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EXTENSIONS OF GM-RINGS

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Abstract. It is shown that a ring R is a GM-ring if and only if there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents such that all e_iRe_i are GM-rings. We also investigate GM-rings for Morita contexts, module extensions and power series rings.

Keywords: GM-ring, module extension, power series ring

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Many authors have studied associative rings with many units and many idempotents (cf. [1]–[6], [8], [9], [12] and [13]). A ring R is said to satisfy the GM-condition provided that for any $x, y \in R$, there exists a $u \in U(R)$ such that $x - u, y - u^{-1} \in U(R)$. In [6], K.R. Goodearl and P. Menal showed that many known rings satisfy the GM-condition. In [8], J. Han and W.K. Nicholson studied extensions of clean rings. A ring R is called a clean ring if for any $x \in R$, there exists $e = e^2 \in R$ such that $x - e \in U(R)$. To extend the GM-condition and clean rings, the first author introduced GM-rings (cf. [5]). We say that a ring R is a GM-ring provided that for any $x, y \in R$ there exist idempotents $e, f \in R$ and $u \in U(R)$ such that $x - eu, y - fu^{-1} \in U(R)$. Clearly, all clean rings and all rings satisfying the GM-condition are GM-rings.

In this paper we show that a ring R is a GM-ring if and only if there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents such that all e_iRe_i are GM-rings. We also investigate GM-rings for Morita contexts, module extensions and power series rings. These give generalizations of [5, Theorem 8] and [8, Theorem].

Throughout, all rings are associative with identity. $GL_n(R)$ stands for the general linear group of R, $U(R)$ stands for the set of units of R and we use $J(R)$ to denote the Jacobson radical of R.

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Let $e_1, e_2, \ldots, e_n \in R$ be idempotents. Clearly,

$$\begin{pmatrix} e_1Re_1 & \ldots & e_1Re_n \\ \vdots & \ddots & \vdots \\ e_nRe_1 & \ldots & e_nRe_n \end{pmatrix} = \left\{ \begin{pmatrix} e_1r_{11}e_1 & \ldots & e_1r_{1n}e_n \\ \vdots & \ddots & \vdots \\ e_1r_{n1}e_1 & \ldots & e_1r_{nn}e_n \end{pmatrix} : r_{ij} \in R \ (1 \leqslant i, j \leqslant n) \right\}$$

forms a ring with the identity $\operatorname{diag}(e_1, \ldots, e_n)$. Now we extend [5, Theorem 8] as follows.

Lemma 1. *Let $e_1, \ldots, e_n$ be idempotents of a ring R. If all e_iRe_i are GM-rings, then so is the ring*

$$\begin{pmatrix} e_1Re_1 & \ldots & e_1Re_n \\ \vdots & \ddots & \vdots \\ e_nRe_1 & \ldots & e_nRe_n \end{pmatrix}.$$

Proof. Clearly, the result holds for $n = 1$. Now assume that the result holds for $m \geqslant 1$. For any $A_1', A_2' \in \begin{pmatrix} e_1Re_1 & \ldots & e_1Re_{m+1} \\ \vdots & \ddots & \vdots \\ e_{m+1}Re_1 & \ldots & e_{m+1}Re_{m+1} \end{pmatrix}$, write $A_1' = \begin{pmatrix} A_1 & B_1 \\ C_1 & d_1 \end{pmatrix}$ and $A_2' = \begin{pmatrix} A_2 & B_2 \\ C_2 & d_2 \end{pmatrix}$, where $A_1, A_2 \in \begin{pmatrix} e_1Re_1 & \ldots & e_1Re_m \\ \vdots & \ddots & \vdots \\ e_mRe_1 & \ldots & e_mRe_m \end{pmatrix}$, B_1, B_2, C_1 and C_2 are m-vectors, and $d_1, d_2 \in e_{m+1}Re_{m+1}$. We can find

$$E_1 = E_1^2, \quad E_2 = E_2^2 \in \begin{pmatrix} e_1Re_1 & \ldots & e_1Re_m \\ \vdots & \ddots & \vdots \\ e_mRe_1 & \ldots & e_mRe_m, \end{pmatrix},$$

$$U, V_1, V_2 \in U\left(\begin{pmatrix} e_1Re_1 & \ldots & e_1Re_m \\ \vdots & \ddots & \vdots \\ e_mRe_1 & \ldots & e_mRe_m \end{pmatrix}\right)$$

