BOUNDEDNESS OF THE SOLUTION OF THE THIRD PROBLEM FOR THE LAPLACE EQUATION

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Abstract. A necessary and sufficient condition for the boundedness of a solution of the third problem for the Laplace equation is given. As an application a similar result is given for the third problem for the Poisson equation on domains with Lipschitz boundary.

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1. General open sets

For $x, y \in \mathbb{R}^m$, $m > 2$, denote

$$h_x(y) = \begin{cases} (m-2)^{-1}A^{-1}|x-y|^{2-m} & \text{for } x \neq y, \\ \infty & \text{for } x = y, \end{cases}$$

where A is the area of the unit sphere in $\mathbb{R}^m$. For the finite real Borel measure ν denote

$$\mathcal{U}\nu(x) = \int_{\mathbb{R}^m} h_x(y)\,\mathrm{d}\nu(y),$$

the single layer potential corresponding to ν, for each x for which this integral has sense.

Suppose that $G \subset \mathbb{R}^m$ $(m > 2)$ is an open set with a non-void compact boundary ∂G such that $\partial G = \partial(\mathbb{R}^m \setminus G)$. Fix a nonnegative element λ of $\mathcal{C}'(\partial G)$ (= the Banach space of all finite signed Borel measures with support in ∂G with the total variation as a norm) and suppose that the single layer potential $\mathcal{U}\lambda$ is bounded and

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continuous on ∂G. It was shown in [26] that $\mathscr{U}\lambda$ is bounded and continuous on ∂G if and only if

$$\lim_{r\to 0+}\sup_{y\in\partial G}\int_{\Omega_r(y)} h_y(x)\,\mathrm{d}\lambda(x)=0.$$

According to [12], Lemma 2.18 this is true if there are constants $\alpha > m-2$ and $k>0$ such that $\lambda(\Omega_r(x))\leqslant kr^\alpha$ for all $x\in\mathbb{R}^m$ and all $r>0$.

Suppose that for λ-a.a. $x\in\partial G$ there is

$$d_G(x)=\lim_{r\searrow 0}\frac{\mathscr{H}_m(G\cap\Omega_r(x))}{\mathscr{H}_m(\Omega_r(x))}>0.$$

Here $\Omega_r(x)$ is the open ball with the centre x and the diameter r, $\mathscr{H}_k$ is the k-dimensional Hausdorff measure normalized so that $\mathscr{H}_k$ is the Lebesgue measure in $\mathbb{R}^k$.

For a Lebesgue measurable function u on a Borel set M and x with $d_M(x)>0$ define

$$\operatorname*{aplimsup}_{\substack{y\to x\\ y\in M}} u(y) = \inf\{t;\ d_{\{z\in M\,;\ u(z)>t\}}(x)=0\},$$
$$\operatorname*{apliminf}_{\substack{y\to x\\ y\in M}} u(y) = \sup\{t;\ d_{\{z\in M\,;\ u(z)<t\}}(x)=0\}.$$

We speak of the approximate limit of u at x over M in case

$$\operatorname*{aplimsup}_{\substack{y\to x\\ y\in M}} u(y) = \operatorname*{apliminf}_{\substack{y\to x\\ y\in M}} u(y),$$

and u is said to be approximately continuous at x with respect to M if

$$\operatorname*{aplim}_{\substack{y\to x\\ y\in M}} u(y) = u(x).$$

If h is a harmonic function on G such that

$$\int_H |\nabla h|\,\mathrm{d}\mathscr{H}_m<\infty$$

for all bounded open subsets H of G we define the weak normal derivative $N^G h$ of h as the distribution

$$\langle\varphi, N^G h\rangle = \int_G \nabla\varphi\cdot\nabla h\,\mathrm{d}\mathscr{H}_m$$

for $\varphi\in\mathscr{D}$ (= the space of all compactly supported infinitely differentiable functions in $\mathbb{R}^m$).

If $H \subset \mathbb{R}^m$ is an open set with a compact smooth boundary, $u \in \mathscr{C}^1(\operatorname{cl} H)$ is a harmonic function on H and

$$\frac{\partial u}{\partial n} + fu = g \quad \text{on } \partial H$$

where $f, g \in \mathscr{C}(\partial H)$ (= the space of all bounded continuous functions on ∂H equipped with the maximum norm) and n is the exterior unit normal of H, then for $\varphi \in \mathscr{D}$ we have

$$\int_{\partial H} \varphi g \,\mathrm{d}\mathscr{H}_{m-1} = \int_H \nabla\varphi \cdot \nabla u \,\mathrm{d}\mathscr{H}_m + \int_{\partial H} \varphi f u \,\mathrm{d}\mathscr{H}_{m-1}. \tag{1}$$

(Here $\operatorname{cl} H$ denotes the closure of H.) If we denote by $\mathscr{H}$ the restriction of $\mathscr{H}_{m-1}$ to ∂H then (1) has the form

$$N^H u + fu\mathscr{H} = g\mathscr{H}. \tag{2}$$

The formula (2) motivates our definition of the solution of the third problem for the Laplace equation

$$\begin{aligned} \Delta u &= 0 \quad \text{in } G,\\ N^G u + u\lambda &= \mu, \end{aligned} \tag{3}$$

where $\mu \in \mathscr{C}'(\partial G)$ (compare [12], [25]).

Let $\mu \in \mathscr{C}'(\partial G)$. *We say that a function u on $\operatorname{cl} G$ is a weak solution of the third problem for the Laplace equation* (3) *if $u \in L_1(\lambda)$, u is harmonic on G, $|\nabla u|$ is integrable over all bounded open subsets of G, $u(x)$ is the approximmative limit of u over G for λ-a.a. $x \in \partial G$, and $N^G u + u\lambda = \mu$.* (If $\lambda = 0$ we say that u is a weak solution of the Neumann problem for the Laplace equation.)

Notation. Let $V \subset \mathbb{R}^m$ be an open set. For $p \geqslant 1$ denote by $W^{1,p}(V)$ the collection of all functions $f \in L_p(V)$ the distributional gradient of which belongs to $[L_p(V)]^m$. By $W^{1,p}_{\mathrm{loc}}(V)$ denote the collection of all functions f such that $f \in W^{1,p}(U)$ for each bounded open set U with $\operatorname{cl} U \subset V$.

Suppose that G has a locally Lipschitz boundary and $u \in W^{1,p}(G)$, $1 < p < \infty$. It is well-known that we can even suppose that $u \in W^{1,p}(\mathbb{R}^m)$ (see [30], Remark 2.5.2). We can choose such a representation of u that u is approximately continuous at $\mathscr{H}_{m-1}$-a.a. points of $\mathbb{R}^m$ (see [30], Theorem 3.3.3, Theorem 2.6.16 and Remark 3.3.5). The restriction of u to ∂G is the trace of u (see [30], p. 190). If $\mathscr{H}$ denotes the restriction of $\mathscr{H}_{m-1}$ to ∂G, then $u \in L_p(\mathscr{H})$ (see [22], Theorem 1.2). If f is a nonnegative bounded Baire function on ∂G and $g \in L_p(\mathscr{H})$, then u is called a weak solution in $W^{1,p}(G)$ of the problem $\Delta u = 0$ in G, $\partial u/\partial n + fu = g$ on ∂G if

$$\int_{\partial G} vg \,\mathrm{d}\mathscr{H}_{m-1} = \int_G \nabla v \cdot \nabla u \,\mathrm{d}\mathscr{H}_m + \int_{\partial G} fvu \,\mathrm{d}\mathscr{H}_{m-1}$$

for each $v \in W^{1,q}(G)$, where $q = p/(p-1)$ (compare [22], Example 2.12). Put $\lambda = f\mathscr{H}$, $\mu = g\mathscr{H}$. Using Hölder's inequality we see that $|\nabla u|$ is integrable over all bounded open subsets of G. Since u is approximately continuous at $\mathscr{H}_{m-1}$-a.a. points of $\mathbb{R}^m$ and λ is absolutely continuous with respect to $\mathscr{H}_{m-1}$, we obtain that $u(x)$ is the approximative limit of u at x over G for λ-a.a. $x \in \partial G$. If u is a weak solution in $W^{1,p}(G)$ of the problem $\Delta u = 0$ in G, $\partial u/\partial n + fu = g$ on ∂G, then u is a weak solution of (3) because $\mathscr{D} \subset W^{1,q}(G)$. Since $\mathscr{D}$ is a dense subset of $W^{1,q}(G)$, u is a weak solution of the third problem for the Laplace equation (3) if and only if u is a weak solution in $W^{1,p}(G)$ of the problem $\Delta u = 0$ in G, $\partial u/\partial n + fu = g$ on ∂G.

It is usual to look for a solution u in the form of the single layer potential $\mathscr{U}\nu$, where $\nu \in \mathscr{C}'(\partial G)$. It was shown in [17] that $\mathscr{U}\nu$ has all the properties of the solution of the third problem with some boundary condition, but our "continuity" on the boundary is replaced by the fine continuity at λ-a.a. points of the boundary. If $\mathscr{U}\nu$ is fine-continuous in $x \in \partial G$ with respect to $\operatorname{cl} G$ then $u(x)$ is the approximative limit of u at x over G (see [11], Theorem 10.15, Corollary 10.5). If $\mathscr{U}\nu$ is a solution of the third problem in the sense of [17] then it is a weak solution of the third problem.

