IMPLICATIVE HYPER K-ALGEBRAS

M. M. Zahedi, Kerman, A. Borumand Saeid, Kerman,
and R. A. Borzooei, Zahedan

(Received August 6, 2002)

Abstract. In this note we first define the notions of (weak, strong) implicative hyper K-algebras. Then we show by examples that these notions are different. After that we state and prove some theorems which determine the relationship between these notions and (weak) hyper K-ideals. Also we obtain some relations between these notions and (weak) implicative hyper K-ideals. Finally, we study the implicative hyper K-algebras of order 3, in particular we obtain a relationship between the positive implicative hyper K-algebras and (weak, strong) implicative hyper K-algebras under a simple condition.

Keywords: hyper K-algebra, hyper K-ideal, (weak, strong) implicative hyper K-algebras, (weak) implicative hyper K-ideal

MSC 2000: 06F35, 03G25

1. Introduction

The hyperalgebraic structure theory was introduced by F. Marty [7] in 1934. Imai and Iseki [5] in 1966 introduced the notion of a BCK-algebra. Recently [3], [6], [11] Borzooei, Jun and Zahedi et al. applied the hyperstructure to BCK-algebras and introduced the concept of the hyper K-algebra which is a generalization of the BCK-algebra. It is well-known [9] that the category of bounded commutative BCK-algebras is equivalent to the category of MV-algebras. In particular, any bounded commutative BCK-algebra is an MV-algebra and vice-versa. On the other hand, an MV-algebra is an algebraic structure of the Lukasiewicz many-valued logic. Hence any bounded commutative BCK-algebra is somehow related to a many-valued logic. Since the concept of the hyper K-algebra is a generalization of the notion of the BCK-algebra, it is natural to search for a logic whose algebraic structure is a hyper K-algebra. To this end, we first need a deeper understanding of hyper K-algebras. Now, in this note we define the notions of (weak, strong) implicative hyper K-algebras, then we obtain some related results which have been mentioned in the abstract.

2. Preliminaries

Definition 2.1 ([3]). Let H be a nonempty set and "$\circ$" a *hyperoperation* on H, that is, "$\circ$" is a function from $H \times H$ to $\mathscr{P}^*(H) = \mathscr{P}(H)\backslash\{\emptyset\}$. Then H is called a *hyper K-algebra* if it contains a constant "0" and satisfies the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x < x$,
(HK4) $x < y,\ y < x \Rightarrow x = y$,
(HK5) $0 < x$

for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists\, a \in A$, $\exists\, b \in B$ such that $a < b$.

Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup\limits_{\substack{a\in A\\ b\in B}} a \circ b$ of H.

Example 2.2 ([3]). Define the hyperoperation "$\circ$" on $H = [0, +\infty)$ as follows:

$$x \circ y = \begin{cases} [0, x] & \text{if } x \leqslant y, \\ (0, y] & \text{if } x > y \neq 0, \\ \{x\} & \text{if } y = 0 \end{cases}$$

for all $x, y \in H$. Then $(H, \circ, 0)$ is a hyper K-algebra.

Theorem 2.3 ([3]). *Let $(H, \circ, 0)$ be a hyper K-algebra. Then for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H the following relations hold:*

(i) $x \circ y < z \Leftrightarrow x \circ z < y$,
(ii) $(x \circ z) \circ (x \circ y) < y \circ z$,
(iii) $x \circ (x \circ y) < y$,
(iv) $x \circ y < x$,
(v) $A \subseteq B \Rightarrow A < B$,
(vi) $x \in x \circ 0$,
(vii) $(A \circ C) \circ (A \circ B) < B \circ C$,
(viii) $(A \circ C) \circ (B \circ C) < A \circ B$,
(ix) $A \circ B < C \Leftrightarrow A \circ C < B$.

Definition 2.4 ([3]). Let I be a nonempty subset of a hyper K-algebra $(H, \circ, 0)$ and $0 \in I$. Then

(i) I is called a *weak hyper K-ideal* of H if $x \circ y \subseteq I$ and $y \in I$ imply that $x \in I$ for all $x, y \in H$;

(ii) I is called a *hyper K-ideal* of H if $x \circ y < I$ and $y \in I$ imply that $x \in I$ for all $x, y \in H$.

Theorem 2.5 ([3]). *Any hyper K-ideal of a hyper K-algebra H is a weak hyper K-ideal.*

Definition 2.6 ([4]). Let I be a nonempty subset of H. Then we say that I satisfies the *additive condition*, if for all $x, y \in H$, $x < y$ and $y \in I$ imply that $x \in I$.

Definition 2.7 ([2]). Let H be a hyper K-algebra. An element $a \in H$ is called a *left* (*right*) *scalar* if $|a \circ x| = 1$ ($|x \circ a| = 1$) for all $x \in H$. If $a \in H$ is both a left and a right scalar, we say that a is a *scalar element*.

Definition 2.8 ([2]). We say that a hyper K-algebra H satisfies the *transitive condition* if for all $x, y, z \in H$, $x < y$ and $y < z$ imply that $x < z$.

Definition 2.9 ([2]). A hyper K-algebra H is called a *positive implicative hyper K-algebra*, if it satisfies $(x \circ z) \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$.

Definition 2.10 ([1]). We say that a hyper K-algebra H satisfies the *strong transitive condition* if for all $A, B, C \subseteq H$, $A < B$ and $B < C$ imply that $A < C$.