such that $A_1 - E_1U = V_1$ and $A_2 - E_2U^{-1} = V_2$. Because $d_1 - C_1V_1^{-1}B_1, d_2 - C_2V_2^{-1}B_2 \in e_{m+1}Re_{m+1}$, we have $e_1 = e_1^2 \in e_{m+1}Re_{m+1}$ and $u, v_1, v_2 \in U(e_{m+1}Re_{m+1})$ such that $d_1 - C_1V_1^{-1}B_1 = e_1u + v_1$ and $d_2 - C_2V_2^{-1}B_2 = e_2u^{-1} + v_2$. Set

$$F_1 = \begin{pmatrix} E_1 & 0 \\ 0 & e_1 \end{pmatrix}, \quad W = \begin{pmatrix} U & 0 \\ 0 & u \end{pmatrix} \quad \text{and} \quad K_1 = \begin{pmatrix} V_1 & B_1 \\ C_1 & v_1 + C_1V_1^{-1}B_1 \end{pmatrix}.$$

It is easy to verify that $F_1 = F_1^2 \in \begin{pmatrix} e_1Re_1 & \dots & e_1Re_{m+1} \\ \vdots & \ddots & \vdots \\ e_{m+1}Re_1 & \dots & e_{m+1}Re_{m+1} \end{pmatrix}$ and

$$\begin{aligned} K_1 &\begin{pmatrix} V_1^{-1} + V_1^{-1}B_1v_1^{-1}C_1V_1^{-1} & -V_1^{-1}B_1v_1^{-1} \\ -v_1^{-1}C_1V_1^{-1} & v_1^{-1} \end{pmatrix} \\ &= \begin{pmatrix} V_1^{-1} + V_1^{-1}B_1v_1^{-1}C_1V_1^{-1} & v_1^{-1}C_1V_1^{-1} \\ -V_1^{-1}B_1v_1^{-1} & v_1^{-1} \end{pmatrix} K_1 \\ &= \operatorname{diag}(e_1, \dots, e_{m+1}). \end{aligned}$$

This means that F_1 is an idempotent and K_1 is a unit. Moreover, $A_1' = F_1W + K_1$ and W is a unit. Analogously, we have an idempotent $F_2 = \begin{pmatrix} E_2 & 0 \\ 0 & e_2 \end{pmatrix}$ and a unit $K_2 = \begin{pmatrix} V_2 & B_2 \\ C_2 & v_2 + C_2V_2^{-1}B_2 \end{pmatrix}$ such that $A_2' = F_2W^{-1} + K_2$. By induction hypothesis, we conclude that $\begin{pmatrix} e_1Re_1 & \dots & e_1Re_n \\ \vdots & \ddots & \vdots \\ e_nRe_1 & \dots & e_nRe_n \end{pmatrix}$ is a GM-ring, as asserted.

$\square$

Theorem 2. *The following conditions are equivalent:*

(1) *R is a GM-ring.*

(2) *There exists a complete orthogonal set $\{e_1, \dots, e_n\}$ of idempotents such that all e_iRe_i are GM-rings.*

P r o o f. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) We construct a map

$$\varphi\colon R \to \begin{pmatrix} e_1Re_1 & \dots & e_1Re_n \\ \vdots & \ddots & \vdots \\ e_nRe_1 & \dots & e_nRe_n \end{pmatrix}$$

given by $\varphi(r) = \begin{pmatrix} e_1re_1 & \dots & e_1re_n \\ \vdots & \ddots & \vdots \\ e_nre_1 & \dots & e_nre_n \end{pmatrix}$. Since $\{e_1, \dots, e_n\}$ is a complete orthogonal set of idempotents, we claim that φ is a ring homomorphism. Assume that $\varphi(r) = 0$. Then e_ire_j are all zero for $1 \leqslant i, j \leqslant n$, hence $r = (e_1re_1 + \dots + e_1re_n) + \dots + (e_nre_1 + \dots + e_nre_n) = 0$. This means that φ is a monomorphism.