The operator $\tau\colon \nu \mapsto N^G\mathscr{U}\nu + (\mathscr{U}\nu)\lambda$ is a bounded linear operator on $\mathscr{C}'(\partial G)$ if and only if $V^G < \infty$, where

$$V^G = \sup_{x \in \partial G} v^G(x),$$
$$v^G(x) = \sup\left\{\int_G \nabla\varphi \cdot \nabla h_x \,\mathrm{d}\mathscr{H}_m\,;\ \varphi \in \mathscr{D},\ |\varphi| \leqslant 1,\ \operatorname{spt}\varphi \subset \mathbb{R}^m - \{x\}\right\}$$

(see [12]). There are more geometrical characterizations of $v^G(x)$ in [12] which ensure that $V^G < \infty$ for G convex or for G with $\partial G \subset \bigcup_{i=1}^{k} L_i$, where L_i are $(m-1)$-dimensional Ljapunov surfaces i.e. of class $C^{1+\alpha}$.

If $z \in \mathbb{R}^m$ and θ is a unit vector such that the symmetric difference of G and the half-space $\{x \in \mathbb{R}^m\,;\ (x-z)\cdot\theta < 0\}$ has m-dimensional density zero at z then $n^G(z) = \theta$ is termed *the exterior normal* of G at z in Federer's sense. If there is no exterior normal of G at z in this sense, we denote by $n^G(z)$ the zero vector in $\mathbb{R}^m$. The set $\{y \in \mathbb{R}^m\,;\ |n^G(y)| > 0\}$ is called the reduced boundary of G and will be denoted by $\widehat{\partial} G$.

If G has a finite perimeter (which is fulfilled if $V^G < \infty$) then $\mathscr{H}_{m-1}(\widehat{\partial} G) < \infty$ and

$$v^G(x) = \int_{\widehat{\partial} G} |n^G(y) \cdot \nabla h_x(y)| \, \mathrm{d}\mathscr{H}_{m-1}(y)$$

for each $x \in \mathbb{R}^m$. Throughout the paper we shall assume that $V^G < \infty$.

If L is a bounded linear operator on the Banach space X we denote by $\|L\|_{\mathrm{ess}}$ the essential norm of L, i.e. the distance of L from the space of all compact linear operators on X. The essential spectral radius of L is defined by

$$r_{\mathrm{ess}} L = \lim_{n \to \infty} (\|L^n\|_{\mathrm{ess}})^{1/n}.$$

Theorem ([17]). *Let $r_{\mathrm{ess}}(\tau - \frac{1}{2} I) < \frac{1}{2}$, where I is the identity operator, $\mu \in \mathscr{C}'(\partial G)$. Then there is a harmonic function u on G, which is a weak solution of the third problem*

$$N^G u + u\lambda = \mu,$$

if and only if $\mu \in \mathscr{C}_0'(\partial G)$ (= the space of such $\nu \in \mathscr{C}'(\partial G)$ that $\nu(\partial H) = 0$ for each bounded component H of $\operatorname{cl} G$ for which $\lambda(\partial H) = 0$). Moreover, if $\mu \in \mathscr{C}_0'(\partial G)$ then there is a solution of this problem in the form of the single layer potential $\mathscr{U}\nu$, where $\nu \in \mathscr{C}'(\partial G)$.

Remark 1. It is well-known that the condition $r_{\mathrm{ess}}\big(\tau - \frac{1}{2} I\big) < \frac{1}{2}$ is fulfilled for sets with a smooth boundary (of class $C^{1+\alpha}$) (see [13]) and for convex sets (see [23]). R. S. Angell, R. E. Kleinman, J. Král and W. L. Wendland proved that rectangular domains (i.e. formed from rectangular parallelepipeds) in $\mathbb{R}^3$ have this property (see [1], [14]). A. Rathsfeld showed in [27], [28] that polyhedral cones in $\mathbb{R}^3$ have this property. (By a polyhedral cone in R^3 we mean an open set Ω whose boundary is locally a hypersurface (i.e. every point of $\partial\Omega$ has a neighbourhood in $\partial\Omega$ which is homeomorphic to $\mathbb{R}^2$) and $\partial\Omega$ is formed by a finite number of plane angles. By a polyhedral open set with bounded boundary in $\mathbb{R}^3$ we mean an open set Ω whose boundary is locally a hypersurface and $\partial\Omega$ is formed by a finite number of polygons). N. V. Grachev and V. G. Maz'ya obtained independently an analogous result for polyhedral open sets with bounded boundary in $\mathbb{R}^3$ (see [9]). (Let us note that there is a polyhedral set in $\mathbb{R}^3$ which does not have a locally Lipschitz boundary.) In [16] it was shown that the condition $r_{\mathrm{ess}}(\tau - \frac{1}{2} I) < \frac{1}{2}$ has a local character. As a conclusion we obtain that this condition is fullfiled for $G \subset \mathbb{R}^3$ such that for each $x \in \partial G$ there are $r(x) > 0$, a domain D_x which is polyhedral or smooth or convex or a complement of a convex domain and a diffeomorphism $\psi_x \colon \mathscr{U}(x; r(x)) \to \mathbb{R}^3$ of class $C^{1+\alpha}$, where $\alpha > 0$, such that $\psi_x(G \cap \mathscr{U}(x; r(x))) = D_x \cap \psi_x(\mathscr{U}(x; r(x)))$. V. G. Maz'ya and N. V. Grachev proved this condition for several types of sets with "piecewise-smooth" boundary in the general Euclidean space (see [7], [8], [10]).

In the rest of the paper we will suppose that $r_{\rm ess}(\tau - \frac{1}{2}I) < \frac{1}{2}$. Since $\tau - N^G\mathscr{U}$ is a compact operator (see [17], Remark 5), this condition is equivalent to the condition $r_{\rm ess}(N^G\mathscr{U} - \frac{1}{2}I) < \frac{1}{2}$. Denote by $\mathscr{H}$ the restriction of $\mathscr{H}_{m-1}$ onto ∂G. Then $\mathscr{H}(\mathbb{R}^m) < \infty$ (see [18], Lemma 2). If $x \in \partial G$ then $d_G(x)$ exists and is strictly positive (see [17], Lemma 14).

Notation. Let us denote by $\mathscr{C}'_b(\partial G)$ the set of all $\mu \in \mathscr{C}'(\partial G)$ for which $\mathscr{U}\mu$ is bounded on $\mathbb{R}^m \setminus \partial G$.

Note that $\mathscr{C}'_b(\partial G)$ is the set of all $\mu \in \mathscr{C}'(\partial G)$ for which there is a polar set M such that $\mathscr{U}\mu(x)$ is meaningful and bounded on $\mathbb{R}^m \setminus M$, because $\mathscr{H}_m(\partial G) = 0$ by [17], Corollary 1 and therefore $\mathbb{R}^m \setminus \partial G$ is finely dense in $\mathbb{R}^m$ (see [2], Chap. VII, §§ 2, 6, [15], Theorem 5.11, Theorem 5.10) and $\mathscr{U}\mu = \mathscr{U}\mu^+ - \mathscr{U}\mu^-$ is finite and fine-continuous outside of a polar set.

Remark 2. Let $m-1 < p < \infty$, $f \in L_p(\mathscr{H})$. Then $\mu = f\mathscr{H} \in \mathscr{C}'_b(\partial G)$ (see [17], Remark 6).

Theorem 1. *Let $\nu, \mu \in \mathscr{C}'(\partial G)$, $N^G\mathscr{U}\nu + (\mathscr{U}\nu)\lambda = \mu$. Then the following assertions are equivalent:*

a) $\nu \in \mathscr{C}'_b(\partial G)$.
b) $\mu \in \mathscr{C}'_b(\partial G)$.
c) *$\mathscr{U}\nu$ is bounded on G.*
d) *$\mathscr{U}\mu$ is bounded on G.*
e) *There are a polar set K and a bounded function f on ∂G such that $\mathscr{U}\nu = f$ on $\partial G \setminus K$.*
f) *There are a polar set K and a bounded function f on ∂G such that $\mathscr{U}\mu = f$ on $\partial G \setminus K$.*

Proof. a) $\Rightarrow$ c) Since $\mathscr{U}\nu$ is bounded in $\mathbb{R}^m \setminus \partial G$ it is bounded in G.

c) $\Rightarrow$ e) Denote $K = \{x \in \partial G;\ \mathscr{U}|\nu|(x) = \infty\}$. Then K is polar and $\mathscr{U}\nu(x)$ is the fine limit of $\mathscr{U}\nu$ for each $x \in \partial G \setminus K$. Put $f(x) = \mathscr{U}\nu(x)$ for each $x \in \partial G \setminus K$, $f(x) = 0$ for $x \in K$. Since the density of G is positive at each point of ∂G by [17], Corollary 1, every fine neighbourhood of $x \in \partial G$ intersects G (see [2], Chap. VII, § 2, § 6, [15], Theorem 5.11, Theorem 5.10), and $\mathscr{U}\nu$ is bounded on G, f is a bounded function.

e) $\Rightarrow$ a) Fix $R > 0$ such that $\partial G \subset \{x;\ |x| < R\}$. Put $H = \{x \in G;\ |x| < R\}$, $M = \{x \in \mathbb{R}^m \setminus \operatorname{cl} G;\ |x| < R\}$. Using [19], Lemma 1 and [19], Lemma 2 for H and M we get

$$\sup_{x\in H} |\mathscr{U}\nu(x)| \leqslant \sup_{x\in\partial H} |f(x)|, \quad \sup_{x\in M} |\mathscr{U}\nu(x)| \leqslant \sup_{x\in\partial M} |f(x)|.$$

Since

$$\lim_{|x|\to\infty} \mathscr{U}\nu(x) = 0,$$

we get for $R \to \infty$

$$\sup_{x\in\mathbb{R}^m\setminus G} |\mathscr{U}\nu(x)| \leqslant \sup_{x\in\partial G} |f(x)| < \infty.$$

b) $\Leftrightarrow$ d) $\Leftrightarrow$ f) We have proved a) $\Leftrightarrow$ c) $\Leftrightarrow$ e). Since we can take arbitrary ν we obtain b) $\Leftrightarrow$ d) $\Leftrightarrow$ f).

