Czechoslovak Mathematical Journal, 55 (130) (2005), 465–470

HOMOMORPHIC IMAGES AND RATIONALIZATIONS BASED ON THE EILENBERG-MACLANE SPACES

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(Received August 21, 2002)

Abstract. Are there any kinds of self maps on the loop structure whose induced homomorphic images are the Lie brackets in tensor algebra? We will give an answer to this question by defining a self map of $\Omega\Sigma K(\mathbb{Z},2d)$, and then by computing efficiently some self maps. We also study the topological rationalization properties of the suspension of the Eilenberg-MacLane spaces. These results will be playing a powerful role in the computation of the same n-type problems and giving us an information about the rational homotopy equivalence.

Keywords: Lie bracket, tensor algebra, rationalization, Steenrod power

MSC 2000: 55S37, 55P62

1. Introduction and results

In this paper, let K denote the Eilenberg-MacLane space $K(\mathbb{Z},2d)$ for a fixed integer $d \geqslant 1$. We will use the usual notations ΣX (or ΩX) for the suspension (or loop space) of X. We know that $\tilde{H}_*(K;\mathbb{Z}) \cong \mathbb{Z}[\beta_0,\beta_1,\beta_2,\ldots,\beta_n,\ldots]$, where $\beta_n \in H_{2dn}(K;\mathbb{Z})$ is the standard generator [5]. The rational homology of $\Omega\Sigma K$ is a tensor algebra $T\langle b_0,b_1,b_2,\ldots,b_n,\ldots\rangle$, where b_n has dimension $2dn$ with diagonal $\Delta(b_n) = \sum\limits_{i+j=n} b_i \otimes b_j$ and $b_n = E_*(\beta_n)$. Here E is the inclusion $E\colon K \hookrightarrow \Omega\Sigma K$.

How can we construct a self map of ΣK? Even though there are so many self maps of ΣK, we need to produce a useful self map to achieve one of the goals of this paper and to solve the problems of the same n-type problems. First of all, let $\chi\colon \Omega\Sigma K \to \Omega\Sigma K$ be the map of loop inverse and I the identity map on $\Omega\Sigma K$ sending b_n to b_n for each n in an integral homology. Secondly, we can consider

This work was supported by grant No. R05-2002-000-00093-0 from the Basic Research Program of the Korea Science & Engineering Foundation.

a map $[(-1)^n]\colon \Omega\Sigma K \to \Omega\Sigma K$ inducing a homomorphism $[(-1)^n]_*\colon H_*(\Omega\Sigma K) \to H_*(\Omega\Sigma K)$ which sends b_n to $(-1)^n b_n$ for each n on the integral homology groups. Indeed, we can construct those kinds of maps by using C. A. McGibbon's result [3].

It is well known that the Hurewicz homomorphism carries the Samelson product (dual of the Whitehead product) into the Lie bracket. Here, however, is an interesting question: Are there any kinds of self maps on the loop structure whose induced homomorphic images can be expressed as the Lie bracket in Lie algebra? We will find one commutator map on the loop structure which gives us an answer to this query right after we define a self map as follows.

Definition 1. Define a self map φ on $\Omega\Sigma K$ as $\varphi = \mathrm{I} \star [(-1)^n]$, where $\star$ is the loop multiplication.

Note that the homomorphism $\varphi_*\colon H_*(\Omega\Sigma K) \to H_*(\Omega\Sigma K)$ induced by φ is equal to the composition $\mu_* \circ (I_* \times [(-1)^n]_*) \circ \Delta_*$ of homomorphisms, where Δ and μ are the diagonal map and the loop multiplication respectively. We, then, define a kind of commutator map on the loop structure to find a map which will give an answer to the question above.

Definition 2. Define a map $[I, \varphi]\colon \Omega\Sigma K \to \Omega\Sigma K$ as the commutator of I and φ, i.e., $[I, \varphi] = I \star \varphi \star \chi \star \varphi^{-1}$.

Conveniently, we will just use the notations 'g' for the induced homomorphisms $g_* = H(g; \mathbb{Z})\colon H_*(\Omega\Sigma K) \to H_*(\Omega\Sigma K)$ on the integral homology groups to avoid abundant notations in Theorem 3 and its proof.