Given any

$$\begin{pmatrix} e_1r_{11}e_1 & \dots & e_1r_{1n}e_n \\ \vdots & \ddots & \vdots \\ e_nr_{n1}e_1 & \dots & e_nr_{nn}e_n \end{pmatrix} \in \begin{pmatrix} e_1Re_1 & \dots & e_1Re_n \\ \vdots & \ddots & \vdots \\ e_nRe_1 & \dots & e_nRe_n \end{pmatrix},$$

we have a $t := (e_1r_{11}e_1 + \ldots + e_1r_{1n}e_n) + \ldots + (e_nr_{n1}e_1 + \ldots + e_nr_{nn}e_n) \in R$ such that

$$\varphi(t) = \begin{pmatrix} e_1r_{11}e_1 & \dots & e_1r_{1n}e_n \\ \vdots & \ddots & \vdots \\ e_nr_{n1}e_1 & \dots & e_nr_{nn}e_n \end{pmatrix}.$$

So φ is an epimorphism, and then

$$\varphi\colon\ R \cong \begin{pmatrix} e_1Re_1 & \dots & e_1Re_n \\ \vdots & \ddots & \vdots \\ e_nRe_1 & \dots & e_nRe_n \end{pmatrix}.$$

By virtue of Lemma 1, R is a GM-ring. □

As an immediate consequence, we show that if R is a GM-ring so also is the matrix ring $M_n(R)$. Furthermore, we can derive the following corollary.

Corollary 3. *Let $M_1, \ldots, M_n$ be right R-modules. If $\mathrm{End}_R(M_1), \ldots, \mathrm{End}_R(M_n)$ are GM-rings, then so is $\mathrm{End}_R(M_1 \oplus \ldots \oplus M_n)$.*

P r o o f. Let $e_1, \ldots, e_n$ be the idempotents for $M = M_1 \oplus \ldots \oplus M_n$. Then they are orthogonal and $1_{\mathrm{End}_R(M)} = e_1 + \ldots + e_n$. That is, we have a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of $\mathrm{End}_R(M)$. Moreover, all $e_i \,\mathrm{End}_R(M) e_i \cong \mathrm{End}_R(M_i)$ are GM-rings. In view of Theorem 2, the result follows. □

A Morita context denoted by (A, B, M, N, ψ, Φ) consists of two rings A, B, two bimodules ${}_AN_B$, ${}_BM_A$ and a pair of bimodule homomorphisms (called pairings) $\psi\colon N \bigotimes_B M \to A$ and $\Phi\colon M \bigotimes_A N \to B$ which satisfy the following associativity: $\psi(n, m)n' = n\Phi(m, n')$, $\Phi(m, n)m' = m\psi(n, m')$ for any $m, m' \in M$, $n, n' \in N$. These conditions ensure that the set T of generalized matrices $\begin{pmatrix} a & n \\ m & b \end{pmatrix}$; $a \in A$, $b \in B$, $m \in M$, $n \in N$ forms a ring, called the ring of the context. A. Haghany studied hopficity and co-hopficity for Morita contexts with zero pairings. Now we give a simple proof of [5, Theorem 8].

Proposition 4. *Let T be the ring of a Morita context (A, B, M, N, ψ, Φ). If A and B are GM-rings, then T is also a GM-ring.*

Proof. Set $e = \operatorname{diag}(1, 0)$. Then $eTe \cong \operatorname{diag}(A, 0)$ and $(1-e)T(1-e) \cong \operatorname{diag}(0, B)$. Since A and B are GM-rings, we directly verify that eTe and $(1-e)T(1-e)$ are GM-rings as well. Clearly, $\{e, 1-e\}$ is a complete orthogonal set of idempotents. Thus we obtain the result by Theorem 2. $\square$

Corollary 5. *Let T be the ring of a Morita context (A, B, M, N, ψ, Φ). If A and B are semiperfect rings, then T is also a GM-ring.*

Proof. Since R is a semiperfect ring, it is a GM-ring. Thus we complete the proof by Proposition 4. $\square$

Let A_1, A_2, A_3 be associative rings with identities, let M_{21}, M_{31}, M_{32} be (A_2, A_1)-, (A_3, A_1)-, (A_3, A_2)-bimodules, respectively. Let $\Phi\colon M_{32} \bigotimes_{A_2} M_{21} \to M_{31}$ be an (A_3, A_1)-homomorphism, and let $T = \begin{pmatrix} A_1 & 0 & 0 \\ M_{21} & A_2 & 0 \\ M_{31} & M_{32} & A_3 \end{pmatrix}$ with the usual matrix operations (see [10]).

Theorem 6. *The following conditions are equivalent:*
(1) *A_1, A_2 and A_3 are GM-rings.*
(2) *The formal triangular matrix ring $T = \begin{pmatrix} A_1 & 0 & 0 \\ M_{21} & A_2 & 0 \\ M_{31} & M_{32} & A_3 \end{pmatrix}$ is a GM-ring.*

Proof. (1) $\Rightarrow$ (2) Let $B = \begin{pmatrix} A_2 & 0 \\ M_{32} & A_3 \end{pmatrix}$ and $M = \begin{pmatrix} M_{21} \\ M_{31} \end{pmatrix}$. Since A_2 and A_3 are GM-rings, so is the ring B by virtue of Theorem 4. In addition, A_1 is a GM-ring. Using Theorem 4 again, we see that $\begin{pmatrix} A_1 & 0 \\ M & B \end{pmatrix}$ is also a GM-ring, as required.

