Czechoslovak Mathematical Journal, 55 (130) (2005), 423–432

WEIGHTED INEQUALITIES FOR INTEGRAL OPERATORS WITH SOME HOMOGENEOUS KERNELS

María Silvina Riveros and Marta Urciuolo, Córdoba

(Received August 1, 2002)

Abstract. In this paper we study integral operators of the form

$$Tf(x) = \int |x - a_1 y|^{-\alpha_1} \dots |x - a_m y|^{-\alpha_m} f(y)\,\mathrm{d}y,$$

$\alpha_1 + \ldots + \alpha_m = n$. We obtain the $L^p(w)$ boundedness for them, and a weighted $(1,1)$ inequality for weights w in A_p satisfying that there exists $c \geqslant 1$ such that $w(a_i x) \leqslant c w(x)$ for a.e. $x \in \mathbb{R}^n$, $1 \leqslant i \leqslant m$. Moreover, we prove $\|Tf\|_{\mathrm{BMO}} \leqslant c\|f\|_\infty$ for a wide family of functions $f \in L^\infty(\mathbb{R}^n)$.

Keywords: weights, integral operators

MSC 2000: 42B25, 42A50, 42B20

1. Introduction

In [7] the authors study the boundedness on $L^2(\mathbb{R})$ of the operator

$$Tf(x) = \int |x - y|^{-\alpha} |x + y|^{\alpha - 1} f(y)\,\mathrm{d}y,$$

$0 < \alpha < 1$.

In [3] the authors study integral operators of the form

$$Tf(x) = \int_{\mathbb{R}^n} |x - y|^{-\alpha} |x + y|^{-n+\alpha} f(y)\,\mathrm{d}y,$$

$0 < \alpha < n$. They obtain the $L^p(\mathbb{R}^n, \mathrm{d}x)$ boundedness and the weak type $(1,1)$ of them.

Partially supported by CONICET, Agencia Córdoba Ciencia and SECYT-UNC.

In this paper we consider integral operators defined for f belonging to the Schwartz class $S(\mathbb{R}^n)$ by

$$Tf(x) = \int_{\mathbb{R}^n} |x - a_1 y|^{-\alpha_1} \dots |x - a_m y|^{-\alpha_m} f(y)\,\mathrm{d}y, \tag{1.1}$$

$\alpha_1 + \ldots + \alpha_m = n$, $\alpha_i > 0$ and $a_i \in \mathbb{R} - \{0\}$ for $i = 1, \ldots, m$.

We take the Hardy-Littlewood maximal function as

$$Mf(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(x)|\,\mathrm{d}x$$

where the supremum is taken along all cubes Q such that x belongs to Q. We recall that a weight w is a measurable, non negative and locally integrable function. It is well known that, for $p > 1$, M is bounded on $L^p(w)$ if and only if there exists $c > 0$ such that

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w\right) \left(\frac{1}{|Q|} \int_Q w^{-1/(p-1)}\right)^{p-1} \leqslant c. \tag{1.2}$$

The class of functions that satisfy (1.2) is denoted by A_p. For $p = 1$, the class A_1 is defined by

$$Mw(x) \leqslant cw(x)$$

for a.e. $x \in \mathbb{R}^n$ and for some positive constant c. The weak type (1,1) of the maximal function is equivalent to $w \in A_1$. These classes A_p have been defined by Muckenhoupt (see [6]) in the one dimensional case and for higher dimensions by Coifmann and Fefferman (see [1]).