a) $\Rightarrow$ b) See [17], Lemma 4.

b) $\Rightarrow$ a) Let $\mathscr{B}$ denote the Banach space of all bounded Baire functions defined on ∂G with the usual supremum norm. The symbol $\mathscr{B}'$ stands for the dual space of $\mathscr{B}$. According to [24], Proposition 8, [13] we may define on $\mathscr{B}$ continuous operators V, W by

$$\begin{aligned} Vf(y) &= \mathscr{U}(f\lambda)(y),\\ Wf(y) &= d_G(y)f(y) + \frac{1}{A}\int\limits_{\partial G} \frac{n^G(x)\cdot(y-x)}{|x-y|^m}\,\mathrm{d}\mathscr{H}_{m-1}(x). \end{aligned}$$

According to [24], Proposition 8 the operator τ is the restiction of $(W+V)'$ (i.e. the adjoint operator of $W+V$) onto $\mathscr{C}'(\partial G)$. Since b) $\Rightarrow$ f), there is $\mathscr{U}_{\mathscr{B}}\mu \in \mathscr{B}$ and a polar set K such that $\mathscr{U}\mu = \mathscr{U}_{\mathscr{B}}\mu$ in $\partial G\setminus K$. We show that $\mathscr{U}_{\mathscr{B}}\mu \in (W+V)(\mathscr{B})$. Let $\sigma \in \operatorname{Ker}(W+V)'$. Since $d_G(x) > 0$ for each $x \in \partial G$, there exists a continuous function $\mathscr{U}_c\sigma$ on $\mathbb{R}^m$ coinciding with $\mathscr{U}\sigma$ on $\mathbb{R}^m\setminus\partial G$ (see [16], Theorem 1.11, [17], Lemma 13). According to [19], Lemma 3 the set G has finitely many components $G_1,\dots,G_n$ and $\operatorname{cl} G_j \cap \operatorname{cl} G_k = \emptyset$ for $j \neq k$. According to [18], Lemma 2 and [17], Lemma 11 there are $c_1,\dots,c_n \in \mathbb{R}$ such that $\mathscr{U}_c\sigma = c_j$ on $\operatorname{cl} G_j$ for $j = 1,\dots,n$ and $c_j = 0$ for each j such that $\lambda(\partial G_j) \neq 0$. Since $\mathscr{U}\sigma(x) \to 0$ as $|x| \to \infty$, we have $c_j = 0$ for G_j unbounded. Since μ, σ have a finite energy (see [18], Lemma 2, [24], Proposition 23, [15], Chapter I, Theorem 1.20), σ, μ do not charge polar sets (see [15], Theorem 2.1, p. 222). Therefore

$$\int_{\partial G} \mathscr{U}_{\mathscr{B}}\mu\,\mathrm{d}\sigma = \int_{\partial G} \mathscr{U}\mu\,\mathrm{d}\sigma = \int_{\partial G} \mathscr{U}\sigma\,\mathrm{d}\mu = \int_{\partial G} \mathscr{U}_c\sigma\,\mathrm{d}\mu = \sum_{j=1}^n c_j\mu(\partial G_j).$$

Fix j such that $c_j \neq 0$. Then G_j is bounded. Choose $\varphi \in \mathscr{D}$ such that $\varphi = 1$ on G_j and $\varphi = 0$ on $G\setminus G_j$. Since $\lambda(\partial G_j) = 0$ we have

$$\mu(\partial G_j) = \langle \tau\nu, \varphi\rangle = \int_G \nabla\varphi\cdot\nabla\mathscr{U}\nu\,\mathrm{d}\mathscr{H}_m = 0.$$

Since $r_{\mathrm{ess}}(W'+V'-\frac{1}{2}I) = r_{\mathrm{ess}}(\tau-\frac{1}{2}I) < \frac{1}{2}$ by [16], Lemma 1.5, the operator $W'+V'$ is Fredholm. Since $\langle\sigma, \mathscr{U}_{\mathscr{B}}\mu\rangle = 0$, we conclude that $\mathscr{U}_{\mathscr{B}}\mu \in (W+V)(\mathscr{B})$ by [29], Chapter VII, Theorem 3.1.

Fix $\alpha > V^G + 1 + \sup \mathscr{U}\lambda$. Put

$$\mu_k = \Big(-\frac{\tau-\alpha I}{\alpha}\Big)^k \frac{\mu}{\alpha}.$$

According to [17], Theorem 2 the series

$$\nu_0 = \sum_{k=0}^{\infty} \mu_k$$

converges and $N^G\mathscr{U}\nu_0 + (\mathscr{U}\nu_0)\lambda = \mu$. According to [26], Lemma 4 the measures $\mu_n \in \mathscr{C}_b'(\partial G)$ and $\mathscr{U}_{\mathscr{B}}\mu_k = [-\alpha^{-1}(W+V)+I]^k\alpha^{-1}\mathscr{U}_{\mathscr{B}}\mu$.

Since $\{\beta\in\mathbb{C};\ |\beta-\frac{1}{2}|<\frac{1}{2}\} \subset \{\beta\in\mathbb{C};\ |\beta-\frac{1}{\alpha}|<\alpha\}$, $r_{\mathrm{ess}}(\tau-\alpha I)<\alpha$. Moreover, if $\beta\in\mathbb{C}$ is an eigenvalue of τ, $|\beta-\alpha|\geqslant\alpha$ then $\beta\geqslant 0$ by [17], Lemma 4, Lemma 11. Since $\|\tau\| < \alpha$ by [17], Lemma 2, there is no eigenvalue $\beta\neq 0$ of τ such that $|\alpha-\beta|\geqslant\alpha$. According to [16], Lemma 1.2, Lemma 1.5 we have $r_{\mathrm{ess}}(W+V-\alpha I) = r_{\mathrm{ess}}(W'+V'-\alpha I) = r_{\mathrm{ess}}(\tau-\alpha I) < \alpha$. If β is an eigenvalue of $W+V$ then β is an eigenvalue of τ', because $W+V$ is the restriction of τ' to $\mathscr{B}$. If $|\alpha-\beta|\geqslant\alpha$ then β is an eigenvalue of τ, because $\tau-\beta I$, $\tau'-\beta I$ are Fredholm operators with index zero. Therefore $\beta = 0$. If 0 is not an eigenvalue of $W+V$ then the spectral radius of $W+V-\alpha I$ is smaller than α (i.e. the spectral radius of $\alpha^{-1}(W+V)-I$ is smaller than 1) and there are constants $M\geqslant 1$, $q\in(0,1)$ such that

$$\big\|\big[\alpha^{-1}(W+V)-I\big]^k f\big\|_{\mathscr{B}} \leqslant Mq^k\|f\|_{\mathscr{B}} \tag{4}$$

for each $f\in\mathscr{B}$ and nonnegative integer k. If 0 is an eigenvalue of $W+V$ then there are constants $M\geqslant 1$, $q\in(0,1)$ such that (4) holds for each $f\in(W+V)(\mathscr{B})$ (see [18], Proposition 3). Since $\mathscr{U}_{\mathscr{B}}\mu\in(W+V)(\mathscr{B})$ and $\mathscr{U}_{\mathscr{B}}\mu_k = [-\alpha^{-1}(W+V)+I]^k\alpha^{-1}\mathscr{U}_{\mathscr{B}}\mu$, (4) gives that $\sum\|\mathscr{U}_{\mathscr{B}}\mu_k\|_{\mathscr{B}}<\infty$. Since moreover $\sum\|\mu_k\|<\infty$, [26], Lemma 3 yields that $\nu_0\in\mathscr{C}_b'(\partial G)$. Since $\tau(\nu-\nu_0)=0$, there is a continuous function $\mathscr{U}_c(\nu-\nu_0)$ on $\mathbb{R}^m$ coinciding with $\mathscr{U}(\nu-\nu_0)$ on $\mathbb{R}^m\setminus\partial G$ (see [17], Lemma 4, Lemma 5, Lemma 10). Therefore $\nu\in\mathscr{C}_b'(\partial G)$. $\square$

Lemma 1. *Let G be bounded, $\mu\in\mathscr{C}'(\partial G)$, $u\in W^{1,1}(\mathbb{R}^m)$ be a weak solution of the Neumann problem for the Laplace equation with the boundary condition μ. Then there is the approximate limit of u at $\mathscr{H}_{m-1}$-a.a. points of ∂G. Suppose moreover that*

$$u(x) = \operatorname*{aplim}_{y\to x} u(y)$$

at any point $x \in \partial G$ where the right-hand side is defined. Then $u \in L_1(\mathscr{H})$ and for each $x \in G$

$$u(x) = \mathscr{U}\mu(x) - \mathscr{D}u(x), \tag{5}$$

where

$$\mathscr{D}u = \int_{\partial G} u(y) n^G(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathscr{H}_{m-1}(y)$$

is the double layer potential corresponding to the density u.

Proof. According to [4] there is a set $E \subset \partial G$ with zero functional capacity of degree 1 such that the approximate limit of u exists at each point of $\partial G \setminus E$. Since $\mathscr{H}_{m-1}(E) = 0$ by [5], Theorem 4.3, the approximate limit of u exists at $\mathscr{H}_{m-1}$-a.a. points of ∂G.