Theorem 3. *Let $\langle -, - \rangle$ be the Lie bracket. Then*
(1) $[I, \varphi](b_1) = 0$,
(2) $[I, \varphi](b_2) = 0$, *and*
(3) $[I, \varphi](b_3) = 2\,\langle b_1, b_2 \rangle$.

As usual, let $\Sigma^k K$, $k \geqslant 1$, denote the k-fold suspensions of the Eilenberg-MacLane space. We shall, once again, use the common notation X_0 as the rationalization of X which is a special case of the L-localization, denoted by X_L, of X [7], where L is a set of primes. We now have the following.

Theorem 4. *There is no map $f\colon \Sigma K \to \Sigma\Omega S^{2d+1}$ inducing an isomorphism on the rational homology groups.*

It is already known that there is a map $h\colon \Sigma\Omega S^{2d+1} \to \Sigma K$ inducing an isomorphism on the rational homology groups. Thus the above theorem says that we can not say generally that the rational homotopy equivalence is an equivalence relation which is, at least for me, one of the most important facts in the rational homotopy theory.

We now get a result about the Whitehead products on the k-fold suspension of the given Eilenberg-MacLane space in the (rational) homotopy theory.

Theorem 5. *The Whitehead products* $[\iota_{2di+k}, \iota_{2dj+k}]$ *in* $\pi_{2di+2dj+2k-1}(\Sigma^k K)$, $k \geqslant 1$, *are rationally non-trivial, where* ι_d *is a generator in dimension* d.

If we take $d = 1$, then $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$. So we have the following simple example.

Example 6. The Whitehead product $[\iota_3, \iota_5]\colon S^7 \to \Sigma\mathbb{C}P^3$ has an infinite order.

2. Proofs

P r o o f of Theorem 3. We now prove the theorem by taking several steps.

Step 1 (computation of χ): Considering the loop inverse map χ, we can compute

$$\begin{aligned}\mu(\mathrm{I}\times\chi)\Delta(b_1) &= \mu(\mathrm{I}\times\chi)(b_1\otimes 1 + 1\otimes b_1)\\ &= \mu(b_1\otimes 1 + 1\otimes\chi(b_1))\\ &= b_1 + \chi(b_1).\end{aligned} \tag{2.1}$$

Since $(\mathrm{I}\star\chi)$ is the trivial map, we have $\chi(b_1) = -b_1$.

Similarly, if we compute

$$\begin{aligned}\mu(\mathrm{I}\times\chi)\Delta(b_2) &= \mu(\mathrm{I}\times\chi)(b_2\otimes 1 + 1\otimes b_2 + b_1\otimes b_1)\\ &= \mu(b_2\otimes 1 + 1\otimes\chi(b_2) - b_1\otimes b_1)\\ &= b_2 + \chi(b_2) - b_1^2,\end{aligned} \tag{2.2}$$

then, by the same reason as above, $\chi(b_2) = -b_2 + b_1^2$.

If we compute once again

$$\begin{aligned}\mu(\mathrm{I}\times\chi)\Delta(b_3) &= \mu(\mathrm{I}\times\chi)(b_3\otimes 1 + b_2\otimes b_1 + b_1\otimes b_2 + 1\otimes b_3)\\ &= b_3 + b_2(-b_1) + b_1(-b_2 + b_1^2) + \chi(b_3),\end{aligned} \tag{2.3}$$

then we finally obtain $\chi(b_3) = -b_3 + b_1b_2 + b_2b_1 - b_1^3$.

Step 2 (computation of φ): We can compute

$$\begin{aligned}\varphi(b_3) &= \mu(\mathrm{I}\times[(-1)^n])\Delta(b_3)\\ &= \mu(\mathrm{I}\times[(-1)^n])(b_3\otimes 1 + b_2\otimes b_1 + b_1\otimes b_2 + 1\otimes b_3)\\ &= b_3 + b_2(-b_1) + b_1b_2 - b_3\\ &= b_1b_2 - b_2b_1.\end{aligned} \tag{2.4}$$

Similarly, we have $\varphi(b_1) = 0$ and $\varphi(b_2) = 2b_2 - b_1^2$.