In this paper we obtain the boundedness of T on $L^p(\mathbb{R}^n, w)$ and a weighted (1,1) inequality for a wide class of weights w in A_p. We prove the following result:

Theorem 1. *Let T be defined by* (1.1). *Suppose there exists $c \geqslant 1$ such that $w(a_i x) \leqslant cw(x)$ for $1 \leqslant i \leqslant m$ and for almost every $x \in \mathbb{R}^n$.*

a) *If $w \in A_p$, $1 < p < \infty$, then T is bounded on $L^p(\mathbb{R}^n, w)$.*

b) *If $w \in A_1$ then there exists $k > 0$ such that, for $\lambda > 0$ and $f \in S(\mathbb{R}^n)$,*

$$w(\{x\colon\ |Tf(x)| > \lambda\}) \leqslant \frac{k}{\lambda} \int |f(x)| w(x)\,\mathrm{d}x.$$

We also analyze the boundedness of the operator T from L^∞ into BMO, the classical space consisting of functions with bounded mean oscillation, defined by

John and Nirenberg in [5]. Precisely, we say that $f \in L^1_{\mathrm{loc}}$ belongs to BMO if there exist $c > 0$ such that

$$\frac{1}{|Q|}\int \left| f(x) - \frac{1}{|Q|}\int f \right| \mathrm{d}x \leqslant c$$

for all cubes $Q \subset \mathbb{R}^n$. The smallest bound c for which the above inequality holds is called $\|f\|_*$. From the techniques used, the following result follows inmediately:

Theorem 2. *Let T be defined by* (1.1). *Then there exists $c > 0$ such that*

$$\|Tf\|_* \leqslant c\|f\|_\infty$$

for all $f \in S(\mathbb{R}^n)$.

If f is a positive constant then $Tf(x) = \infty$ for all $x \in \mathbb{R}^n$, so we cannot expect a general boundedness from L^∞ into BMO. With techniques similar to those developed in [8], we obtain

Theorem 3. *Let T be defined by* (1.1).

a) *If $f \in L^\infty$ and $T|f|(x_0) < \infty$ for some $x_0 \in \mathbb{R}^n$ then $Tf(x)$ is well defined for all $x \neq 0$ and $Tf \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.*

b) *There exists $c > 0$ such that*

$$\|Tf\|_* \leqslant c\|f\|_\infty$$

for all f as in a).

By c we denote a positive constant, not the same at each occurrence.

Proof of the main results

We follow the argument developed in [2, p. 144] where the case of the Calderón-Zygmund operators is treated. As there we define, for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the sharp maximal function by

$$M^{\#}(f)(x) = \sup_{Q\colon\, x\in Q} \frac{1}{|Q|}\int_Q |f - f_Q|(y)\,\mathrm{d}y$$

with $f_Q = |Q|^{-1}\int_Q f$.

We denote $D = \max\limits_{1\leqslant i\leqslant m} |a_i^{-1}|$ and $d = \min\limits_{1\leqslant i\leqslant m} |a_i^{-1}|$. We need the following result:

Lemma 1.3. *If T is defined by* (1.1) *and $s > 1$ then there exists $c > 0$ such that for all $f \in S(\mathbb{R}^n)$,*

$$M^{\#}(Tf)(x) \leqslant c[(Mf^s(a_1^{-1}x))^{1/s} + \ldots + (Mf^s(a_m^{-1}x))^{1/s}].$$

P r o o f. We first observe that T is a bounded operator on $L^p(\mathbb{R}^n, \mathrm{d}x)$, $1 < p < \infty$ (see [4]), so for $f \in S(\mathbb{R}^n)$, $Tf \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $M^{\#}(Tf)(x)$ is well defined for all $x \in \mathbb{R}^n$. We take $x \in \mathbb{R}^n$ such that $T|f|(x) < \infty$ and Q a cube that contains x. We set $l(Q)$ as the length of the side of Q, denote by $\overline{Q}$ the cube with the same center as Q, such that $l(\overline{Q}) \geqslant 2D/d \cdot l(Q)$ and, for $1 \leqslant i \leqslant m$, we also set $\overline{Q}_i = a_i^{-1}\overline{Q}$. We decompose $f = f_1 + f_2$, $f_1 = f\chi_{\bigcup_{1\leqslant k\leqslant m} \overline{Q}_k}$ and take $a = Tf_2(x)$. Then

$$\frac{1}{|Q|}\int_Q |Tf(y) - a|\,\mathrm{d}y \leqslant \frac{1}{|Q|}\int_Q |Tf_1(y)|\,\mathrm{d}y + \frac{1}{|Q|}\int_Q |Tf_2(y) - Tf_2(x)|\,\mathrm{d}y.$$

