be a *fixed point* for f if $f(a) = a$. If every isotone map of P into itself has a fixed point (a least fixed point), then P is said to have the *fixed point property* (the *least fixed point property*) and we express it by writing P has the *FPP* (P has the *least FPP*). It is known (see [5]) that every complete trellis has the FPP.

A poset $\langle P, \leqslant \rangle$ is said to be *chain-complete* if every chain in P, including the empty chain, has a join. This notion and its applications were discussed by G. Markowsky in [3]. It was proved that a poset P is chain-complete if and only if P has the least FPP.

The aim of this paper is to introduce the notion of weak chain-completeness for psosets as a generalization of the notion of chain-completeness of posets and to prove that every weakly chain-complete psoset has the least FPP.

Any psoset $\langle P, \sqsubseteq \rangle$ can be represented by a digraph (possibly infinite) whose points are the elements of P while for distinct points a and b there is a unique directed line from a to b if and only if $a \sqsubset b$. The digraph of Fig. 1 represents the psoset $\langle A, \sqsubseteq \rangle$ where $A = \{0, a, b, c\}$ with $0 \sqsubset x$ for every $x \in \{a, b, c\}$ while $a \sqsubset b \sqsubset c \sqsubset a$. $\langle A, \sqsubseteq \rangle$ is a trellis having the minimum element 0 which is not complete.

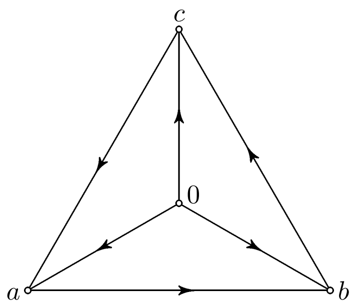

Figure 1

2. Weakly chain-complete psosets

Let $\langle P, \sqsubseteq \rangle$ be a psoset. A subset C of P, including $C = \emptyset$, is called a *chain in P* if the restriction of $\sqsubseteq$ to C is a complete order (i.e. it is a partial order on C such that every pair of elements of C are comparable). A subset $C = \{a_i \mid i = 0, 1, 2, \ldots\}$ of P is said to be a *descending chain* in P if $a_i \sqsupset a_j$ whenever $i < j$. A psoset P is said to satisfy the *descending chain condition* if there exists no infinite descending chain in P or equivalently if every nonempty chain in P has a minimum element. A chain C in P is said to be a *well-ordered chain* if it satisfies the descending chain condition or equivalently if every nonempty subset of C has a minimum element. A psoset $\langle P, \sqsubseteq \rangle$ is said to be *weakly chain-complete* if every well-ordered chain in P has a join.

We know that weak chain-completeness and chain-completeness are equivalent for posets and that a chain-complete lattice is a complete lattice. However, a weakly chain-complete trellis need not be a complete trellis since the trellis A of Fig. 1 is weakly chain-complete but not complete. In this trellis the only (well-ordered) chains are the proper subsets of A excluding $\{a, b, c\}$.

The following lemma is a major step in proving our main theorem. The proof of the lemma is an adaptation, with several modifications, of the proof of Zorn's lemma (see J. Lewin [2]).

Lemma. *Every weakly chain-complete psoset has the FPP.*

Proof. Let $\langle P, \sqsubseteq \rangle$ be a weakly chain-complete psoset and $f\colon P \to P$ be an isotone map. If C, D are chains in P, then C is said to be a *section* of D determined by some $d \in D$ denoted by $C = S_d(D)$, if there exists $d \in D$ such that $C = \{x \in D \mid x \sqsubset d\}$. Call a subset A of P *admissible* if it satisfies the following three conditions:

(1) A is a well-ordered chain in P.

(2) For $a \in A$, if $S_a(A)$ has a maximum element, say b, then $f(b) = a$; if $S_a(A)$ does not have a maximum element, then $\bigvee S_a(A) = a$.

(3) If $a \in A$ and b is an upper bound of $S_a(A)$ in P, the $f(b)$ is also an upper bound of $S_a(A)$.

Let $\mathcal{A}$ denote the collection of all admissible subsets of P. We now make the following observations.

(i) $\mathcal{A}$ is nonempty since the empty chain $\emptyset$ lies in $\mathcal{A}$. Also, if $\bigvee \emptyset = 0$, then $\{0\} \in \mathcal{A}$.

(ii) If $A \in \mathcal{A}$ and $a \in A$, then $S_a(A) \in \mathcal{A}$.

In fact, for any $x \in S_a(A)$, clearly $S_x(S_a(A)) = S_x(A)$.

(iii) Let $A \in \mathcal{A}$. If b is an upper bound of A in P, then $f(b)$ is also an upper bound of A.

In fact, for any $a \in A$, clearly b is an upper bound of $S_a(A)$ so that, by (3), $f(b)$ is also an upper bound of $S_a(A)$. If $S_a(A)$ has a maximum element, say c, then by (2), $a = f(c) \sqsubseteq f(b)$, since $c \sqsubset b$ and f is isotone. If $S_a(A)$ does not have a maximum element, then, by (2), $a = \bigvee S_a(A) \sqsubseteq f(b)$.

(iv) If $A \in \mathcal{A}$ and A has the maximum element, say a, then $A \cup \{f(a)\} \in \mathcal{A}$.

In fact, by (iii), $f(a)$ is an upper bound of A.

(v) If $A \in \mathcal{A}$, then $A \cup \{\bigvee A\} \in \mathcal{A}$.

(vi) If $A \in \mathcal{A}$, then $a \sqsubseteq f(a)$ for every $a \in A$.

In fact, if $a \in A$, then $S_a(A) \in \mathcal{A}$. By using (2), (iv) and (v), $S_a(A) \cup \{a\} \in \mathcal{A}$. Hence, by (iii), $f(a)$ is an upper bound of $S_a(A) \cup \{a\}$.