Definition 2.11 ([1]). Let H be a hyper K-algebra, then a nonempty subset I of H is called

(a) a *weak implicative hyper K-ideal* if it satisfies
  (i) $0 \in I$,
  (ii) $(x \circ z) \circ (y \circ x) \subseteq I$ and $z \in I$ imply $x \in I$ for all $x, y, z \in H$,

(b) an *implicative hyper K-ideal* if it satisfies
  (i) $0 \in I$,
  (ii) $(x \circ z) \circ (y \circ x) < I$ and $z \in I$ imply $x \in I$ for all $x, y, z \in H$.

Theorem 2.12 ([1]). *Let I be a weak hyper K-ideal of H. Then the following statements hold:*

(i) *If for all $x, y, z \in H$, $x \circ (y \circ x) \subseteq I$ implies $x \in I$, then I is a weak implicative hyper K-ideal.*

(ii) *Let $0 \in H$ be a right scalar element and I a weak implicative hyper K-ideal. Then for all $x, y \in H$, $x \circ (y \circ x) \subseteq I$ implies that $x \in I$.*

Theorem 2.13 ([1]). *Let I be a hyper K-ideal of H. Then I is an implicative hyper K-ideal if and only if*

$$x \circ (y \circ x) < I \quad \textit{implies that } x \in I \quad \textit{for any } x, y \in H.$$

Definition 2.14 ([10]). Let $H = \{0, 1, 2\}$ be a hyper K-algebra of order 3. We say that H satisfies the *simple condition* if $1 \not< 2$ and $2 \not< 1$.

Definition 2.15 ([10]). Let $H = \{0, 1, 2\}$ be a hyper K-algebra of order 3. We say that H satisfies the *normal condition* if $1 < 2$ or $2 < 1$.

3. Implicative hyper K-algebra

From now on H is a hyper K-algebra, unless stated otherwise.

Definition 3.1. H is said to be
(i) *weak implicative* if $x < x \circ (y \circ x)$ for all $x, y \in H$,
(ii) *implicative* if $x \in x \circ (y \circ x)$ for all $x, y \in H$,
(iii) *strong implicative* if $x \circ 0 \subseteq x \circ (y \circ x)$ for all $x, y \in H$.

Example 3.2. Let $H = \{0, 1, 2, 3\}$. Then the following table shows a hyper K-algebra structure on H:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{2\}$
3	$\{2,3\}$	$\{1,2\}$	$\{0,2,3\}$	$\{0,1,2\}$

It can be checked that H is a weak implicative, implicative and strong implicative hyper K-algebra.

Theorem 3.3.
(i) *Any implicative hyper K-algebra is a weak implicative hyper K-algebra.*
(ii) *Any strong implicative hyper K-algebra is an implicative hyper K-algebra.*

P r o o f. The proof is trivial. □

The following example shows that the notions given in Definition 3.1 are not equivalent.

Example 3.4. (i) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1,2\}$	$\{0,2\}$	$\{0,2\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

We can see that H is a weak implicative hyper K-algebra. However it is not an implicative hyper K-algebra, because $1 \notin 1 \circ (2 \circ 1) = \{0, 2\}$.

(ii) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1\}$	$\{1,2\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

Now, H is an implicative hyper K-algebra. However it is not a strong implicative one because $0 \circ 0 = \{0, 1\} \nsubseteq 0 \circ (1 \circ 0) = \{0\}$.

Proposition 3.5. *Let $0 \in H$ be a right scalar element. Then the notions of implicative and strong implicative hyper K-algebras are equivalent.*

Proof. The proof follows from the fact that $x \circ 0 = x$. □

Proposition 3.6. *H is a weak implicative hyper K-algebra if and only if $x \circ 0 < x \circ (y \circ x)$ for all $x, y \in H$.*

Proof. Let $x \circ 0 < x \circ (y \circ x)$ for all $x, y \in H$. Then we have $x \circ (x \circ (y \circ x)) < 0$. Thus there exists $t \in x \circ (x \circ (y \circ x))$ such that $t < 0$. Hence $t = 0$, therefore $x < x \circ (y \circ x)$. The proof of the converse is trivial. □

Theorem 3.7. *Let H be a hyper K-algebra of order 3 that satisfies the simple condition. Then H is implicative if and only if it is weak implicative.*

Proof. Let H be a weak implicative hyper K-algebra. We show that $x \in x \circ (y \circ x)$ for all $x, y \in H$. If $x = 0$, then $0 \in 0 \circ (y \circ 0)$ for all $y \in H$. Let $x \neq 0$ and $x \notin x \circ (y \circ x)$. Since $x < x \circ (y \circ x)$, there exists $t \in x \circ (y \circ x)$ such that $x < t$. Clearly since $x \neq 0$, we must have $t \neq 0$ and $t \neq x$. Since H satisfies the simple condition, $x < t$ is impossible. Thus $x \in x \circ (y \circ x)$ for all $x, y \in H$. For the converse see Theorem 3.3 (i). □

Example 3.8. This example shows that in the above theorem, the simple condition can not be omitted. Indeed let $H = \{0,1,2\}$. Then the following table shows a hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,2\}$

Then H is weak implicative while it is not implicative, since $2 \notin 2 \circ (1 \circ 2)$.