If $s > 1$ then T is bounded on $L^s(\mathbb{R}^n, \mathrm{d}x)$ (see [4]), so

$$\begin{aligned}\frac{1}{|Q|}\int_Q |Tf_1(y)|\,\mathrm{d}y &\leqslant \left(\frac{1}{|Q|}\int_Q |Tf_1(y)|^s\,\mathrm{d}y\right)^{1/s}\\ &\leqslant c\left(\left(\frac{1}{|Q|}\int_{\overline{Q}_1} |f(y)|^s\,\mathrm{d}y\right)^{1/s} + \ldots + \left(\frac{1}{|Q|}\int_{\overline{Q}_m} |f(y)|^s\,\mathrm{d}y\right)^{1/s}\right)\\ &\leqslant c'\big[(Mf^s(a_1^{-1}x))^{1/s} + \ldots + (Mf^s(a_m^{-1}x))^{1/s}\big].\end{aligned}$$

On the other hand,

$$\frac{1}{|Q|}\int_Q |Tf_2(y) - Tf_2(x)|\,\mathrm{d}y \leqslant \frac{1}{|Q|}\int_Q \left|\int_{\left(\bigcup_{1\leqslant k\leqslant m} \overline{Q}_k\right)^c} (K(y,z) - K(x,z))f(z)\,\mathrm{d}z\right|\mathrm{d}y$$

where we denote by $K(x,y)$ the kernel $|x - a_1y|^{-\alpha_1} \ldots |x - a_my|^{-\alpha_m}$.

We now estimate $|K(y,z) - K(x,z)|$.

Case $l(Q) \geqslant 2|x|$. In this situation $\bigcup\limits_{1\leqslant k\leqslant m} \overline{Q}_k \supset \{y\colon |y| < 3D|x|\}$. Indeed, if $z \in \left(\bigcup_{1\leqslant k\leqslant m} \overline{Q}_k\right)^c$, then $|z| \geqslant |z - a_1^{-1}x| - |a_1^{-1}x| \geqslant l(\overline{Q_1}) - D|x| \geqslant dl(\overline{Q}) - D|x| \geqslant 3D|x|$. Moreover, in this case $|x - a_1z| \leqslant |x| + |a_1z| \leqslant \big(|a_1| + \frac{1}{3D}\big)|z|$ then

$$\begin{aligned}|x - a_iz| &\geqslant |a_iz| - |x| \geqslant \Big(|a_i| - \frac{1}{3d}\Big)|z|\\ &\geqslant \Big(\frac{3|a_i|D - 1}{3|a_1|D + 1}\Big)\frac{1}{2}|x - a_1z|.\end{aligned} \tag{1.4}$$

Thus we apply the mean value theorem to obtain, for $x, y \in Q$ and $z \in \Big(\bigcup\limits_{1\leqslant k\leqslant m} \overline{Q}_k\Big)^c$,

$$|K(y,z) - K(x,z)| \leqslant |x-y| \sum_{i=1}^{m} \frac{\alpha_i}{|\xi - a_i z|^{\alpha_i+1} \prod\limits_{l\neq i} |\xi - a_l z|^{\alpha_l}}$$

for some ξ between x and y. But $|a_i^{-1}\xi - z| \geqslant |a_i^{-1}x - z| - |a_i^{-1}\xi - a_i^{-1}x| \geqslant \frac{1}{2}|a_i^{-1}x - z|$, so (1.4) implies