Define $u^+(x) = \max(u(x), 0)$, $u^-(x) = \max(-u(x), 0)$. Acoording to [30], Corollary 2.1.8 the functions $u^+, u^- \in W^{1,1}(\mathbb{R}^m)$. Since there is a positive constant M such that $\mathscr{H}(\Omega_r(x)) \leqslant M r^{m-1}$ for each $x \in \mathbb{R}^m$, $r > 0$ (see [12], Corollary 2.17 and [17], Corollary 1), [30], Theorem 5.12.4 yields that $u^+, u^- \in L_1(\mathscr{H})$. Since $u(y) = u^+(y) - u^-(y)$ for $\mathscr{H}$-a.a. y (see [30], Theorem 5.9.6) we have $u \in L_1(\mathscr{H})$.

Fix $x \in G$. Choose a sequence G_j of open sets with C^∞ boundary such that $\operatorname{cl} G_j \subset G_{j+1} \subset G$, $x \in G_1$ and $\bigcup G_j = G$. Fix $r > 0$ such that $\Omega_{2r}(x) \subset G_1$. Choose infinitely differentiable function ψ such that $\psi = 0$ on $\Omega_r(x)$ and $\psi = 1$ on $\mathbb{R}^m \setminus \Omega_{2r}(x)$. According to Green's identity

$$\begin{aligned} u(x) &= \lim_{j\to\infty} \left[\int_{\partial G_j} h_x(y) \frac{\partial u(y)}{\partial n} \, \mathrm{d}\mathscr{H}_{m-1}(y) - \int_{\partial G_j} u(y) n(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathscr{H}_{m-1}(y) \right] \\ &= \lim_{j\to\infty} \left[\int_{G_j} \nabla u(y) \cdot \nabla (h_x(y)\psi(y)) \, \mathrm{d}\mathscr{H}_m(y) \right. \\ &\qquad\qquad \left. - \int_{G_j} \nabla (u(y)\psi(y)) \cdot \nabla h_x(y) \, \mathrm{d}\mathscr{H}_m(y) \right] \\ &= \int_G \nabla u(y) \cdot \nabla (h_x(y)\psi(y)) \, \mathrm{d}\mathscr{H}_m(y) - \int_G \nabla (u(y)\psi(y)) \cdot \nabla h_x(y) \, \mathrm{d}\mathscr{H}_m(y) \\ &= \mathscr{U}\mu(x) - \int_G \nabla (u(y)\psi(y)) \cdot \nabla h_x(y) \, \mathrm{d}\mathscr{H}_m(y). \end{aligned}$$

According to [30], Theorem 2.3.2 there is a sequence of infinitely differentiable functions $u_n \in W^{1,1}(\mathbb{R}^m)$ such that $u_n \to u\psi$ in $W^{1,1}(\mathbb{R}^m)$. According to [12], § 2

$$u(x) = \mathscr{U}\mu(x) - \lim_{n\to\infty} \int_G \nabla u_n(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathscr{H}_m(y) = \mathscr{U}\mu(x) - \lim_{n\to\infty} \mathscr{D}u_n(x).$$

For a Borel set $M \subset \mathbb{R}^m$ put

$$\nu_1(M) = \int_{\partial G \cap M} \max(0, n^G(y) \cdot \nabla h_x(y)) \, \mathrm{d}\mathscr{H}_{m-1}(y),$$
$$\nu_2(M) = \int_{\partial G \cap M} \min(0, n^G(y) \cdot \nabla h_x(y)) \, \mathrm{d}\mathscr{H}_{m-1}(y).$$

According to [30], Theorem 5.12.4 there is a positive constant K such that

$$\left| \int (u\psi - u_n) \, \mathrm{d}\nu_j \right| \leqslant K |u\psi - u_n|_{W^{1,1}(\mathbb{R}^m)},$$

for $j = 1, 2$. Since $u_n \to u\psi$ in $W^{1,1}(\mathbb{R}^m)$, we have

$$\lim_{n\to\infty} \mathscr{D}u_n(x) = \lim_{n\to\infty} \int u_n \, \mathrm{d}\nu_1 + \lim_{n\to\infty} \int u_n \, \mathrm{d}\nu_2 = \int u \, \mathrm{d}\nu_1 + \int u \, \mathrm{d}\nu_2 = \mathscr{D}u(x).$$

$\square$

Lemma 2. *Let G be unbounded, $\mu \in \mathscr{C}'(\partial G)$, $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^m)$ be a weak solution of the Neumann problem for the Laplace equation with the boundary condition μ. Suppose moreover that*

$$u(x) = \operatorname*{aplim}_{y\to x} u(y)$$

at any point $x \in \partial G$ where the right-hand side is defined. Then $u \in L_1(\mathscr{H})$. If $|u(x)| = O(1)$ as $|x| \to \infty$ then there exists

$$u(\infty) = \lim_{|x|\to\infty} u(x),$$

and for each $x \in G$

$$u(x) = u(\infty) + \mathscr{U}\mu(x) - \mathscr{D}u(x). \tag{6}$$

P r o o f. Since $u(y) = o(|y|)$ as $|y| \to \infty$, [20], Lemma 3 yields that there exists

$$u(\infty) = \lim_{|y|\to\infty} u(y).$$

Choose $r > 0$ such that $\partial G \subset \Omega_r(x)$. Put $G_r = G \cap \Omega_r(x)$,

$$\mu_r(M) = \mu(M) + \int_{M\cap\partial G_r} \frac{\partial u}{\partial n} \, \mathrm{d}\mathscr{H}_{m-1}$$

for each Borel set M. Then u is a weak solution of the Neumann problem for the Laplace equation on G_r with the boundary condition μ_r. According to Lemma 1

$$\begin{aligned} u(x) &= \mathscr{U}\mu_r(x) - \int_{\partial G_r} u(y) n(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathscr{H}_{m-1}(y) \\ &= \mathscr{U}\mu(x) - \mathscr{D}u(x) + \frac{1}{A(m-2)} \int_{\partial\Omega_r(x)} \frac{\partial u}{\partial n} r^{2-m} \, \mathrm{d}\mathscr{H}_{m-1} \\ &\quad + \frac{1}{A} \int_{\partial\Omega_r(x)} [u(y) - u(\infty)] r^{1-m} \, \mathrm{d}\mathscr{H}_{m-1} + \frac{1}{A} \int_{\partial\Omega_r(x)} u(\infty) r^{1-m} \, \mathrm{d}\mathscr{H}_{m-1}. \end{aligned}$$

Since $|u(y) - u(\infty)| = o(1)$ as $|y| \to \infty$, [20], Lemma 3 yields that $\partial u(y)/\partial n = O(|y|^{1-m})$. For $r \to \infty$ we get

$$u(x) = \mathscr{U}\mu(x) - \mathscr{D}u(x) + u(\infty).$$

□

Definition. Let $H \subset \mathbb{R}^m$ be an open set, $1 \leqslant p < \infty$. We say that H is $W^{1,p}$-extendible if there is a bounded linear operator $P\colon W^{1,p}(H) \to W^{1,p}(\mathbb{R}^m)$ such that $Pf = f$ on H for each $f \in W^{1,p}(H)$.

Remark that G is $W^{1,1}$-extendible if ∂G is locally a graph of a Lipschitz function. (See [30], Remark 2.5.2.)

Theorem 2. *Let $\mu \in \mathscr{C}_0'(\partial G)$. Then the following assertions are equivalent:*

a) $\mu \in \mathscr{C}_b'(\partial G)$.

b) *There is $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^m)$, bounded in G, which is a weak solution of the third problem for the Laplace equation* (3).

If G is $W^{1,1}$-extendible then these assertions are equivalent to

c) *There is a bounded function on G which is a weak solution of the third problem for the Laplace equation* (3).

P r o o f. a) ⇒ b) According to Theorem 1 there is $\nu \in \mathscr{C}_b'(\partial G)$ such that $\mathscr{U}\nu$ is a solution of (3). But $\mathscr{U}\nu \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^m)$ and bounded on G.

b) ⇒ a) Let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^m)$, bounded in G, be a weak solution of the third problem for the Laplace equation (3). Put $\tilde{\mu} = \mu - u\lambda$. Then u is a weak solution of the Neumann problem for the Laplace equation on G with the boundary condition $\tilde{\mu}$. Fix a constant K such that $|u| \leqslant K$ in G. Put $v(x) = \max(\min(K, u(x)), -K)$ for $x \in \mathbb{R}^m \setminus \partial G$,

$$v(x) = \operatorname*{aplim}_{y \to x} v(y) \quad \text{for } x \in \partial G.$$

Then $v \in W^{1,1}_{\rm loc}(\mathbb{R}^m)$ (see [30], Corollary 2.1.8). According to Lemma 1 and Lemma 2 there is a constant c such that

$$\mathscr{U}\tilde{\mu}(x) = v(x) + \mathscr{D}v(x) + c$$

for each $x \in G$. Since

$$|\mathscr{U}\tilde{\mu}(x)| \leqslant K + Kv^G(x) + |c| \leqslant K + K\Big(V^G + \frac{1}{2}\Big) + |c|$$

for $x \in G$ by [12], Theorem 2.16, we have $\tilde{\mu} \in \mathscr{C}'_b(\partial G)$ by Theorem 1. Since $|u| \leqslant K$ λ-a.e., $u^+\lambda, u^-\lambda \in \mathscr{C}'_b(\partial G)$ by [25], Proposition 6 and $\mu = \tilde{\mu} + u^+\lambda - u^-\lambda \in \mathscr{C}'_b(\partial G)$.

c) $\Rightarrow$ b) Let u be a weak solution of the third problem for the Laplace equation (3), bounded in G. Then $u\varphi \in W^{1,1}(G)$ for each $\varphi \in \mathscr{D}$. Since G is $W^{1,1}$-extendible we can extend u to $\mathbb{R}^m$ so that $u \in W^{1,1}_{\rm loc}(\mathbb{R}^m)$. □

Theorem 3. *Let G be unbounded, $\lambda(\partial H) > 0$ for the unbounded component H of G. Put*

$$\nu_0 = \sum_{n=0}^{\infty} \Big(-\frac{\tau - \alpha I}{\alpha}\Big)^n \frac{\lambda}{\alpha}, \tag{7}$$

where

$$\alpha > \frac{1}{2}\Big(V^G + 1 + \sup_{x \in \partial G} \mathscr{U}\lambda(x)\Big).$$

Then $u = (\mathscr{U}\nu_0 - 1) \in W^{1,1}_{\rm loc}(\mathbb{R}^m)$ is a bounded weak solution of the third problem for the Laplace equation with zero boundary condition, which is nonconstant on H.