Example 3.9. (i) It is not necessary that a (weak, strong) implicative hyper K-algebra be a positive implicative hyper K-algebra. Example 3.2 shows a hyper K-algebra which is strong implicative while it is not a positive implicative hyper K-algebra. Indeed $(3 \circ 2) \circ (1 \circ 2) = \{0,1,2,3\} \neq (3 \circ 1) \circ 2 = \{0,1,2\}$.

(ii) In general it is not needed that a positive implicative hyper K-algebra be a (weak, strong) implicative hyper K-algebra. Because let $H = \{0,1,2\}$. Then the following table shows a positive implicative hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$

but H is not a (weak, strong) implicative hyper K algebra. Indeed $1 \not< 1 \circ (2 \circ 1)$.

Theorem 3.10. *Let H be a positive implicative hyper K-algebra of order 3 that satisfies the simple condition. Then H is a (weak, strong) implicative hyper K-algebra.*

Proof. Since H satisfies the simple condition, we know that $1 \circ 0 = \{1\}$, $2 \circ 0 = \{2\}$, $1 \circ 2 \neq \{2\}$ and $2 \circ 1 \neq \{1\}$ by Theorem 3.17 of [10]. Now we show that H is a strong implicative hyper K-algebra, that is $x \circ 0 \subseteq x \circ (y \circ x)$ for all $x, y \in H$. To do this we consider three different cases:

(i) If $x = 0$, then we must show that $0 \circ 0 \subseteq 0 \circ (y \circ 0)$ for all $y \in H$. If $y = 0$, then we are done. We know that $0 \in 0 \circ 0$, so $0 \circ 0 = \{0\}$, $\{0,1\}$, $\{0,2\}$ or $\{0,1,2\}$. If $0 \circ 0 = \{0\}$, then clearly $0 \in 0 \circ 1$ and $0 \in 0 \circ 2$, and so we are done. Now let $0 \circ 0 = \{0,1\}$. If $y = 1$, then we must show that $0 \circ 0 \subseteq 0 \circ (1 \circ 0) = 0 \circ 1$. We have $(0 \circ 0) \circ 0 = \{0,1\} \circ 0 = (0 \circ 0) \cup (1 \circ 0) = \{0,1\} \cup \{1\} = \{0,1\}$. On the other hand, since H is positive implicative then $\{0,1\} = (0 \circ 0) \circ 0 = (0 \circ 0) \circ (0 \circ 0) = \{0,1\} \circ \{0,1\} =$

$(0 \circ 0) \cup (1 \circ 0) \cup (0 \circ 1) \cup (1 \circ 1)$. Thus we conclude that $(0 \circ 1)$ and $(1 \circ 1) \subseteq \{0, 1\}$. If $1 \notin (0 \circ 1)$, we get that $0 \circ 1 = \{0\}$. So $(0 \circ 1) \circ 1 = 0 \circ 1 = \{0\}$ and on the other hand, since H is positive implicative we have $\{0\} = (0 \circ 1) \circ 1 = (0 \circ 1) \circ (1 \circ 1) \supseteq 0 \circ 0 = \{0, 1\}$, which is a contradiction. Thus $0 \circ 1 = \{0, 1\}$, and hence $0 \circ 0 = 0 \circ 1$. Now let $y = 2$. Since $0 \in 0 \circ 2$ then $0 \circ 2 = \{0\}$, $\{0, 1\}$, $\{0, 2\}$ or $\{0, 1, 2\}$. If $0 \circ 2 = \{0\}$, then $(0 \circ 2) \circ 2 = 0 \circ 2 = \{0\}$ and on the other hand, since H is positive implicative we have $\{0\} = (0 \circ 2) \circ 2 = (0 \circ 2) \circ (2 \circ 2) \supseteq 0 \circ 0 = \{0, 1\}$, which is a contradiction. Hence $0 \circ 2 \neq \{0\}$. Let $0 \circ 2 = \{0, 2\}$. Since $1 \not< 2$, then $0 \notin 1 \circ 2$. So $1 \circ 2 = \{1\}$ or $\{1, 2\}$. If $1 \circ 2 = \{1\}$, then $\{0, 2\} = 0 \circ 2 \subseteq (1 \circ 1) \circ 2 = (1 \circ 2) \circ 1 = 1 \circ 1 \subseteq \{0, 1\}$, which is a contradiction. Hence $1 \circ 2 = \{1, 2\}$. Now we have $0 \in 2 \circ 2 \subseteq (1 \circ 2) \circ (0 \circ 2) = (1 \circ 0) \circ 2 = 1 \circ 2 = \{1, 2\}$, which is a contradiction. Hence $0 \circ 2 = \{0, 1\}$ or $\{0, 1, 2\}$. Thus in the case $0 \circ 0 = \{0, 1\}$, we conclude that $0 \circ 0 \subseteq 0 \circ 2$. The proof for the case $0 \circ 0 = \{0, 2\}$ is similar as above. If $0 \circ 0 = \{0, 1, 2\}$, then since H is a positive implicative we have $\{1\} = (1 \circ 0) = (1 \circ 0) \circ 0 = (1 \circ 0) \circ (0 \circ 0) = 1 \circ \{0, 1, 2\} = (1 \circ 0) \cup (1 \circ 1) \cup (1 \circ 2)$, thus we must have $1 \circ 1 = \{1\}$ and this is a contradiction with (HK3). Hence $0 \circ 0 \neq \{0, 1, 2\}$. Thus if $x = 0$, then $0 \circ 0 \subseteq 0 \circ (y \circ 0)$ for all $y \in H$.