$$|K(y,z) - K(x,z)| \leqslant c\,\frac{|x-y|}{|x - a_1 z|^{n+1}}. \tag{1.5}$$

Thus

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \left| \int_{\left(\bigcup_{1\leqslant k\leqslant m} \overline{Q}_k\right)^c} (K(y,z) - K(x,z)) f(z)\,\mathrm{d}z \right| \mathrm{d}y \\
&\quad \leqslant \frac{c}{|Q|} \int_Q \sum_{k=1}^{\infty} \int_{2^k D l(Q) \leqslant |a_1^{-1}x - z| < 2^{k+1} D l(Q)} \frac{|x-y|}{|a_1^{-1}x - z|^{n+1}} |f(z)|\,\mathrm{d}z\,\mathrm{d}y \\
&\quad \leqslant c l(Q) \sum_{k=1}^{\infty} \frac{1}{2^k D l(Q)} \frac{1}{(2^k D l(Q))^n} \int_{|a_1^{-1}x - z| < 2^{k+1} D l(Q)} |f(z)|\,\mathrm{d}z \\
&\quad \leqslant cMf(a_1^{-1}x) \leqslant c(Mf^s(a_1^{-1}x))^{1/s}.
\end{aligned}$$

Case $l(Q) < 2|x|$. We decompose

$$\int_{\left(\bigcup_{1\leqslant k\leqslant m} \overline{Q}_k\right)^c} (K(y,z) - K(x,z)) f(z)\,\mathrm{d}z = \int_{|z|\geqslant 3D|x|} + \int_{\{|z|<3D|x|\}\cap\left(\bigcup_{1\leqslant k\leqslant m} \overline{Q}_k\right)^c}.$$

To estimate the first integral, we proceed as before and we obtain (1.5) for $x, y \in Q$ and $|z| \geqslant 3D|x|$, then

$$\frac{1}{|Q|}\int_Q \left| \int_{|z|\geqslant 3D|x|} (K(y,z) - K(x,z)) f(z)\,\mathrm{d}z \right| \mathrm{d}y \leqslant c(Mf^s(a_1^{-1}x))^{1/s}.$$

We now study the second integral. For $1 \leqslant i \leqslant m$, $x, y \in Q$ and $z \in \{z\colon |z| < 3D|x|\} \cap \Big(\bigcup\limits_{1\leqslant k\leqslant m} \overline{Q}_k\Big)^c$, we have

$$|a_i^{-1}y - z| \geqslant |a_i^{-1}x - z| - |a_i^{-1}y - a_i^{-1}x| \geqslant \frac{|a_i^{-1}x - z|}{2},$$

hence

$$|K(y,z) - K(x,z)| \leqslant c|K(x,z)|.$$

So

$$\int_{\{|z|<3D|x|\}\cap\left(\bigcup_{1\leqslant k\leqslant m}\overline{Q}_k\right)^c}(K(y,z)-K(x,z))f(z)\,\mathrm{d}z$$
$$\leqslant c\int_{\{z\colon\,|z|<3D|x|\}}\frac{|f(z)|}{|x-a_1z|^{\alpha_1}\dots|x-a_mz|^{\alpha_m}}\,\mathrm{d}z.$$

We define $b=\frac{1}{2}\min\limits_{1\leqslant l,\,j\leqslant m}(|a_l^{-1}-a_j^{-1}|)$. We set $A_i=\{z\colon\,|a_i^{-1}x-z|\leqslant b|x|\}$, $1\leqslant i\leqslant m$, and $A_{m+1}=\Big(\bigcup\limits_{i=1}^{m}A_i\Big)^c$ and decompose

$$\int_{\{z\colon\,|z|<3D|x|\}}\frac{|f(z)|}{|x-a_1z|^{\alpha_1}\dots|x-a_mz|^{\alpha_m}}\,\mathrm{d}z$$
$$=\int_{A_1}+\ldots+\int_{A_m}+\int_{A_{m+1}\cap\{z\colon\,|z|<3D|x|\}}.$$