P r o o f. According to [17], Theorem 2 the function $\mathscr{U}\nu_0$ is a weak solution of the third problem for the Laplace equation with the boundary condition λ. Since $\lambda \in \mathscr{C}'_b(\partial G)$, the function $\mathscr{U}\nu_0$ is bounded by Theorem 1. Therefore u is a bounded weak solution of the third problem for the Laplace equation with zero boundary condition. Suppose now that u is constant on H. Since $u(x) \to -1$ as $|x| \to \infty$ we have $u = -1$ on H. Since $\operatorname{cl} H \cap \operatorname{cl}(G \setminus H) = \emptyset$ by [19], Lemma 3 we can choose $\varphi \in \mathscr{D}$ such that $\varphi = 0$ on $G \setminus H$ and $\varphi = 1$ on ∂H. Then

$$0 = \int_G \nabla\varphi \cdot \nabla u \,\mathrm{d}\mathscr{H}_m + \int_{\partial G} \varphi u \,\mathrm{d}\lambda = -\lambda(\partial H) < 0,$$

what is a contradiction. □

2. Lipschitz domains

In the rest of the paper we will suppose that ∂G is locally a graph of a Lipschitz function.

Theorem 4. *Denote by $G_1, \dots, G_k$ all components of G. Let $\mu \in \mathscr{C}_0'(\partial G)$. Then there is a bounded weak solution of the Neumann problem for the Laplace equation with the boundary condition μ if and only if $\mu \in \mathscr{C}_b'(\partial G)$. The general form of this solution is*

$$u = \mathscr{U}\nu + \sum_{j=1}^{k} c_j \chi_{G_j}, \tag{8}$$

where

$$\nu = \mu + 2\sum_{j=0}^{\infty} (I - 2N^G\mathscr{U})^j (I - N^G\mathscr{U})\mu, \tag{9}$$

χ_{G_j} are characteristic functions of G_j, and c_j are arbitrary constants.

Proof. According to Theorem 2 there is a bounded function on G which is a weak solution of the Neumann problem for the Laplace equation with the boundary condition μ if and only if $\mu \in \mathscr{C}_b'(\partial G)$.

Suppose now that $\mu \in \mathscr{C}_b'(\partial G)$. According to Theorem 1 and [16], Theorem 1 the function u given by (8) is a bounded weak solution of the Neumann problem for the Laplace equation with the boundary condition μ, which is in $W^{1,1}(\mathbb{R}^m)$. Let v be a bounded weak solution of the Neumann problem for the Laplace equation with the boundary condition μ. Since $v \in W^{1,1}(H)$ for each bounded open subset H of G and G is $W^{1,1}$ extendible, we can suppose that $v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^m)$. The function $w = v - \mathscr{U}\nu$ is a bounded weak solution of the Neumann problem for the Laplace equation with zero boundary condition. Put $\tilde{w} = w$ for G bounded and $\tilde{w} = w - w(\infty)$ for G unbounded (see Lemma 2). According to Lemma 1 and Lemma 2 we have $\tilde{w} = -\mathscr{D}\tilde{w}$ in G. Put

$$W^G f(x) = d_G(x) f(x) + \int_{\partial G} f(y) n^G(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathscr{H}_{m-1}(y),$$
$$W^{\mathbb{R}^m \setminus G} f(x) = d_{\mathbb{R}^m \setminus G}(x) f(x) - \int_{\partial G} f(y) n^G(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathscr{H}_{m-1}(y)$$

for $x \in \partial G$ and $f \in \mathscr{B}$, the space of all bounded Baire functions on ∂G. Since $\tilde{w} = -\mathscr{D}\tilde{w}$ in G we obtain $\tilde{w} = W^{\mathbb{R}^m \setminus G}\tilde{w}$ on ∂G (see [21], Lemma 3) and therefore $W^G\tilde{w} = 0$. Let $G_1, \dots, G_n$ be all bounded components of G. Then $W^G \chi_{\partial G_j} = 0$ for $j = 1, \dots, n$ (see [16], Lemma 1.13). (Here $\chi_{\partial G_j}$ denotes the characteristic function of ∂G_j.) According to [16], Lemma 1.5 the operator W^G is a bounded Fredholm operator with index 0 on $\mathscr{B}$. Since $N^G\mathscr{U}$ is the restriction of the adjoint operator of W^G to $\mathscr{C}'(\partial G)$ (see [24], Proposition 8) and the kernel of the adjoint operator of W^G is a subset of $\mathscr{C}'(\partial G)$ (see [16], Theorem 1.12), the dimension of the kernel of W^G is equal to the dimension of the kernel of $N^G\mathscr{U}$. Since $N^G\mathscr{U}$ is a Fredholm operator with index 0, the dimension of the kernel of W^G is equal to the codimension of the range of $N^G\mathscr{U}$. Since the codimension of the range of $N^G\mathscr{U}$ is equal to n by [16], Theorem 1.14, the functions $\chi_{\partial G_1}, \ldots, \chi_{\partial G_n}$ form a basis of the kernel of W^G. Since $W^G\tilde{w} = 0$ and $\tilde{w} = -\mathscr{D}\tilde{w}$ in G, there are constants $a_1, \ldots, a_n$ such that $\tilde{w} = -a_1\mathscr{D}\chi_{\partial G_1} - \ldots - a_n\mathscr{D}\chi_{\partial G_n}$ in G. Since $\chi_{G_j} = -\mathscr{D}\chi_{\partial G_j}$ for $j = 1, \ldots, n$ by Lemma 1 and Lemma 2, we obtain $\tilde{w} = a_1\chi_{G_1} + \ldots a_n\chi_{G_n}$ in G.

□

Theorem 5. *Denote by $G_1, \ldots, G_k$ all components of G such that $\lambda(\partial G_j) = 0$. Let $\mu \in \mathscr{C}_0'(\partial G)$. Then there is a bounded weak solution of the third problem for the Laplace equation* (3) *if and only if $\mu \in \mathscr{C}_b'(\partial G)$.*

a) *If $G \setminus (G_1 \cup \ldots \cup G_k)$ is bounded then the general form of this solution is*

$$u = \mathscr{U}\nu + \sum_{j=1}^{k} c_j\chi_{G_j}, \tag{10}$$

where

$$\nu = \sum_{n=0}^{\infty}\Big(-\frac{\tau - \alpha I}{\alpha}\Big)^n \frac{\mu}{\alpha}, \tag{11}$$

$$\alpha > \frac{1}{2}\Big(V^G + 1 + \sup_{x\in\partial G} \mathscr{U}\lambda(x)\Big), \tag{12}$$

and c_j are arbitrary constants.

b) *If $G \setminus (G_1 \cup \ldots \cup G_k)$ is unbounded then the general form of this solution is*

$$u = \mathscr{U}\nu + \sum_{j=1}^{k} c_j\chi_{G_j} + c_{k+1}(\mathscr{U}\nu_0 - 1), \tag{13}$$

where ν is given by (11), *ν_0 is given by* (7) *and c_j are arbitrary constants;* (10) *is a general form of a bounded weak solution v of the third problem for the Laplace equation with the boundary condition μ for which $v(x) \to 0$ as $|x| \to \infty$.*

P r o o f. Since G is $W^{1,1}$-extendible by [30], Remark 2.5.2, there is a bounded function on G which is a weak solution of the third problem for the Laplace equation (3) if and only if $\mu \in \mathscr{C}_b'(\partial G)$. (See Theorem 2.)

Suppose now that $\mu \in \mathscr{C}_b'(\partial G)$. According to Theorem 1, Theorem 3 and [17], Theorem 2 the function u given by (10) or (13) is a bounded weak solution of the third problem for the Laplace equation with the boundary condition μ. If $G\setminus(G_1\cup\ldots\cup G_k)$ is unbounded and u is given by (10) then $u(x)\to 0$ as $|x|\to\infty$.