(ii) If $x = 1$, then we must show that $1 \in 1 \circ (y \circ 1)$ for all $y \in H$. If $y = 0$ or 1 it is trivial, so let $y = 2$. Since $2 \not< 1$, then $0 \notin 2 \circ 1$ and $2 \circ 1 \neq \{1\}$. Thus we conclude that $2 \circ 1 = \{2\}$ or $\{1, 2\}$. Since $1 \not< 2$, then $0 \notin 1 \circ 2$ and $1 \circ 2 \neq \{2\}$. Therefore $1 \circ 2 = \{1\}$ or $\{1, 2\}$. Hence in all cases by some manipulations we can get that $1 \in 1 \circ (2 \circ 1)$.

(iii) If $x = 2$, then by the same argument as in (ii) we can show that $2 \in 2 \circ (y \circ 2)$ for all $y \in H$. □

Remark 3.11. The following example shows that in the above theorem the simple condition can not be omitted. Let $H = \{0, 1, 2\}$. Then the following table shows a positive implicative hyper K-algebra structure on H where H does not satisfy the simple condition:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0, 2\}$

and H is not an implicative hyper K-algebra, either, because $2 \notin 2 \circ (1 \circ 2) = \{0\}$.

Theorem 3.12. *Let H be a weak implicative hyper K-algebra. Then each hyper K-ideal of H is a weak implicative hyper K-ideal.*

Proof. Let I be a hyper K-ideal and $(x \circ z) \circ (y \circ x) \subseteq I$, $z \in I$. Then for all $t \in x \circ (y \circ x)$ we have $t \circ z \subseteq (x \circ (y \circ x)) \circ z = (x \circ z) \circ (y \circ x) \subseteq I$ and $z \in I$. Thus

$t \in I$ and hence $x \circ (y \circ x) \subseteq I$. Since H is weak implicative, then $x < x \circ (y \circ x) \subseteq I$. So there exists $r \in I$ such that $x < r$. Thus $0 \in x \circ r$, hence $x \circ r < I$ and $r \in I$ which implies that $x \in I$. □

Remark 3.13. (i) The following example shows that in the above theorem we can not use "weak hyper K-ideal" instead of "hyper K-ideal". Let $H = \{0, 1, 2\}$. Then the following table shows a weak implicative hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1, 2\}$	$\{2\}$
2	$\{2\}$	$\{0, 1, 2\}$	$\{0, 1\}$

Now $I = \{0, 1\}$ is a weak hyper K-ideal and $(2 \circ 0) \circ (1 \circ 2) = \{0, 1\} \subseteq I$ and $0 \in I$, but $2 \notin I$. Hence I is not a weak implicative hyper K-ideal.

(ii) The following example shows that in the above theorem, if we use "weak hyper K-ideal" instead of "hyper K-ideal", we can not conclude that "any weak hyper K-ideal is implicative". Let $H = \{0, 1, 2\}$. Then the following table shows a weak implicative hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0, 1\}$	$\{0, 1, 2\}$	$\{0, 1, 2\}$
1	$\{1\}$	$\{0, 1\}$	$\{1, 2\}$
2	$\{1, 2\}$	$\{0, 1, 2\}$	$\{0, 1, 2\}$

Then $I = \{0\}$ is a weak hyper K-ideal and $1 \circ (0 \circ 1) = \{0, 1, 2\} < I$, but $1 \notin I$. Hence I is not an implicative hyper K-ideal.

(iii) The following example shows that the conditions of the above theorem do not imply that any hyper K-ideal is implicative. Let $H = \{0, 1, 2\}$. Then the following table shows a weak implicative hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0, 1\}$	$\{0, 1, 2\}$

We see that $I = \{0\}$ is a hyper K-ideal and $2 \circ (2 \circ 2) = \{0, 1, 2\} < I$, but $2 \notin I$. Hence I is not an implicative hyper K-ideal.

Theorem 3.14. *Let H be an implicative hyper K-algebra. Then each weak hyper K-ideal of H is a weak implicative hyper K-ideal.*

Proof. Let I be a weak hyper K ideal and $(x \circ z) \circ (y \circ x) \subseteq I$, $z \in I$. Then $(x \circ (y \circ x)) \circ z \subseteq I$. Since H is implicative, we have $x \in (x \circ (y \circ x))$. Therefore $x \circ z \subseteq (x \circ (y \circ x)) \circ z \subseteq I$ and since $z \in I$, we conclude that $x \in I$. $\square$

Corollary 3.15. *Let H be an implicative hyper K-algebra. Then each hyper K ideal of H is a weak implicative hyper K-ideal.*

Remark 3.16. The following example shows that the conditions of the above corollary do not imply that any hyper K-ideal is implicative. Let $H = \{0, 1, 2\}$. Then the following table shows an implicative hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$

Now $I = \{0\}$ is a hyper K-ideal while it is not an implicative hyper K-ideal, since $(2 \circ 0) \circ (2 \circ 2) = \{0, 2\} < I$ and $0 \in I$, but $2 \notin I$.

Note that the following theorem says that if we restrict ourselves to the hyper K-algebras of order 3, then the above corollary holds even if H is not implicative.