For $z\in A_i$ and $l\neq i$ we have $|a_l^{-1}x-z|\geqslant b|x|$, hence

$$\int_{A_i}\frac{|f(z)|}{|x-a_1z|^{\alpha_1}\dots|x-a_mz|^{\alpha_m}}\,\mathrm{d}z$$
$$\leqslant\frac{c}{|x|^{n-\alpha_i}}\sum_{j=0}^{\infty}\int_{2^{-j-1}b|x|\leqslant|a_i^{-1}x-z|\leqslant2^{-j}b|x|}\frac{|f(z)|}{|a_i^{-1}x-z|^{\alpha_i}}\,\mathrm{d}z$$
$$\leqslant c\sum_{j=1}^{\infty}2^{j(\alpha_i-n)}\frac{1}{(2^{-j}b|x|)^n}$$
$$\times\int_{|z-a_i^{-1}x|\leqslant2^{-j}b|x|}|f(z)|\,\mathrm{d}z\leqslant cMf(a_i^{-1}x)\leqslant c(Mf^s(a_i^{-1}x))^{1/s}.$$

Now

$$\int_{A_{m+1}\cap\{z\colon\,|z|<3D|x|\}}\frac{|f(z)|}{|x-a_1z|^{\alpha_1}\dots|x-a_mz|^{\alpha_m}}\,\mathrm{d}z\leqslant c|x|^{-n}\int_{\{z\colon\,|z|<3D|x|\}}|f(z)|\,\mathrm{d}z$$
$$\leqslant cMf(a_1^{-1}x)\leqslant c(Mf^s(a_1^{-1}x))^{1/s},$$

and the lemma follows. □

Lemma 1.6. *Let T be defined by* (1.1), $1<p<\infty$, $w\in A_p$ *and* $f\in L^p(w)$. *Then* $Tf\in L^p(w)$.

P r o o f. If $\operatorname{supp}f\subset B(0,R)$ and $|x|>2R$ then $|K(x,y)|\leqslant c/|x|^n$ and so in this case $|Tf(x)|\leqslant c_R/|x|^n$. The proof follows as in Theorem 7.18 in [2], since T is a bounded operator on $L^p(\mathbb{R}^n,\mathrm{d}x)$ (see [4]). □

P r o o f of Theorem 1. a) Taking account of Lemmas 1.3 and 1.6, we proceed as in the proof of Theorem 7.18 in [2] to obtain, for $f \in S(\mathbb{R}^n)$,

$$\begin{aligned}\int |Tf(x)|^p w(x)\,\mathrm{d}x &\\ &\leqslant c\int |(Mf^s(a_1^{-1}x))^{1/s} + \ldots + (Mf^s(a_m^{-1}x))^{1/s}|^p w(x)\,\mathrm{d}x\\ &\leqslant c\int |Mf^s(x)|^{p/s} w(a_1x)\,\mathrm{d}x + \ldots + \int |Mf^s(x)|^{p/s} w(a_mx)\,\mathrm{d}x\\ &\leqslant c\int |Mf^s(x)|^{p/s} w(x)\,\mathrm{d}x.\end{aligned}$$

The last inequality follows from the hypothesis about the weight w. The rest of the proof is as in Theorem 7.18 in [2].

b) For $\lambda > 0$ we perform the Calderón-Zygmund decomposition for f to obtain a sequence of disjoint $\{Q_j\}_{j\in\mathbb{N}}$ such that $f(x) \leqslant \lambda$ for almost every $x \notin \bigcup\limits_{j\in\mathbb{N}} Q_j$. We take

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \bigcup\limits_{j\in\mathbb{N}} Q_j,\\ \dfrac{1}{|Q_j|}\displaystyle\int_{Q_j} f & \text{if } x \in Q_j\end{cases}$$

and write $f = g + b$.