Let v be a bounded weak solution of the third problem for the Laplace equation with the boundary condition μ. Then $w=v-\mathscr{U}\nu$ is a bounded weak solution of the third problem for the Laplace equation with zero boundary condition. Then w is a bounded weak solution of the Neumann problem for the Laplace equation with the boundary condition $-w\lambda$. Let $G_1,\ldots,G_n$ be all components of G. According to Theorem 4 there are $\tilde\nu\in\mathscr{C}'(\partial G)$ and constants $c_1,\ldots,c_n$ such that $w=\mathscr{U}\tilde\nu+c_1\chi_{\partial G_1}+\ldots+c_n\chi_{\partial G_n}$. Let f be the characteristic function of the unbounded component of G for G unbounded; $f\equiv 0$ for G bounded. Since for each bounded component H of G there is $\nu_H\in\mathscr{C}'(\partial G)$ such that $\mathscr{U}\nu_H=1$ on H and $\mathscr{U}\nu_H=0$ on $G\setminus H$ (see [20], Lemma 1), there are $\nu'\in\mathscr{C}'(\partial G)$ and a constant a such that $w=\mathscr{U}\nu'+af$. If $G\setminus(G_1\cup\ldots\cup G_k)$ is bounded then $\mathscr{U}\nu'=w-af$ is a weak solution of the third problem for the Laplace equation with zero boundary condition. Then $\mathscr{U}\nu'=a_1\chi_{\partial G_1}+\ldots+a_k\chi_{\partial G_k}$ for some constants $a_1,\ldots,a_k$ by [16], Theorem 1.12. Suppose now that $G\setminus(G_1\cup\ldots\cup G_k)$ is unbounded. Theorem 3 yields that $\tilde w=w+a(\mathscr{U}\nu_0-1)$ is a bounded weak solution of the third boundary problem with zero boundary condition and $\tilde w(x)\to 0$ as $|x|\to\infty$. As was shown there are $\nu''\in\mathscr{C}'(\partial G)$ and a constant b such that $\tilde w=\mathscr{U}\nu''+bf$. Since $\tilde w(x)\to 0$ as $|x|\to\infty$ we obtain $b=0$. Therefore $\mathscr{U}\nu''=a_1\chi_{\partial G_1}+\ldots+a_k\chi_{\partial G_k}$ for some constants $a_1,\ldots,a_k$ by [16], Theorem 1.12. □

Lemma 3. *Let u be a bounded weak solution of the third problem for the Laplace equation with the boundary condition $\mu\in\mathscr{C}'(\partial G)$. Then $|\nabla u|\in L_2(G)$. If G is bounded then $u\in W^{1,2}(G)$. If G is unbounded and $m>4$ then $u\in W^{1,2}(G)$ if and only if $u(x)\to 0$ as $|x|\to\infty$. Let now $m\leqslant 4$ and H be an unbounded component of G. Denote by $\tilde\lambda$ the restriction of λ to ∂G. If $\mathscr{U}\tilde\lambda$ is constant on ∂H (for example if $\tilde\lambda=0$) then $u\in W^{1,2}(G)$ if and only if $u(x)\to 0$ as $|x|\to\infty$ and $\mu(\partial H)=0$.*

Proof. According to Theorem 5 the function u has the form (10) or (13). Since $\nu,\nu_0\in\mathscr{C}_b'(\partial G)$ by Theorem 1 and Theorem 3, $|\nabla\mathscr{U}\nu|,|\nabla\mathscr{U}\nu_0|\in L_2(\mathbb{R}^m)$ by [26], Proposition 23. Therefore $|\nabla u|\in L_2(G)$. If G is bounded then $u\in W^{1,2}(G)$, because u is bounded. If G is unbounded and $m>4$ then $u\in L_2(G)$ if and only if $u(x)\to 0$ as $|x|\to\infty$ by [20], Lemma 3. Suppose now that H is an unbounded component of G, $m\leqslant 4$ and $\mathscr{U}\tilde\lambda$ is equal to a constant c on ∂H. If $u\in W^{1,2}(G)$ then $u(x)\to 0$ as $|x|\to\infty$ by [20], Lemma 3. Suppose now that $u(x)\to 0$ as $|x|\to\infty$. Denote by $\tilde\mu$ the restriction of μ to ∂H. Then $N^Hu+u\tilde\lambda=\tilde\mu$. Since $V^H<\infty$, $r_{\mathrm{ess}}(N^H\mathscr{U}-\frac12 I)<\frac12$

(see [16], Theorem 2.3), Theorem 5 yields that $u = \mathscr{U}\tilde{\nu}$ on H, where

$$\tilde{\nu} = \sum_{n=0}^{\infty} \Big(-\frac{\tau^H - \alpha I}{\alpha}\Big)^n \frac{\tilde{\mu}}{\alpha}.$$

$u \in W^{1,2}(H)$ if and only if $\tilde{\nu}(\mathbb{R}^m) = 0$, because $\mathscr{U}\tilde{\nu}(x) = \tilde{\nu}(\mathbb{R}^m)|x|^{2-m} + O(|x|^{1-m})$ for $|x| \to \infty$. If $\tilde{\nu}(\partial H) = 0$ then Fubini's theorem and [18], Lemma 9 yield $\mu(\partial H) = \tilde{\mu}(\partial H) = \tau^H\tilde{\nu}(\partial H) = N^H\mathscr{U}\tilde{\nu}(\partial H) + \int \mathscr{U}\tilde{\nu}\,\mathrm{d}\tilde{\lambda} = 0 + \int \mathscr{U}\tilde{\lambda}\,\mathrm{d}\tilde{\nu} = c\tilde{\nu}(\partial H) = 0$. On the other hand, if $\mu(\partial H) = 0$ we get by induction $(I - \alpha^{-1}\tau^H)^n\tilde{\mu}(\partial H) = 0$ and therefore $\tilde{\nu}(\partial H) = \alpha^{-1}\sum(I - \alpha^{-1}\tau^H)^n\tilde{\mu}(\partial H) = 0$. □

Example 1. Let $G = \mathbb{R}^3 \setminus \operatorname{cl}\Omega_1([2,0,0]) \setminus \operatorname{cl}\Omega_1([-2,0,0])$. For fixed constants $c \in (1/2, 1\rangle$, $a \in (0,\infty)$ put $u(x) = 1/|x - [2,0,0]| - c/|x - [-2,0,0]|$,

$$\lambda(M) = \int_{\partial\Omega_1([-2,0,0])\cap M} a/|u|\,\mathrm{d}\mathscr{H}_2,$$
$$\mu(M) = \int_{\partial G\cap M} \frac{\partial u}{\partial n}\,\mathrm{d}\mathscr{H}_2 - a\mathscr{H}_2(M \cap \partial\Omega_1([-2,0,0]))$$

for any Borel set M. Then u is a weak bounded solution of the third problem for the Laplace equation with the boundary condition μ. If $c < 1$ and $a = 1 - c$ then $u \notin W^{1,2}(G)$ but $\mu(\partial G) = \mathscr{H}_2(\Omega_1(0))[1 - c - (1-c)] = 0$. If $c = 1$ then $u \in W^{1,2}(G)$ but $\mu(\partial G) = -a\mathscr{H}_2(\Omega_1(0)) \neq 0$.

Definition. Let $f \in L_\infty(\mathscr{H})$ be a nonnegative function. Let L be a bounded linear functional on $W^{1,2}(G)$ such that $L(\varphi) = 0$ for each $\varphi \in \mathscr{D}(G) = \{\varphi \in \mathscr{D};\ \operatorname{spt}\varphi \subset G\}$. We say that $u \in W^{1,2}(G)$ is a weak solution in $W^{1,2}(G)$ of the third problem

$$\begin{aligned} \Delta u &= 0 \quad \text{on } G, \\ \frac{\partial u}{\partial n} + uf &= L \quad \text{on } \partial G, \end{aligned} \tag{14}$$

if

$$\int_G \nabla u \cdot \nabla v\,\mathrm{d}\mathscr{H}_m + \int_{\partial G} ufv\,\mathrm{d}\mathscr{H} = L(v)$$

for each $v \in W^{1,2}(G)$.

Remark 3. Let u be a weak solution in $W^{1,2}(G)$ of (14). If there is $\mu \in \mathscr{C}'(G)$ such that $L(\varphi) = \int \varphi\,\mathrm{d}\mu$ for each $\varphi \in \mathscr{D}$ then u is a weak solution of (3) with $\lambda = f\mathscr{H}$.

Lemma 4. *Let $\mu \in \mathscr{C}_b'(\partial G)$. Then there is a unique bounded linear functional L_μ on $W^{1,2}(G)$ such that*

$$L_\mu(\varphi) = \int_{\partial G} \varphi \, \mathrm{d}\mu$$

for each $\varphi \in \mathscr{D}$.

P r o o f. Let $G_1, \ldots, G_n$ are all components of G. Fix real numbers $c_1, \ldots, c_n$ such that $\mu(\partial G_j) - c_j \mathscr{H}(\partial G_j) = 0$ for $j = 1, \ldots, n$. Put

$$\tilde{\mu}(M) = \mu - \sum_{j=1}^{n} c_j \mathscr{H}(M \cap \partial G_j)$$

for each Borel set M. Since $\tilde{\mu} \in \mathscr{C}_b'(\partial G)$ by [17], Remark 6, there is $\nu \in \mathscr{C}_b'(\partial G)$ such that $N^G \mathscr{U} \nu = \tilde{\mu}$ by Theorem 5 and Theorem 1. Fix $\psi \in \mathscr{D}$ such that $\psi = 1$ in a neighbourhood of ∂G. If $\varphi \in \mathscr{D}$ then Hölder's inequality yields

$$\begin{aligned}
\int_{\partial G} \varphi \, \mathrm{d}\tilde{\mu} &= \int_{\partial G} \psi\varphi \, \mathrm{d}N^G \mathscr{U} \nu = \int_G \nabla(\psi\varphi) \cdot \nabla \mathscr{U} \nu \, \mathrm{d}\mathscr{H}_m \\
&\leqslant \sup|\psi| \left(\int_{G \cap \operatorname{spt} \psi} |\nabla\varphi|^2 \, \mathrm{d}\mathscr{H}_m \right)^{1/2} \left(\int_{G \cap \operatorname{spt} \psi} |\nabla \mathscr{U} \nu|^2 \, \mathrm{d}\mathscr{H}_m \right)^{1/2} \\
&\quad + \sup|\nabla\psi| \left(\int_{G \cap \operatorname{spt} \psi} |\varphi|^2 \, \mathrm{d}\mathscr{H}_m \right)^{1/2} \left(\int_{G \cap \operatorname{spt} \psi} |\nabla \mathscr{U} \nu|^2 \, \mathrm{d}\mathscr{H}_m \right)^{1/2} \\
&\leqslant C \|\varphi\|_{W^{1,2}(G)},
\end{aligned}$$

where

$$C = 2(\sup|\psi| + \sup|\nabla\psi|) \left(\int_{G \cap \operatorname{spt} \psi} |\nabla \mathscr{U} \nu|^2 \, \mathrm{d}\mathscr{H}_m \right)^{1/2} < \infty$$

by Lemma 3. According to the Hahn-Banach theorem there is a bounded linear functional $L_{\tilde{\mu}}$ on $W^{1,2}(G)$ such that

$$L_{\tilde{\mu}}(\varphi) = \int_{\partial G} \varphi \, \mathrm{d}\tilde{\mu}$$

for each $\varphi \in \mathscr{D}$. If we define

$$L_\mu(v) = L_{\tilde{\mu}}(v) + \sum_{j=1}^{n} c_j \int_{G_j} v \, \mathrm{d}\mathscr{H}$$

for $v \in W^{1,2}(G)$, then L_μ is a bounded linear operator on $W^{1,2}(G)$ satisfying $L_\mu(\varphi) = \int \varphi \, \mathrm{d}\mu$ for each $\varphi \in \mathscr{D}$. Since $\mathscr{D}$ is dense in $W^{1,2}(G)$ by [30], Remark 2.5.2 and [30], Lemma 2.1.3, the functional L_μ is unique. $\square$