Step 3 (computation of φ^{-1}): Since $\varphi \star \varphi^{-1} = 0$ and $\varphi(b_1) = 0$, we have $\varphi^{-1}(b_1) = 0$. We can also compute

$$\begin{aligned} \mu(\varphi \times \varphi^{-1})\Delta(b_2) &= \mu(\varphi \times \varphi^{-1})(b_2 \otimes 1 + b_1 \otimes b_1 + 1 \otimes b_2) \\ &= \mu(\varphi(b_2) \otimes 1 + 0 + 1 \otimes \varphi^{-1}(b_2)) \\ &= \varphi(b_2) + \varphi^{-1}(b_2). \end{aligned} \tag{2.5}$$

This computation forces to be $\varphi^{-1}(b_2) = -\varphi(b_2) = -(2b_2 - b_1^2)$.

Similarly, we have $\varphi^{-1}(b_3) = -(b_1b_2 - b_2b_1)$. We can make sure about these facts again by using the equation $\varphi^{-1} = \chi \circ \varphi$.

Step 4 (computation of $\mathrm{I} \star \varphi$ and $\chi \star \varphi^{-1}$): We can compute this map by the same method as follows:

$$(\mathrm{I} \star \varphi)(b_i) = \begin{cases} b_1 & \text{for } i = 1, \\ 3b_2 - b_1^2 & \text{for } i = 2, \\ b_3 + 3b_1b_2 - b_2b_1 - b_1^3 & \text{for } i = 3 \end{cases} \tag{2.6}$$

and

$$(\chi \star \varphi^{-1})(b_i) = \begin{cases} -b_1 & \text{for } i = 1, \\ -3b_2 + 2b_1^2 & \text{for } i = 2, \\ -b_3 + 2b_2b_1 + 2b_1b_2 - 2b_1^3 & \text{for } i = 3. \end{cases} \tag{2.7}$$

Final step (computation of the commutator): By using the above steps, we finally have $[I, \varphi](b_1) = 0$, $[I, \varphi](b_2) = 0$ and

$$\begin{aligned} [I, \varphi](b_3) &= \{(I \star \varphi) \star (\chi \star \varphi^{-1})\}(b_3) \\ &= \mu\{(I \star \varphi) \times (\chi \star \varphi^{-1})\}\Delta(b_3) \\ &= \mu\{(I \star \varphi) \times (\chi \star \varphi^{-1})\}(b_3 \otimes 1 + b_2 \otimes b_1 + b_1 \otimes b_2 + 1 \otimes b_3) \\ &= b_3 + 3b_1b_2 - b_2b_1 - b_1^3 + (3b_2 - b_1^2)(-b_1) \\ &\quad + b_1(-3b_2 + 2b_1^2) - b_3 + 2b_2b_1 + 2b_1b_2 - 2b_1^3 \\ &= 2(b_1b_2 - b_2b_1) = 2\,\langle b_1, b_2 \rangle \end{aligned} \tag{2.8}$$

which complete the proof. □

The above theorem will be helpful in the computation of the homotopy same n-type problems for certain suspensions [1].

P r o o f of Theorem 4. We know that ΣK is rationally homotopy equivalent to the wedge product of spheres, i.e., $\Sigma K \simeq_0 S^{2d+1} \vee S^{4d+1} \vee \ldots$, where $\simeq_0$ means the rational homotopy equivalence.

Suppose the map $f\colon \Sigma K \to \Sigma\Omega S^{2d+1}$ induces an isomorphism on the rational homology groups.

We firstly consider a map $f_1\colon \Sigma K \to S^{2d+1}$ inducing the non-zero degree λ in the $(2d+1)$-dimensional integral homology groups. In other words, we can, by assumption, choose an induced homomorphism

$$f_{1*}\colon H_{2d+1}(\Sigma K;\mathbb{Z})/\text{torsion} \cong \mathbb{Z} \to H_{2d+1}(S^{2d+1};\mathbb{Z}) \cong \mathbb{Z} \tag{2.9}$$

given by sending the generator x to $f_{1*}(x) = \lambda x$, for $\lambda \neq 0$.

Secondly, we can also select a prime p such that p and λ are relatively prime.