As usual, from a), we obtain

$$w\{x\colon\ |Tg(x)| > \lambda\} \leqslant \frac{c}{\lambda}\int |f(x)|w(x)\,\mathrm{d}x.$$

For each $i = 1, \ldots, m$ and $j \in \mathbb{N}$ we denote by $\overline{Q_j}$ the cube with the same center as Q_j and such that $l(\overline{Q_j}) \geqslant 2D/d \cdot l(Q_j)$, and $\overline{Q_{j,i}} = a_i\overline{Q_j}$. We obtain

$$\begin{aligned} w\Big(\bigcup_{j\in\mathbb{N}} \overline{Q_{j,i}}\Big) &\leqslant \sum_{j\in\mathbb{N}} w(\overline{Q_{j,i}}) \leqslant c\sum_{j\in\mathbb{N}} \frac{w(\overline{Q_{j,i}})}{|\overline{Q_{j,i}}|}|\overline{Q_{j,i}}|\\ &\leqslant c\sum_{j\in\mathbb{N}} |Q_j|\frac{w(\overline{Q_{j,i}})}{|\overline{Q_{j,i}}|} \leqslant \sum_{j\in\mathbb{N}} \frac{c}{\lambda}\int_{Q_j} |f|\frac{w(\overline{Q_{j,i}})}{|\overline{Q_{j,i}}|}\\ &\leqslant \frac{c}{\lambda}\sum_{j\in\mathbb{N}} \int_{Q_j} |f(y)|Mw(a_iy)\,\mathrm{d}y\\ &\leqslant \frac{c}{\lambda}\int |f(y)|w(a_iy)\,\mathrm{d}y \leqslant \frac{c}{\lambda}\int |f(y)|w(y)\,\mathrm{d}y.\end{aligned}$$

Then

$$w\Big(\bigcup_{j\in\mathbb{N},\ i=1,\ldots,m} \overline{Q_{j,i}}\Big) \leqslant \frac{c}{\lambda}\int |f(y)|w(y)\,\mathrm{d}y.$$

Now for each fixed $i = 1, \dots, m$, if c_j denotes the center of Q_j, we have

$$\begin{aligned} &w\bigg(\{x\colon |Tb(x)| > \lambda\} \cap \bigg(\bigcup_{j\in\mathbb{N}} \overline{Q_{j,i}}\bigg)^c\bigg) \\ &\quad\leqslant \frac{c}{\lambda} \sum_{j\in\mathbb{N}} \int_{(\overline{Q_{j,i}})^c} \bigg|\int_{Q_j} b_j(y)(K(x,y) - K(x,c_j))\,\mathrm{d}y\bigg| w(x)\,\mathrm{d}x \\ &\quad\leqslant \frac{c}{\lambda} \sum_{j\in\mathbb{N}} \int_{Q_j} |b_j(y)| \int_{(\overline{Q_{j,i}})^c} |K(x,y) - K(x,c_j)| w(x)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Now we observe that $K(x,y) = c\widetilde{K}(y,x)$ where $\widetilde{K}(x,y) = |x - a_1^{-1}y|^{-\alpha_1} \dots |x - a_m^{-1}y|^{-\alpha_m}$. Reasoning as in a) with $\widetilde{K}$ instead of K and using the hypothesis on w, we get

$$\int_{(\overline{Q_{j,i}})^c} |K(x,y) - K(x,c_j)| w(x)\,\mathrm{d}x \leqslant cMw(a_i y) \leqslant cw(y).$$

So

$$\begin{aligned} &w\bigg(\{x\colon |Tb(x)| > \lambda\} \cap \bigg(\bigcup_{j\in\mathbb{N},\, i=1,\dots,m} \overline{Q_{j,i}}\bigg)^c\bigg) \\ &\quad\leqslant \frac{c}{\lambda} \int |b(y)| w(y)\,\mathrm{d}y \leqslant \frac{c}{\lambda} \int |f(y)| w(y)\,\mathrm{d}y. \end{aligned}$$