Lemma 5. *Let $f \in L_\infty(\mathscr{H})$ be a nonnegative function, $\lambda = f\mathscr{H}$. Let $\mu \in \mathscr{C}_0'(\partial G)$. If $u, v \in W^{1,2}(G)$ are weak solutions of* (3) *then $w \equiv u - v$ is locally constant in G and $w = 0$ on the unbounded component of G and on each component H of G for which $\lambda(\partial H) > 0$.*

Proof. Fix a sequence $\varphi_n \in \mathscr{D}$ such that $\varphi_n \to w$ in $W^{1,2}(G)$ (see [30], Remark 2.5.2 and [30], Lemma 2.1.3). Then

$$0 = \lim_{n\to\infty} \left[\int_G \nabla w \cdot \nabla \varphi_n \,\mathrm{d}\mathscr{H}_m + \int_{\partial G} w f \varphi_n \,\mathrm{d}\mathscr{H} \right] = \int_G |\nabla w|^2 \,\mathrm{d}\mathscr{H}_m + \int_{\partial G} w^2 f \,\mathrm{d}\mathscr{H}.$$

Since $\int |\nabla w|^2 \,\mathrm{d}\mathscr{H}_m \geqslant 0$, $\int f w^2 \,\mathrm{d}\mathscr{H} \geqslant 0$, we have $\int |\nabla w|^2 \,\mathrm{d}\mathscr{H}_m = 0$ and therefore w is locally constant on G. Since $\int f w^2 \,\mathrm{d}\mathscr{H} = 0$ we obtain that $w = 0$ on each component H of G for which $\lambda(\partial H) > 0$. Since $w \in W^{1,2}(G)$ and w is constant on the unbounded component of G, $w = 0$ on this component. $\square$

Theorem 6. *Let $f \in L_\infty(\mathscr{H})$ be a nonnegative function, $\lambda = f\mathscr{H}$. Let $\mu \in \mathscr{C}_0'(\partial G) \cap \mathscr{C}_b'(\partial G)$, and let L be a bounded linear functional on $W^{1,2}(G)$ such that $L(\varphi) = \int \varphi \,\mathrm{d}\mu$ for each $\varphi \in \mathscr{D}$. If G is unbounded and $m \leqslant 4$ suppose moreover that $\mu(\partial H) = 0$ and $f = 0$ on ∂H, where H is the unbounded component of G. Then there is a bounded weak solution u in $W^{1,2}(G)$ of the third problem for the Laplace equation* (14). *If $G_1, \dots, G_k$ are all components of G such that $\lambda(\partial G_j) = 0$, then the general solution of this problem has the form* (10), *where ν is given by* (11) *and $c_j = 0$ for G_j unbounded and c_j is an arbitrary constant for G_j bounded.*

Proof. Let ν be given by (11). Then $\mathscr{U}\nu$ is a bounded weak solution of (3) by Theorem 5. According to Lemma 3 we have $\mathscr{U}\nu \in W^{1,2}(G)$. For fixed $v \in W^{1,2}(G)$ choose $\varphi_n \in \mathscr{D}$ such that $\varphi_n \to v$ in $W^{1,2}(G)$ as $n \to \infty$ (see [30], Remark 2.5.2 and [30], Lemma 2.1.3). Then

$$\begin{aligned} L(v) &= \lim_{n\to\infty} \int \varphi_n \,\mathrm{d}\mu = \lim_{n\to\infty} \left[\int_G \nabla \varphi_n \cdot \nabla \mathscr{U}\nu \,\mathrm{d}\mathscr{H}_m + \int_{\partial G} \varphi_n f \mathscr{U}\nu \,\mathrm{d}\mathscr{H} \right] \\ &= \int_G \nabla v \cdot \nabla \mathscr{U}\nu \,\mathrm{d}\mathscr{H}_m + \int_{\partial G} v f \mathscr{U}\nu \,\mathrm{d}\mathscr{H}. \end{aligned}$$

$\mathscr{U}\nu$ is a weak solution in $W^{1,2}(G)$ of the third problem (14). If u has the form (10), where $c_j = 0$ for G_j unbounded, then u is a weak solution of this third problem.

Let $u \in W^{1,2}(G)$ be a weak solution in $W^{1,2}(G)$ of the third problem (14). Lemma 5 yields that u has the form (10) with $c_j = 0$ for G_j unbounded. $\square$

Theorem 7. *Let $f \in L_\infty(\mathscr{H})$ be a nonnegative function. Let L be a bounded linear functional on $W^{1,2}(G)$ and $\mu \in \mathscr{C}'(\partial G)$ be such that $L(\varphi) = \int \varphi \,\mathrm{d}\mu$ for each $\varphi \in \mathscr{D}$. If $u \in W^{1,2}(G)$ is a weak solution in $W^{1,2}(G)$ of the third problem for the Laplace equation* (14) *then u is bounded in G if and only if $\mu \in \mathscr{C}'_b(\partial G)$.*

Proof. Put $\lambda = f\mathscr{H}$. Since $N^G u + u\lambda = \mu$, [17], Theorem 1 yields that $\mu \in \mathscr{C}'_0(\partial G)$. If the function u is bounded then $\mu \in \mathscr{C}'_b(\partial G)$ by Theorem 2, because G is $W^{1,1}$-extendible by [30], Remark 2.5.2. Suppose now that $\mu \in \mathscr{C}'_b(\partial G)$. If G is bounded put $\tilde G = G$. If G is unbounded fix $R > 0$ such that $\partial G \subset \Omega_R(0)$ and put $\tilde G = G \cap \Omega_R(0)$, $\tilde\mu = \mu + \partial u/\partial n(\mathscr{H}_{m-1}/\partial\Omega_R(0))$, $f = 0$ on $\partial\Omega_R(0)$. Since $V^G < \infty$ we have $V^{\tilde G} < \infty$. Since $r_{\mathrm{ess}}(N^G\mathscr{U} - \frac12 I) < \frac12$ and $(N^H\mathscr{U} - \frac12 I)$ is compact for each bounded open set H with a smooth boundary (see [12], Theorem 4.1, Proposition 2.20, [29], Theorem 4.1), [16], Theorem 2.3 yields that $r_{\mathrm{ess}}(N^{\tilde G}\mathscr{U} - \frac12 I) < \frac12$. Since $N^{\tilde G}u + u\lambda = \tilde\mu$, [17], Theorem 1 yields that $\tilde\mu \in \mathscr{C}'_0(\partial G)$. If G is unbounded then $\partial u/\partial n(\mathscr{H}_{m-1}/\partial\Omega_R(0)) \in \mathscr{C}'_b(\partial\tilde G)$ by [17], Remark 6 and therefore $\tilde\mu \in \mathscr{C}'_b(\partial\tilde G)$. According to Theorem 6 there is a bounded $v \in W^{1,2}(G)$ which is a weak solution in $W^{1,2}(G)$ of the third problem for the Laplace equation on $\tilde G$ with the boundary condition $L_{\tilde\mu}$

$$\begin{aligned} \Delta v &= 0 \quad \text{in } \tilde G,\\ \frac{\partial v}{\partial n} + fv &= L_{\tilde\mu} \quad \text{on } \partial\tilde G. \end{aligned}$$

Since $u - v$ is locally constant in $\tilde G$ by Lemma 5, the function u is bounded in $\tilde G$. Since $u \in W^{1,2}(G)$, $u(x) \to 0$ as $|x| \to \infty$ (see [20], Lemma 3). Therefore u is bounded in G. $\square$

Definition. Let $f \in L_\infty(\mathscr{H})$ be a nonnegative function. Let $g \in L_2(G)$ and let L be a bounded linear functional on $W^{1,2}(G)$ such that $L(\varphi) = 0$ for each $\varphi \in \mathscr{D}(G)$. We say that $u \in W^{1,2}(G)$ is a weak solution in $W^{1,2}(G)$ of the third problem for the Poisson equation

$$\begin{aligned} \Delta u &= g \quad \text{on } G,\\ \frac{\partial u}{\partial n} + uf &= L \quad \text{on } \partial G, \end{aligned} \tag{15}$$

if

$$\int_G \nabla u \cdot \nabla v \,\mathrm{d}\mathscr{H}_m + \int_{\partial G} ufv \,\mathrm{d}\mathscr{H} = L(v) - \int_G gv \,\mathrm{d}\mathscr{H}_m$$

for each $v \in W^{1,2}(G)$.