(vii) If $A, B \in \mathcal{A}$, $a \in A$ and $b \in B$ such that $S_a(A) = S_b(B)$, then $a = b$. This follows from (2).

(viii) If $A, B \in \mathcal{A}$, then either $A = B$ or one of A, B is a section of the other.

Before proving this statement we prove the following special case of it:

$(*)$ If $B \subset A$ (where $\subset$ denotes proper inclusion), then B is a section of A.

Given $B \subset A$, let $x = \min(A - B)$. Then $S_x(A) \subseteq B$. We assert that $S_x(A) = B$. If $S_x(A) \neq B$, then let $y = \min(B - S_x(A))$. Then $S_y(B) \subseteq S_x(A)$. Clearly $x, y \in A$ and $x \neq y$ since $x \notin B$ whereas $y \in B$. Also $y \not\sqsubset x$ since $y \notin S_x(A)$. Thus $x \sqsubset y$. But then $S_x(A) \subseteq S_y(B)$. For, if $t \in S_x(A)$, then $t \in B$ and $t \sqsubset x \sqsubset y$ so that $t \sqsubset y$ since $t, x, y \in A$, a chain; and therefore $t \in S_y(B)$. Thus $S_x(A) = S_y(B)$. Hence, by (vii), $x = y$, a contradiction. This proves our assertion and hence $(*)$ holds.

In general, given $A, B \in \mathcal{A}$, either $A \subseteq B$ or $A \nsubseteq B$. In the first case, by $(*)$, it follows that either $A = B$ or A is a section of B. If $A \nsubseteq B$, then let $x = \min(A - B)$. Then $S_x(A) \subseteq B$. If $S_x(A) \neq B$, then since $S_x(A) \in \mathcal{A}$, by $(*)$, $S_x(A) = S_y(B)$ for some $y \in B$. But then $x = y$ by (vii), a contradiction since $x \notin B$ whereas $y \in B$. Thus $B = S_x(A)$, a section of A.

(ix) Let $U = \cup\mathcal{A}$. Then $U \in \mathcal{A}$.

To prove this, we first observe by (viii) that U is a chain. Consider any $u \in U$. Then $u \in A$ for some $A \in \mathcal{A}$. We assert that

$$(**) \qquad S_u(U) = S_u(A).$$

Clearly, $S_u(A) \subseteq S_u(U)$. On the other hand if $v \in S_u(U)$, then $v \in B$ for some $B \in \mathcal{A}$ and $v \sqsubset u$. If $B = A$ or B is a section of A, the clearly $v \in A$. Otherwise, $A = S_w(B)$ for some $w \in B$. But then $v \sqsubset u \sqsubset w$ since $u \in A = S_w(B)$. Thus $v \sqsubset w$ since $v, u, w \in B$, a chain. Therefore $v \in S_w(B) = A$. Hence $(**)$ holds. Using $(**)$ it is easy to verify that $U \in \mathcal{A}$.

Now, by (v), $U \cup \{\bigvee U\} \in \mathcal{A}$ so that $U \supseteq U \cup \{\bigvee U\}$ and hence $\bigvee U \in U$. Let $m = \bigvee U$. Then $m = \max U$. Hence, using (iv), $U \cup \{f(m)\} \in \mathcal{A}$, and this implies $f(m) \in U$. Consequently, $f(m) \sqsubseteq \max U = m$. Already, by (vi), $m \sqsubseteq f(m)$. Thus $f(m) = m$. Hence f has a fixed point. $\square$

We now present the main theorem of this paper.

Theorem. *Every weakly chain-complete psoset has the least FPP.*

P r o o f. Let $\langle P, \sqsubseteq \rangle$ be a weakly chain-complete psoset and $f\colon P \to P$ be an isotone map. Let $F = \{x \in P \mid f(x) = x\}$ and F^* denote the set of all lower bounds of F in P. Then F^* is also a weakly chain-complete psoset with respect to $\sqsubseteq$. In fact, any well-ordered chain C in F^*, being a well-ordered chain in P, has a join, $\bigvee C = a$, in P and clearly $a \in F^*$. Further, the restriction of f to F^* is an isotone map of F^* into itself. In fact, for any $a \in F^*$ we have $a \sqsubseteq x$ for every $x \in F$ so that $f(a) \sqsubseteq f(x) = x$, for every $x \in F$ and therefore $f(a) \in F^*$. Now an application of our lemma to F^* yields a fixed point $y \in F^*$ for f. Then clearly $y \in F \cap F^*$ so that y is the least element of F. Thus y is the least fixed point for f in P. $\square$

In view of the above theorem the following problems remain unsolved.

Problem 1. Is it necessary that in a weakly chain-complete psoset every chain has a join?

Problem 2. Does the least FPP imply weak chain-completeness for psosets in general?

It may be recalled that for psosets, these problems have an affirmative solution (see [1] and [3]).

References

[1] *P. Crawley and R. P. Dilworth*: Algebraic Theory of Lattices. Prentice-Hall, Englewood Cliffs, 1973.
[2] *J. Lewin*: A simple proof of Zorn's lemma. Amer. Math. Monthly *98* (1991), 353–354.
[3] *G. Markowski*: Chain-complete posets and directed sets with applications. Algebra Universalis *6* (1976), 54–69.
[4] *H. L. Skala*: Trellis theory. Algebra Universalis *1* (1971), 218–233.
[5] *H. Skala*: Trellis Theory. Mem. Amer. Math. Soc. 121, Providence, 1972.

Author's address: Department of Mathematics, Mangalore University, Mangalagangothri, 574 199 Karnataka, India, e-mail: s_p_bhatta@yahoo.co.in.