Finally, we can deduce a contradiction by using the cohomology argument and the Steenrod power ([6], [2]) in the following commutative diagram

$$\begin{array}{ccc} H^{2d+1}(\Sigma K;\mathbb{Z}/p) & \xleftarrow{\ \lambda\ } & H^{2d+1}(S^{2d+1};\mathbb{Z}/p) \\ \mathscr{P}^1\big\downarrow & & \mathscr{P}^1\big\downarrow \\ H^{2d+2p-1}(\Sigma K;\mathbb{Z}/p) & \xleftarrow{\ f_1^*\ } & H^{2d+2p-1}(S^{2d+1};\mathbb{Z}/p). \end{array} \tag{2.10}$$

Indeed, for a generator $\langle x\rangle$ in $H^{2d+1}(S^{2d+1};\mathbb{Z}/p)$, we have

$$\begin{aligned} 0 &\neq \mathscr{P}^1(\lambda\langle x\rangle) \ \ (\text{since } \lambda \neq 0, \text{ and } \mathscr{P}^1(\langle x\rangle) \neq 0) \\ &= f_1^*(\mathscr{P}^1(\langle x\rangle)) \ \ (\text{commutativity}) \\ &= f_1^*(0) \\ &= 0 \end{aligned} \tag{2.11}$$

which is a contradiction. Actually, we can choose infinitely many different primes. We thus conclude that $\lambda = 0$. □

The above theorem says that all maps $f\colon \Sigma K \to \Sigma\Omega S^{2d+1}$ induce the trivial homomorphism on the rational homology groups. One can verify the above theorem by considering the phantom map theory. Let X be a pointed CW-complex. A map $f\colon X \to Y$ is called a *phantom map* [4] if its restriction to each n-skeleton X_n of X is inessential. Since the map $g\colon K(\mathbb{Z},2d) \to \Omega S^{2d+1}$ is an example of a phantom map, the map $\Sigma g = f\colon \Sigma K(\mathbb{Z},2d) \to \Sigma\Omega S^{2d+1}$ of suspension is also phantom. We thus get the above theorem because the rationalized map $\Sigma g_0 = f_0\colon \Sigma K(\mathbb{Q},2d) \to \Sigma\Omega S_0^{2d+1}$ of a phantom map is always trivial, where $\mathbb{Q}$ is the set of all rational numbers.

P r o o f of Theorem 5. Since the k-fold suspension $\Sigma^k K$, $k \geqslant 1$, of the Eilenberg-MacLane space is rationally homotopy equivalent to the wedge product of spheres as mentioned before, we can take generators, say, $\iota_{2di+k} \in \mathbb{Z} \subset \pi_{2di+k}(\Sigma^k K)$, where $i = 1, 2, \ldots$.

Suppose $[\iota_{2di+k}, \iota_{2dj+k}]$ has a finite order in $\pi_{2di+2dj+2k-1}(S^{2di+k} \vee S^{2dj+k})$, then, from the cofibration sequence

$$\begin{aligned} S^{2di+2dj+2k-1} \longrightarrow S^{2di+k} \vee S^{2dj+k} &\longrightarrow S^{2di+k} \times S^{2dj+k} \\ &\longrightarrow S^{2di+2dj+2k} \longrightarrow \ldots, \end{aligned} \tag{2.12}$$

we have

$$S^{2di+k} \times S^{2dj+k} \simeq (S^{2di+k} \vee S^{2dj+k}) \cup_{[\iota_{2di+k}, \iota_{2dj+k}]} \mathrm{e}^{2di+2dj+2k}. \tag{2.13}$$

Since the rationalized Whitehead product $[\iota_{2di+k}, \iota_{2dj+k}]_0$ is trivial, we also have

$$(S^{2di+k} \times S^{2dj+k}) \simeq_0 S^{2di+k} \vee S^{2dj+k} \vee S^{2di+2dj+2k} \tag{2.14}$$

which is a contradiction by using the cohomology argument. Indeed, we have non-zero cup products in $H^*(S^{2di+k} \times S^{2dj+k})$ while all cup products on the right-hand side of (2.14) are always trivial. □

In fact, we have

$$\iota_{2di+k} \in \mathbb{Z} \subset \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \oplus T \cong \pi_{2di+k}(\Sigma^k K), \tag{2.15}$$

where T is torsion. Note that in this case $\pi_{2di+k}(\Sigma^k K)/T \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ (n-times). Here n is equal to the sum of the number of indecomposable (1) and the number of decomposable $(n-1)$ parts (the Whitehead products) of generators.

Acknowledgement. I would like to thank Professor C. A. McGibbon for encouragement and introducing me to the self maps in Theorem 3 which play an important role and a conclusive step in solving the same n-type conjectures for certain suspensions [1].

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