□

P r o o f of Theorem 2. It follows straightforward from Lemma 1.3. □

P r o o f of Theorem 3. a) Let $f \in L^\infty(\mathbb{R}^n)$ and let x_0 be such that $T|f|(x_0) < \infty$. We take $R = 4D|x_0|$, denote $B = B(0,R) = \{x \in \mathbb{R}^n\colon |x| \leqslant R\}$, define $f_1 = |f|\chi_B$ and decompose $|f| = f_1 + f_2$. Then

$$\begin{aligned} Tf_1(x) &\leqslant \int_B |x - a_1 y|^{-\alpha_1} \dots |x - a_m y|^{-\alpha_m} f(y)\,\mathrm{d}y \\ &\leqslant \|f\|_\infty \int_B |x - a_1 y|^{-\alpha_1} \dots |x - a_m y|^{-\alpha_m}\,\mathrm{d}y. \end{aligned}$$

If $x \neq 0$ we choose $r > 0$ such that $r = \frac{1}{4} \min\limits_{1\leqslant i,\, k\leqslant m} |a_i^{-1} - a_k^{-1}|\,|x|$. For $1 \leqslant i \leqslant m$, we define $B_i = B(a_i^{-1}x, r)$. We have

$$\begin{aligned} &\int_B |x - a_1 y|^{-\alpha_1} \dots |x - a_m y|^{-\alpha_m}\,\mathrm{d}y \\ &\quad\leqslant \sum_{1\leqslant i\leqslant m} \int_{B_i} |x - a_1 y|^{-\alpha_1} \dots |x - a_m y|^{-\alpha_m}\,\mathrm{d}y \\ &\qquad + \int_{B\cap\left(\bigcup_{1\leqslant i\leqslant m} B_i\right)^c} |x - a_1 y|^{-\alpha_1} \dots |x - a_m y|^{-\alpha_m}\,\mathrm{d}y. \end{aligned}$$

Now

$$\int_{B_i} |x-a_1y|^{-\alpha_1}\dots|x-a_my|^{-\alpha_m}\,\mathrm{d}y$$
$$\leqslant c\prod_{k\neq i} r^{-\alpha_k}\int_{B_i}|x-a_iy|^{-\alpha_i}\,\mathrm{d}y \leqslant c\prod_{k\neq i} r^{-\alpha_k}r^{-\alpha_i+n} = c.$$

If $|a_i^{-1}x| < 2R$ for some $1\leqslant i\leqslant m$, then, for $y\in B\cap(B_i)^c$, we have $r<|a_i^{-1}x-y|\leqslant 3R$ and so

$$\begin{aligned}\int_{B\cap\left(\bigcup_{1\leqslant i\leqslant m}B_i\right)^c}&|x-a_1y|^{-\alpha_1}\dots|x-a_my|^{-\alpha_m}\,\mathrm{d}y\\ &\leqslant c\prod_{k\neq i} r^{-\alpha_k}\int_{B\cap(B_i)^c}|x-a_iy|^{-\alpha_i}\,\mathrm{d}y\\ &\leqslant c\prod_{k\neq i} r^{-\alpha_k}\int_r^{3R} t^{-\alpha_i+n-1}\,\mathrm{d}t\\ &= c\prod_{k\neq i} r^{-\alpha_k}[(3R)^{\alpha_i+n}-r^{-\alpha_i+n}] = c\left(|x|^{\sum_{k\neq i}-\alpha_k}+1\right),\end{aligned}$$

so for $x\neq 0$ and such that $|a_i^{-1}x|<2R$ we obtain

$$|Tf_1(x)|\leqslant c\|f\|_\infty\left(1+|x|^{\sum_{k\neq i}-\alpha_k}\right). \tag{1.7}$$

