A POLYNOMIAL OF DEGREE FOUR NOT SATISFYING ROLLE'S THEOREM IN THE UNIT BALL OF l_2

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Abstract. We give an example of a fourth degree polynomial which does not satisfy Rolle's Theorem in the unit ball of l_2.

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1. Introduction

S. A. Shkarin gave in [2] an example of a fourth degree continuous polynomial which does not satisfy Rolle's Theorem in the unit ball of $L_2[0,1]$. This polynomial was given by the function $P(x) = (1-\|x\|^2)Q(x)$, where $Q(x) = \langle Ax, x\rangle + 2\langle\varphi, x\rangle + k$, with A being the positive operator given by $Ax(t) = t\,x(t)$, $x \in L_2[0,1]$, $\varphi(t) = t(1-t)$, $t \in [0,1]$, and $k = 4/27$. Clearly, for $\|x\| = 1$, $P(x) = 0$ and Shkarin showed that, for $\|x\| < 1$, the Fréchet derivative $P'(x) = 2[(1-\|x\|^2)(Ax+\varphi) - Q(x)x] \neq 0$.

Since Rolle's Theorem is an isometric invariant, it is clear that there exist continuous polynomials of degree four in l_2 for which the result fails. Now, the task of constructing explicitly one of such polynomials has turned out to be a not so easy one. This note is devoted to giving one of such constructive counterexamples.

The polynomial that we give in the following is easily seen to be in the class of Shkarin polynomials which we introduced in [1]. Indeed, what we do here is to guarantee, by means of convenient restrictions, that the inequality given in Theorem 1 of [1] is fulfilled so that the polynomial will not satisfy Rolle's Theorem.

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2. The polynomial

In order to obtain an appropriate positive multiplication operator A, we consider the set $S = \bigcup_{n=1}^{\infty} S_n$, where, for each n, the set S_n is formed by all rationals in $]0,1[$ with exactly n significant decimals, i.e., $S_n = \{0.d_1d_2\dots d_n\colon\ d_i \in \{0,1,2,\dots,9\},\ 1 \leqslant i \leqslant n,\ d_n \neq 0\}$. A well-ordering in S can be defined by setting that, for each n, the elements of S_n are prior to those of S_{n+1}, and inside each S_n the order considered is the usual one. Hence, we can represent the set S by means of the sequence (r_n) following the order just defined. Recall that, if $r_p = 0.d_1d_2\dots d_n$, then $p < 10^n$. Now, we define the operator A as, if $x = (x_n) \in l_2$, $Ax = (r_nx_n)$. Notice that A is bounded and $\|A\| = 1$.

Following Shkarin's construction, we proceed to find an appropriate vector $\varphi = (\varphi_n) \in l_2$. For this purpose, let q be a positive number such that

$$\sigma := \sum_{n=1}^{\infty} \frac{1}{(n+q)^{4/3}} < \frac{1}{4}.$$

For each n, let $a_n := (n+q)^{-2/3}$, and let $\varphi_n := a_nr_n(1-r_n)$. It follows that $\varphi = (\varphi_n) \in l_2$ and $\|\varphi\|^2 < \sigma$.

Finally, let $k \in (\sigma, 1-3\sigma]$. We show that, if $Q(x) = \langle Ax, x\rangle + 2\langle \varphi, x\rangle + k$, $x \in l_2$, the polynomial $P(x) = (1-\|x\|^2)Q(x)$ has non-zero derivative when $\|x\| < 1$. Notice first that $Q(x) > 0$, $x \in l_2$, since, considering the sequence $\psi = (\psi_n) := ((r_n-1)a_n)$, we have that $A\psi = -\varphi$ and so, since A is positive,

$$Q(x) = \langle A(x-\psi), x-\psi\rangle + k - \langle A\psi, \psi\rangle \geqslant k - \langle A\psi,\psi\rangle \geqslant k - \sigma > 0.$$

Proceeding by contradiction, let us assume that, for some vector x with $\|x\| < 1$, $P'(x) = 0$. Then, there would be a real number λ such that

$$(I - \lambda A)x = \lambda\varphi, \quad \lambda = \frac{1-\|x\|^2}{Q(x)} > 0. \tag{1}$$

We show next that $\lambda \leqslant 1$. Assuming $\lambda > 1$, after the first equality in (1), we obtain

$$x_n = \frac{a_nr_n(1-r_n)}{\lambda^{-1} - r_n}, \quad n \geqslant 1. \tag{2}$$

We may suppose that $\lambda^{-1} \notin S$, otherwise $\lambda^{-1} = r_n$, for some n, and x_n would not be defined. Hence, considering the decimal expansion

$$\lambda^{-1} = 0.d_1d_2\dots d_n\dots,$$

we know that it has infinitely many non-zero decimals. We want to show that the sequence $x = (x_n)$ is not bounded, thus contradicting that $x \in l_2$. With this in mind, let α be such that $0 < \alpha < \lambda^{-1}$. We find positive integers m_1, p_1 such that

$$\alpha < r_{p_1} = 0.d_1 d_2 \dots d_{m_1}, \quad d_{m_1} \neq 0.$$

Now, for an arbitrary value $M > 0$, we find a positive integer $m_2 > m_1$ such that the corresponding decimal $d_{m_2} \neq 0$ and

$$\alpha(1 - \lambda^{-1}) \frac{10^{m_2}}{(10^{m_2} + q)^{2/3}} > M.$$

Then, if p_2 is such that $r_{p_2} = 0.d_1 d_2 \dots d_{m_2}$, it follows that $p_2 > p_1$, $r_{p_2} > r_{p_1}$ and so

$$x_{p_2} > \frac{a_{p_2} \alpha (1 - \lambda^{-1})}{\lambda^{-1} - r_{p_2}} > \alpha(1 - \lambda^{-1}) \frac{10^{m_2}}{(10^{m_2} + q)^{2/3}} > M.$$

We have then that $\lambda \in \left]0, 1\right]$. After (2) it follows that $|x_n| \leqslant a_n r_n$, $n \geqslant 1$, and so $\|x\|^2 < \sigma$. Finally, making use of the first equality in (1),

$$\|x\|^2 + \lambda Q(x) = 2\|x\|^2 + \lambda(\langle \varphi, x \rangle + k) < 3\sigma + k \leqslant 1,$$

which contradicts the second equality in (1).

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References

[1] *J. Ferrer*: Rolle's Theorem for polynomials of degree four in a Hilbert space. J. Math. Anal. App. *265* (2002), 322–331 Zbl pre01718988.

[2] *S. A. Shkarin*: On Rolle's Theorem in infinite-dimensional Banach spaces. Mat. Zametki *51* (1992), 128–136 Zbl 0786.46044.

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