Theorem 8. *Let $f \in L_\infty(\mathscr{H})$ be a nonnegative function. Let $g \in L_p(\mathbb{R}^m)$, where $p > m$, be a compactly supported function. Put $\lambda = f\mathscr{H}$. Denote by $G_1, \ldots, G_k$ all bounded components of G such that $\lambda(\partial G_j) = 0$. Let $\mu \in \mathscr{C}_b'(\partial G)$ be such that*

$$\mu(\partial G_j) = \int_{G_j} g \,\mathrm{d}\mathscr{H}_m$$

for $j = 1, \ldots, k$. If G is unbounded and $m \leqslant 4$ suppose moreover that

$$\int_{\mathbb{R}^m} g \,\mathrm{d}\mathscr{H}_m = 0,$$
$$\mu(\partial H) = \int_H g \,\mathrm{d}\mathscr{H}_m,$$

$\lambda(\partial H) = 0$ for the unbounded component H of G. Then there is $u \in W^{1,2}(G)$ which is a weak solution in $W^{1,2}(G)$ of the third problem for the Poisson equation (15) with the boundary condition $L \equiv L_\mu$. The general form of this solution is

$$u = \mathscr{U}\nu - \mathscr{U}(g\mathscr{H}_m) + \sum_{j=1}^{k} c_j \chi_{G_j}, \tag{16}$$

where

$$\nu = \sum_{n=0}^{\infty} \Bigl(-\frac{\tau - \alpha I}{\alpha}\Bigr)^n \frac{\tilde{\mu}}{\alpha}, \tag{17}$$

$$\tilde{\mu} = \mu + [n^G \cdot \nabla \mathscr{U}(g\mathscr{H}_m)]\mathscr{H} + \mathscr{U}(g\mathscr{H}_m)\lambda, \tag{18}$$
$$\alpha > \frac{1}{2}\Bigl(V^G + 1 + \sup_{x \in \partial G} \mathscr{U}\lambda(x)\Bigr).$$

Proof. Put

$$\varphi(x) = \begin{cases} C \exp[-1/(1 - |x|^2)] & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geqslant 1, \end{cases}$$

where C is chosen so that $\int \varphi = 1$. For $\varepsilon > 0$ put $\varphi_\varepsilon(x) = \varepsilon^{-m}\varphi(x\varepsilon)$. Since $\mathscr{U}(g\mathscr{H}_m) \in \mathscr{C}^1(\mathbb{R}^m)$ (see [6], Theorem A.6, Theorem A.11), $\varphi_\varepsilon * \mathscr{U}(g\mathscr{H}_m) \to \mathscr{U}(g\mathscr{H}_m)$, $\varphi_\varepsilon * \nabla \mathscr{U}(g\mathscr{H}_m) \to \nabla \mathscr{U}(g\mathscr{H}_m)$ locally uniformly as $\varepsilon \searrow 0$ (see [30], Theorem 1.6.1, [27], § 12). The Divergence Theorem (see [12], p. 49) and [6], Theorem A.16 yield for $j \in \{1, \ldots, k\}$

$$\begin{aligned}
\tilde{\mu}(\partial G_j) &= \mu(\partial G_j) + \int_{\partial G_j} n^G(y) \cdot \nabla \mathscr{U}(g\mathscr{H}_m)(y) \, \mathrm{d}\mathscr{H}(y) \\
&= \mu(\partial G_j) + \lim_{\varepsilon \to 0_+} \int_{\partial G_j} n^G(y) \cdot (\varphi_\varepsilon * \nabla \mathscr{U}(g\mathscr{H}_m))(y) \, \mathrm{d}\mathscr{H}(y) \\
&= \mu(\partial G_j) + \lim_{\varepsilon \to 0_+} \int_{\partial G_j} n^G(y) \cdot \nabla[\varphi_\varepsilon * (h_0 * g)](y) \, \mathrm{d}\mathscr{H}(y) \\
&= \mu(\partial G_j) + \lim_{\varepsilon \to 0_+} \int_{\partial G_j} n^G(y) \cdot \nabla[h_0 * (\varphi_\varepsilon * g)](y) \, \mathrm{d}\mathscr{H}(y) \\
&= \mu(\partial G_j) + \lim_{\varepsilon \to 0_+} \int_{G_j} \Delta \mathscr{U}[(\varphi_\varepsilon * g)\mathscr{H}_m] \, \mathrm{d}\mathscr{H}_m \\
&= \mu(\partial G_j) - \lim_{\varepsilon \to 0_+} \int_{G_j} (\varphi_\varepsilon * g) \, \mathrm{d}\mathscr{H}_m = \mu(\partial G_j) - \int_{G_j} g \, \mathrm{d}\mathscr{H}_m = 0.
\end{aligned}$$

If G is unbounded and $m \leqslant 4$ then [6], Theorem A.16 and the Divergence Theorem (see [12], p. 49) yield

$$\begin{aligned}
\tilde{\mu}(\partial H) = \; & \lim_{R \to \infty} \Big\{ \lim_{\varepsilon \to 0_+} \int_{\partial(H \cap \Omega_R(0))} n^{H \cap \Omega_R(0)} \cdot [\varphi_\varepsilon * \nabla \mathscr{U}(g\mathscr{H}_m)] \, \mathrm{d}\mathscr{H}_{m-1} \\
& - \int_{\partial \Omega_R(0)} n^{\Omega_R(0)}(y) \cdot \nabla \mathscr{U}(g\mathscr{H}_m)(y) \, \mathrm{d}\mathscr{H}_{m-1}(y) \Big\} + \mu(\partial H) \\
= \; & \lim_{R \to \infty} \lim_{\varepsilon \to 0_+} \int_{\partial(H \cap \Omega_R(0))} n^{H \cap \Omega_R(0)} \cdot \nabla[h_0 * (\varphi_\varepsilon * g)] \, \mathrm{d}\mathscr{H}_{m-1} + \mu(\partial H) \\
= \; & \lim_{R \to \infty} \lim_{\varepsilon \to 0_+} \int_{H \cap \Omega_R(0)} \Delta \mathscr{U}[(\varphi_\varepsilon * g)\mathscr{H}_m] \, \mathrm{d}\mathscr{H}_m + \mu(\partial H) \\
= \; & - \lim_{R \to \infty} \lim_{\varepsilon \to 0_+} \int_{H \cap \Omega_R(0)} (\varphi_\varepsilon * g) \, \mathrm{d}\mathscr{H}_m + \mu(\partial H) \\
= \; & - \int_H g \, \mathrm{d}\mathscr{H}_m + \mu(\partial H) = 0.
\end{aligned}$$

According to Theorem 6,

$$\mathscr{U}\nu + \sum_{j=1}^{k} c_j \chi_{G_j}$$

is a weak solution in $W^{1,2}(G)$ of the third problem for the Laplace equation (14) with the boundary condition $L \equiv L_{\tilde{\mu}}$. If u has the form (16) then [20], Lemma 5 yields that u is a weak solution in $W^{1,2}(G)$ of the third problem for the Poisson equation (15) with the boundary condition $L \equiv L_\mu$.

Let now $u \in W^{1,2}(G)$ be a weak solution of the third problem for the Poisson equation (15) with the boundary condition $L \equiv L_\mu$. Then

$$w = u - \mathscr{U}\nu + \mathscr{U}(g\mathscr{H}_m)$$

is a weak solution in $W^{1,2}(G)$ of the third problem for the Laplace equation with the zero boundary condition. According to Lemma 5 the function w is locally constant and vanishes on $G \setminus (G_1 \cup \ldots \cup G_k)$. □

Theorem 9. *Let $f \in L_\infty(\mathscr{H})$ be a nonnegative function. Let $g \in L_p(\mathbb{R}^m)$, where $p > m$, be a compactly supported function. Let L be a bounded linear functional on $W^{1,2}(G)$ and $\mu \in \mathscr{C}'(\partial G)$ be such that $L(\varphi) = \int \varphi\,\mathrm{d}\mu$ for each $\varphi \in \mathscr{D}$. If $u \in W^{1,2}(G)$ is a weak solution in $W^{1,2}(G)$ of the third problem for the Poisson equation* (15) *then u is bounded in G if and only if $\mu \in \mathscr{C}_b'(\partial G)$.*

P r o o f. Changing g on $\mathbb{R}^m \setminus G$ we can suppose that

$$\int_{\mathbb{R}^m} g\,\mathrm{d}\mathscr{H}_m = 0.$$

Put $\lambda = f\mathscr{H}$, $\varrho \equiv -[n^G \cdot \nabla\mathscr{U}(g\mathscr{H}_m)]\mathscr{H} - \mathscr{U}(g\mathscr{H}_m)\lambda$. Then [20], Lemma 5 yields that $u + \mathscr{U}(g\mathscr{H}_m)$ is a weak solution in $W^{1,2}(G)$ of the Neumann problem for the Laplace equation with the boundary condition $L - L_\varrho$. Since $\mathscr{U}(g\mathscr{H}_m) \in C^1(\mathbb{R}^m)$ (see [6], Theorem A.6 and Theorem A.11) and $\mathscr{U}(g\mathscr{H}_m)(x) \to 0$ as $|x| \to \infty$, the function $\mathscr{U}(g\mathscr{H}_m)$ is bounded. Therefore u is bounded if and only if $u + \mathscr{U}(g\mathscr{H}_m)$ is bounded. According to Theorem 7 the function $u + \mathscr{U}(g\mathscr{H}_m)$ is bounded if and only if $\mu - \varrho \in \mathscr{C}_b'(\partial G)$. Since $\varrho \in \mathscr{C}_b'(\partial G)$ by [20], Lemma 5, the function u is bounded in G if and only if $\mu \in \mathscr{C}_b'(\partial G)$. □

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