Theorem 3.17. *If H is a hyper K-algebra of order 3, then each nonzero hyper K-ideal is a weak implicative hyper K-ideal.*

Proof. Let $H = \{0, 1, 2\}$. Without loss of generality let $I = \{0, 1\}$ be a hyper K-ideal of H. By Theorem 2.11 it is enough to show that $x \circ (y \circ x) \subseteq I$ implies that $x \in I$. If $x = 0, 1$ then we are done. Now let $x = 2$, then $2 \circ (y \circ 2) \subseteq I$ for all $y \in H$ and we will get a contradiction. To obtain it, consider three different cases:

(i) Let $y = 0$. Then $2 \in 2 \circ (0 \circ 2) \subseteq I$, and this is a contradiction.

(ii) Let $y = 1$. If $1 < 2$, then $0 \in 1 \circ 2$. Therefore $2 \in 2 \circ 0 \subseteq 2 \circ (1 \circ 2) \subseteq I$, and this is a contradiction. If $1 \not< 2$, then $0 \notin 1 \circ 2$, so we must have $1 \circ 2 = \{1\}$, $\{2\}$ or $\{1, 2\}$. If $1 \circ 2 = \{1\}$, then $2 \circ 1 = 2 \circ (1 \circ 2) \subseteq I$ and $1 \in I$ imply that $2 \in I$, which is a contradiction. If $1 \circ 2 = \{2\}$, then $0 \in 0 \circ 2 \subseteq (1 \circ 1) \circ 2 = (1 \circ 2) \circ 1 = 2 \circ 1$. Hence $2 \circ 1 < I$ and $1 \in I$ imply that $2 \in I$, which is a contradiction. If $1 \circ 2 = \{1, 2\}$, consider $(2 \circ 1) \cup (2 \circ 2) = 2 \circ \{1, 2\} = 2 \circ (1 \circ 2) \subseteq I$, therefore $2 \circ 1 \subseteq I$ and $1 \in I$ imply that $2 \in I$, which is a contradiction.

(iii) If $y = 2$ then $2 \in 2 \circ (2 \circ 2) \subseteq I$, which is a contradiction. $\square$

Remark 3.18. (i) The converse of the above theorem is not correct. Indeed let $H = \{0, 1, 2, 3\}$. Then the following table shows a hyper K-algebra structure on H:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$	$\{2\}$
3	$\{2,3\}$	$\{1,2,3\}$	$\{0,1,3\}$	$\{0,1,2,3\}$

Then $I = \{0, 1\}$ is a weak implicative hyper K-ideal, which is not a hyper K-ideal, since $3 \circ 1 = \{1, 2, 3\} < I$ and $1 \in I$, but $3 \notin I$.

(ii) The following example shows that the condition “nonzero hyper K-ideal” in the above theorem can not be omitted. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1\}$	$\{0,1\}$

Now it is easy to see that $I = \{0\}$ is a hyper K-ideal while it is not a weak implicative one since $(1 \circ 0) \circ (2 \circ 1) \subseteq I$ and $0 \in I$, but $1 \notin I$.

Lemma 3.19. *Let H be a positive implicative hyper K-algebra of order 3 that satisfies the normal condition. Then the following statements hold:*

(i) $1 \circ 0 = \{1\}$,

(ii) $2 \circ 0 = \{2\}$.

Proof. (i) We know that $1 \in 1 \circ 0$ and $0 \notin 1 \circ 0$, thus $1 \circ 0 = \{1\}$ or $\{1, 2\}$. Let $1 \circ 0 = \{1, 2\}$. Since H satisfies the normal condition, then $1 < 2$ or $2 < 1$. Now we consider the following two cases.

Case 1: Let $1 < 2$. Then $0 \notin 2 \circ 1$. Since $0 \in 2 \circ 2 \subseteq (2 \circ 0) \circ \{1, 2\} = (2 \circ 0) \circ (1 \circ 0) = (2 \circ 1) \circ 0$, thus $2 \circ 1 < 0$. So there is $x \in 2 \circ 1$ such that $x < 0$, therefore $x = 0$. Hence $0 \in 2 \circ 1$, which is a contradiction.

Case 2: Let $2 < 1$. Then $0 \notin 1 \circ 2$. Since $0 \in 2 \circ 2 \subseteq \{1, 2\} \circ (2 \circ 0) = (1 \circ 0) \circ (2 \circ 0) = (1 \circ 2) \circ 0$, thus there is $x \in 1 \circ 2$ such that $x < 0$, so $x = 0$. Hence $0 \in 2 \circ 1$, which is a contradiction. Thus we must have $1 \circ 0 = \{1\}$.

(ii) The proof is similar to the proof of (i). □

Theorem 3.20. *Let H be a hyper K-algebra of order 3 and $I \subset H$. Then*

(i) *If H satisfies the simple condition, then I is a weak implicative hyper K-ideal if and only if I is a weak hyper K-ideal;*

(ii) *if H is positive implicative and satisfies the normal condition then $I \neq \{0\}$ is a weak implicative hyper K-ideal if and only if I is a weak hyper K-ideal.*

Proof. (i) Let $I = \{0\}$ be a weak hyper K-ideal and $(x \circ z) \circ (y \circ x) \subseteq I$ and $z \in I$. Then $x \circ (y \circ x) \subseteq (x \circ 0) \circ (y \circ x) = \{0\}$. We must show that $x = 0$. On the contrary, let $x = 1$. Then $1 \circ (y \circ 1) = \{0\}$. If $y = 0$ or 1, we get the contradiction $1 \in \{0\}$. If $y = 2$, since H satisfies the simple condition, then $1 \circ (2 \circ 1) \neq \{0\}$, which is a contradiction, hence $x = 1$ is impossible. By a similar argument we show that $x = 2$ is also impossible. Thereby $x = 0 \in I$. Note that since $I = \{0\}$ is always a weak hyper K-ideal the converse is trivial. For the case $I \neq \{0\}$ see Theorem 4.11 of [1].