Now if $|a_i^{-1}x|\geqslant 2R$ for all $1\leqslant i\leqslant m$, then $|a_i^{-1}x-y|\geqslant R$ for $y\in B(0,R)$ and so

$$|Tf_1(x)|\leqslant\|f\|_\infty.$$

So (1.7) holds for all $x\neq 0$. Then $Tf_1(x)<\infty$ for all $x\neq 0$ and it belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Now $Tf_2(x_0)<\infty$ so we write, for $x\in\mathbb{R}^n$, $Tf_2(x)=Tf_2(x)-Tf_2(x_0)+Tf_2(x_0)$. Then we have to study

$$\int_{B^c}|K(x,y)-K(x_0,y)|\,|f|(y)\,\mathrm{d}y.$$

For $x\neq 0$ we have

$$\begin{aligned}\int_{B^c}|K(x,y)-K(x_0,y)|\,|f|(y)\,\mathrm{d}y\leqslant&\int_{B^c\cap B(0,4D|x|)^c}|K(x,y)-K(x_0,y)|\,|f|(y)\,\mathrm{d}y\\ &+\int_{B^c\cap B(0,4D|x|)}|K(x,y)|\,|f|(y)\,\mathrm{d}y+c.\end{aligned}$$

To estimate the first integral, we proceed as in the proof of Lemma 1.3 to obtain that, for $y \in B^c \cap B(0, 4D|x|)^c$,

$$|K(x,y) - K(x_0,y)| \leqslant c\frac{|x-x_0|}{|x-a_1y|^{n+1}},$$

so

$$\begin{aligned}\int_{B^c\cap B(0,4D|x|)^c} |K(x,y) - K(x_0,y)|\,|f|(y)\,\mathrm{d}y &\leqslant c|x-x_0| \int_{B^c} \frac{|f|(y)}{|x-a_1y|^{n+1}}\,\mathrm{d}y \\ &\leqslant c|x-x_0|\|f\|_\infty.\end{aligned}$$

To study the second integral, we observe that it appears only if $D|x| \geqslant R/4$, so we proceed as in the previous estimate for Tf_1 to obtain that, for x in this region,

$$\int_{B^c\cap B(0,4D|x|)} |K(x,y)|\,|f|(y)\,\mathrm{d}y \leqslant c\|f\|_\infty.$$

So, for $x \neq 0$, $Tf_2(x) < \infty$ and it belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

b) If f satisfies the hypothesis of a) we obtain that $M^\#(Tf)(x)$ is well defined for all $x \in \mathbb{R}^n$, so Lemma 1.3 still holds for these functions, and b) follows. □

References

[1] *R. Coifmann and C. Fefferman*: Weighted norm inequalities for maximal functions and singular integrals. Studia Math. *51* (1974), 241–250.

[2] *J. Duoandikoetxea*: Análisis de Fourier. Ediciones de la Universidad Autónoma de Madrid, Editorial Siglo XXI, 1990.

[3] *T. Godoy and M. Urciuolo*: About the L^p boundedness of some integral operators. Revista de la UMA *38* (1993), 192–195.

[4] *T. Godoy and M. Urciuolo*: On certain integral operators of fractional type. Acta Math. Hungar. *82* (1999), 99–105.

[5] *F. John and L. Nirenberg*: On functions of bounded mean oscillation. Comm. Pure Appl. Math. *14* (1961), 415–426.

[6] *B. Muckenhoupt*: Weighted norm inequalities for the Hardy maximal function. Trans. Amer. Math. Soc. *165* (1972), 207–226.

[7] *F. Ricci and P. Sjögren*: Two parameter maximal functions in the Heisenberg group. Math. Z. *199* (1988), 565–575.

[8] *A. de la Torre and J. L. Torrea*: One-sided discrete square function. Studia Math. *156* (2003), 243–260.

Authors' address: María Silvina Riveros, Marta Urciuolo, FaMAF Universidad Nacional de Córdoba, Ciem-CONICET, Ciudad Universitaria 5000 Córdoba, e-mails: sriveros@mate.uncor.edu, urciuolo@mate.uncor.edu.