(ii) Without loss of generality let $I = \{0, 1\}$ be a weak hyper K-ideal. By Theorem 2.11 (i), it is enough to show that if $x \circ (y \circ x) \subseteq I$ for $x, y \in H$, then $x \in I$. If $x = 0$ or 1 we are done. Now let $x = 2$. So $2 \circ (y \circ 2) \subseteq I$ for arbitrary $y \in H$. We consider three cases for y and show that none of these cases holds.

(a) Let $y = 0$. Then $2 \in 2 \circ 0 \subseteq 2 \circ (0 \circ 2) \subseteq I$, which is a contradiction.

(b) Let $y = 1$. If $1 < 2$, then $0 \in 1 \circ 2$, hence $2 \in 2 \circ 0 \subseteq 2 \circ (1 \circ 2) \subseteq I$, which is a contradiction. If $1 \not< 2$, then $1 \circ 2 = \{1\}$, $\{2\}$ or $\{1, 2\}$. If $1 \circ 2 = \{1\}$, then $2 \circ 1 = 2 \circ (1 \circ 2) \subseteq I$. Since $1 \in I$, we get that $2 \in I$, which is a contradiction. If $1 \circ 2 = \{2\}$, then we have $2 \circ 2 = 2 \circ (1 \circ 2) \subseteq I = \{0, 1\}$. Since H is positive implicative we have $2 \in 2 \circ 0 \subseteq 2 \circ (2 \circ 2) = (1 \circ 2) \circ (2 \circ 2) = (1 \circ 2) \circ 2 = 2 \circ 2 \subseteq I = \{0, 1\}$ which is a contradiction. If $1 \circ 2 = \{1, 2\}$, then $(2 \circ 1) \cup (2 \circ 2) = 2 \circ \{1, 2\} = 2 \circ (1 \circ 2) \subseteq I$. Hence $2 \circ 1 \subseteq I$. Since I is a weak hyper K-ideal and $1 \in I$, so $2 \in I$, which is a contradiction.

(c) Let $y = 2$. Then $2 \in 2 \circ 0 \subseteq 2 \circ (2 \circ 2) \subseteq I$, which is a contradiction. Hence $x = 2$ is impossible and therefore I is a weak implicative hyper K-ideal.

Conversely, let $I = \{0, 1\}$ be a weak implicative hyper K-ideal and $x \circ y \subseteq I$ and $y \in I$. We must show that $x \in I$. If $x = 0$ or 1 then $x \in I$ and we are done. Now we show that $x = 2$ is impossible. If $x = 2$ then we have $2 \circ y \subseteq I$ and $y \in I$. We consider three different cases for y:

(a′) If $y = 0$, then $2 \in 2 \circ 0 \subseteq I$, which is a contradiction.

(b′) The case $y = 2$ never occurs since we must have $y \in I$.

(c′) If $y = 1$, since $2 \circ 1 \subseteq I$ we conclude that $2 \circ 1 = \{0\}$, $\{1\}$ or $\{0, 1\}$. Now consider the following cases:

(c$'$1) If $2\circ 1=\{0\}$, then $0\notin 1\circ 2$, therefore $1\circ 2=\{1\}$, $\{2\}$ or $\{1,2\}$. Thus we have to consider the following three subcases:

(c$'$1.1) If $1\circ 2=\{1\}$, since by Lemma 3.19, $2\circ 0=\{2\}$, hence $(2\circ 0)\circ(1\circ 2)=2\circ 1=\{0\}\subseteq I$. Since I is a weak implicative hyper$K$-ideal and $0\in I$, we have $2\in I$, which is a contradiction.

(c$'$1.2) If $1\circ 2=\{2\}$, since H is positive implicative we have $\{0\}=2\circ 1=(1\circ 2)\circ 1=(1\circ 1)\circ 2=(1\circ 2)\circ(1\circ 2)=2\circ 2$. Since by Lemma 3.19, $2\circ 0=\{2\}$, $(2\circ 0)\circ(1\circ 2)=2\circ(1\circ 2)=2\circ 2=\{0\}\subseteq I$ and $0\in I$, we conclude that $2\in I$, which is a contradiction.

(c$'$1.3) If $1\circ 2=\{1,2\}$, since $1\circ 0=\{1\}$ and $2\circ 0=\{2\}$ hence $(1\circ 0)\circ 2=1\circ 2=\{1,2\}$ and $(1\circ 0)\circ 2=(1\circ 2)\circ(0\circ 2)=\{1,2\}\circ(0\circ 2)$. If 1 or 2 belongs to $0\circ 2$, then $0\in(1\circ 0)\circ 2=\{1,2\}$ which is a contradiction, thus $0\circ 2=\{0\}$. We know that $0\in 2\circ 2$; if $2\circ 2=\{0\}$ then $(2\circ 0)\circ(1\circ 2)=2\circ(1\circ 2)=2\circ\{1,2\}=(2\circ 1)\cup(2\circ 2)=\{0\}\subseteq I$. Since $0\in I$ and I is weak implicative, we get that $2\in I$, which is a contradiction. Thus $2\circ 2\neq\{0\}$. Since H is positive implicative we have $\{0\}=0\circ 2=(2\circ 1)\circ 2=(2\circ 2)\circ(1\circ 2)=(2\circ 2)\circ\{1,2\}\neq\{0\}$, hence this case is impossible.

(c$'$2) If $2\circ 1=\{1\}$, we know that $1\circ 0=\{1\}$. Then $\{1\}=1\circ 0=(2\circ 1)\circ 0=(2\circ 0)\circ 1=(2\circ 1)\circ(0\circ 1)=1\circ(0\circ 1)$, therefore $0\circ 1=\{0\}$. We have $0\in 0\circ 2$. If $0\circ 2=\{0\}$, then by considering $(2\circ 1)\circ(0\circ 2)=1\circ 0=\{1\}\subseteq I$, $1\in I$ and I is weak implicative, we conclude that $2\in I$, which is a contradiction. Hence $0\circ 2\neq\{0\}$. Consider $(0\circ 2)\circ 1=(0\circ 1)\circ(2\circ 1)=0\circ\{1\}=\{0\}$. But if $0\circ 2\neq\{0\}$ then $(0\circ 2)\circ 1\neq\{0\}$, hence this case is impossible.

(c$'$3) If $2\circ 1=\{0,1\}$, since $0\in 2\circ 1$, hence $0\notin 1\circ 2$. Therefore $1\circ 2=\{1\}$, $\{2\}$ or $\{1,2\}$. Now we discuss the following three different subcases.

(c$'$3.1) If $1\circ 2=\{1\}$, since $(2\circ 0)\circ(1\circ 2)=2\circ 1\subseteq I$, $0\in I$ and I is weak implicative, we get that $2\in I$, which is impossible.

(c$'$3.2) Suppose $1\circ 2=\{2\}$. Consider $(2\circ 0)\circ(1\circ 2)=2\circ(1\circ 2)=2\circ 2=(1\circ 2)\circ(1\circ 2)=(1\circ 1)\circ 2=(1\circ 2)\circ 1=2\circ 1=\{0,1\}\subseteq I$. Since $0\in I$ and I is weak implicative, we get that $2\in I$, which is a contradiction.

(c$'$3.3) If $1\circ 2=\{1,2\}$, then since H is positive implicative we have $\{1,2\}=1\circ 2=(1\circ 0)\circ 2=(1\circ 2)\circ(0\circ 2)=\{1,2\}\circ(0\circ 2)$. If 1 or $2\in 0\circ 2$, then $0\in(1\circ 0)\circ 2=\{1,2\}$ which is a contradiction, hence $0\circ 2=\{0\}$. Consider $\{1\}=1\circ 0=(1\circ 0)\circ 0=(1\circ 0)\circ(0\circ 0)=1\circ(0\circ 0)$. Then we conclude that $0\circ 0=\{0\}$. Then $(2\circ 1)\circ(0\circ 2)=\{0,1\}\circ 0=(0\circ 0)\cup(1\circ 0)=\{0,1\}\subseteq I$. Since $1\in I$ we get that $2\in I$, which is a contradiction.

Thus the above arguments show that $x=2$ is impossible, hence I is a weak hyper K-ideal of H.

Remark 3.21. (i) In part (ii) of the above theorem the condition "positive implicative" can not be omitted. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H which satisfies the normal condition:

$\circ$	0	1	2
0	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{1,2\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

We can check that $I = \{0, 1\}$ is a weak implicative hyper K-ideal, but it is not a weak hyper K-ideal, since $2\circ 1 \subseteq I$ and $1 \in I$ but $2 \notin I$. Note that H is not a positive implicative hyper K-algebra, since $\{1, 2\} = (1 \circ 2) \circ 0 \neq (1 \circ 0) \circ (2 \circ 0) = \{0, 1, 2\}$.

(ii) In part (ii) of above theorem the condition "$I \neq \{0\}$" can not be omitted, since hyper K-algebra H of Example 3.9 (ii) is positive implicative and normal and $I = \{0\}$ is weak hyper K-ideal but is not weak implicative hyper K-ideal, since $2 \circ (1 \circ 2) = \{0\} \subseteq I$ but $2 \notin I$.

Theorem 3.22. *Let H be an implicative hyper K-algebra that satisfies the strong transitive condition and let I be a hyper K-ideal of H. Then I is an implicative hyper K-ideal.*

Proof. Let $x \circ (y \circ x) < I$. Since H is implicative, then $x \in x \circ (y \circ x)$. Hence $x < I$ and I is a hyper K-ideal, so $x \in I$. Thus by Theorem 2.13, I is an implicative hyper K-ideal. $\square$

Note that the example given in Remark 3.16 shows that the "strong transitive condition" is necessary in the above proposition.

Theorem 3.23. *Let H be a hyper K-algebra of order 3 and $0 \in H$ a right scalar element. If $I = \{0\}$ is an implicative hyper K-ideal, then H is a strong implicative hyper K-algebra.*

Proof. Since 0 is a right scalar element, it is enough to show that $x \in x \circ (y \circ x)$ for all $x, y \in H$. To do this consider the following cases:

(i) If $x = 0$, then it is clear that $0 \in 0 \circ (y \circ 0)$ for all $y \in H$.

(ii) If $x = 1$, we consider three cases: (a) if $y = 0$, then $1 \in 1 \circ 0 \subseteq 1 \circ (0 \circ 1)$. (b) if $y = 1$, then $1 \in 1 \circ 0 \subseteq 1 \circ (1 \circ 1)$. (c) Let $y = 2$, consider two cases $2 < 1$ and $2 \not< 1$. If $2 < 1$, then $0 \in 2 \circ 1$. Therefore $1 \in 1 \circ 0 \subseteq 1 \circ (2 \circ 1)$. If $2 \not< 1$, then $0 \notin 2 \circ 1$. Thus $2 \circ 1 = \{1\}$, $\{2\}$ or $\{1, 2\}$. If $2 \circ 1 = \{1\}$, then $0 \in 1 \circ (2 \circ 1)$ and therefore $1 \circ (2 \circ 1) < I$. Since I is implicative, by Theorem 2.13 we conclude that $1 \in I$, which is a contradiction. If $2 \circ 1 = \{2\}$, we show that $1 \circ 2 = \{1\}$.

To do this, we show that $0 \notin 1 \circ 2$ and $2 \notin 1 \circ 2$. If $0 \in 1 \circ 2$, then we have $0 \in 1 \circ 2 = 1 \circ (2 \circ 1)$, so $1 \circ (2 \circ 1) < I$. Since I is implicative, by Theorem 2.13 we conclude that $1 \in I$, which is a contradiction. If $2 \in 1 \circ 2$, then $0 \in 2 \circ (1 \circ 2)$, therefore $2 \circ (1 \circ 2) < I$. Since I is implicative, by Theorem 2.12 we conclude that $2 \in I$, which is a contradiction. Therefore $1 \circ 2 = \{1\}$, so $1 \in 1 \circ (2 \circ 1)$. Now, let $2 \circ 1 = \{1, 2\}$. Hence $0 \in (1 \circ 1) \cup (1 \circ 2) = 1 \circ \{1, 2\} = 1 \circ (2 \circ 1)$, thus $1 \circ (2 \circ 1) < I$. Since I is implicative, by Theorem 2.13 we conclude that $1 \in I$, which is a contradiction.

(iii) If $x = 2$ then by the same argument as in the case (ii) we can obtain that $2 \in 2 \circ (y \circ 2)$ for all $y \in H$. $\square$

Remark 3.24. If in the above theorem we replace $I = \{0\}$ by $I = \{0, 1\}$, then the theorem does not hold. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper K-algebra structure on H:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

Here $0 \in H$ is a scalar element and $I = \{0, 1\}$ is an implicative hyper K-ideal, but H is not an implicative hyper$K$-algebra since $1 \notin 1 \circ (2 \circ 1)$.

References

[1] *A. Borumand Saeid, R. A. Borzooei and M. M. Zahedi*: (Weak) implicative hyper K-ideals. Bull. Korean Math. Soc. *40* (2003), 123–137.

[2] *R. A. Borzooei, P. Corsini and M. M. Zahedi*: Some kinds of positive implicative hyper K-ideals. Journal of Discrete Mathematical Sciences and Cryptography *6* (2003), 97–108.

[3] *R. A. Borzooei, A. Hasankhani, M. M. Zahedi and Y. B. Jun*: On hyper K-algebras. Math. Japon. *52* (2000), 113–121.

[4] *R. A. Borzooei and M. M. Zahedi*: Positive implicative hyper K-ideals. Scientiae Mathematicae Japonicae *53* (2001), 525–533.

[5] *Y. Imai and K. Iseki*: On axiom systems of propositional calculi. XIV Proc. Japan Academy *42* (1966), 19–22.

[6] *Y. B. Jun, M. M. Zahedi, X. L. Xin and R. A. Borzooei*: On hyper BCK-algebras. Ital. J. Pure Appl. Math. (2000), 127–136.

[7] *F. Marty*: Sur une generalization de la notion de groups. 8th Congress Math. Scandinaves, Stockholm. 1934, pp. 45–49.

[8] *J. Meng and Y. B. Jun*: BCK-Algebras. Kyung Moonsa, Seoul, 1994.

[9] *D. Mundici*: MV-algebras are categorically equivalent to bounded commutative BCK-algebras. Math. Japon. *31* (1986), 889–894.

[10] *M. M. Zahedi, R. A. Borzooei and H. Rezaei*: Some classification of hyper K-algebras of order 3. Scientiae Mathematicae Japonicae *53* (2001), 133–142.

[11] *M. M. Zahedi, R. A. Borzooei, Y. B. Jun and A. Hasankhani*: Some results on hyper K-algebra. Scientiae Mathematicae *3* (2000), 53–59.

Authors' addresses: M. M. Zahedi, Dept. of Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran, e-mail: zahedi_mm@mail.uk.ac.ir; A. Borumand Saeid, Dept. of Mathematics, Islamic Azad University, Kerman, Iran, e-mail: arsham@iauk.ac.ir; R. A. Borzooei, Dept. of Mathematics, Sistan and Baluchestan University, Zahedan, Iran, e-mail: borzooei@hamoon.usb.ac.ir.