(2) $\Rightarrow$ (1) For any $x, y \in A_2$, we have $\begin{pmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T$. Since T is a GM-ring, we have idempotents

$$\begin{pmatrix} e_1 & 0 & 0 \\ * & e_2 & 0 \\ * & * & e_3 \end{pmatrix}, \begin{pmatrix} f_1 & 0 & 0 \\ * & f_2 & 0 \\ * & * & f_3 \end{pmatrix} \in T,$$

and a unit $\begin{pmatrix} u_1 & 0 & 0 \\ * & u_2 & 0 \\ * & * & u_3 \end{pmatrix} \in T$ such that

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} e_1 & 0 & 0 \\ * & e_2 & 0 \\ * & * & e_3 \end{pmatrix} \begin{pmatrix} u_1 & 0 & 0 \\ * & u_2 & 0 \\ * & * & u_3 \end{pmatrix} \in U(T)$$

and

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} f_1 & 0 & 0 \\ * & f_2 & 0 \\ * & * & f_3 \end{pmatrix} \begin{pmatrix} u_1 & 0 & 0 \\ * & u_2 & 0 \\ * & * & u_3 \end{pmatrix}^{-1} \in U(T).$$

One easily checks that $e_2 = e_2^2$, $f_2 = f_2^2$ and $u_2 \in U(R)$. Furthermore, we have $x - e_2u_2, y_2 - f_2u_2^{-1} \in U(R)$. Therefore A_2 is a GM-ring. Likewise, we claim that A_1 and A_3 are GM-rings, as asserted. □

Corollary 7. *A ring R is a GM-ring if and only if so is the ring of all $n \times n$ lower triangular matrices over R is a GM-ring.*

Proof. According to Theorem 6, the result follows. □

Analogously, we deduce that a ring R is a GM-ring if and only if the ring of all $n \times n$ upper triangular matrices over R is a GM-ring.

Recall that a ring R is called an exchange ring if for every right R-module A and any two decompositions $A = M' \oplus N = \bigoplus_{i \in I} A_i$, where $M'_R \cong R_R$ and the index set I is finite, there exist submodules $A'_i \subseteq A_i$ such that $A = M' \oplus \big(\bigoplus_{i \in I} A'_i\big)$. The class of exchange rings includes local rings, semiperfect rings, semiregular rings, π-regular rings, strongly π-regular rings and C^*-algebras with real rank one (cf. [1], [14] and [16]).

Corollary 8. *Let R be an exchange ring with artinian primitive factors. Then the ring of all $n \times n$ lower (upper) triangular matrices over R is a GM-ring.*

Proof. Applying Corollary 7, we get the result. □

As every exchange ring of bounded index has artinian primitive factors, we deduce the following result.

Corollary 9. *Let R be an exchange ring of bounded index. Then the ring of all $n \times n$ lower (upper) triangular matrices over R is a GM-ring.*

Let $TM_2(R)$ be the ring of all 2×2 lower triangular matrices over R. Define $QM_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a+c = b+d,\ \ a,b,c,d \in R \right\}$. Then $QM_2(R)$ is a ring with the identity $\operatorname{diag}(1,1)$.

Corollary 10. *A ring R is a GM-ring if and only if so is $QM_2(R)$.*

Proof. Construct a map $\psi\colon QM_2(R) \to TM_2(R)$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+c & 0 \\ c & d-c \end{pmatrix}$ for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in QM_2(R)$. For any $\begin{pmatrix} x & 0 \\ z & y \end{pmatrix} \in TM_2(R)$, we have

$$\psi\left(\begin{pmatrix} x-z & x-y-z \\ z & y+z \end{pmatrix}\right) = \begin{pmatrix} x & 0 \\ z & y \end{pmatrix}.$$

Thus ψ is an epimorphism. It is easy to verify that ψ is a monomorphism; hence, it is a ring isomorphism. Therefore we complete the proof by Corollary 7. $\square$

If M is a R-R-bimodule, then the module extension of R by M is the ring $R \bowtie M$ with the usual addition and multiplication defined by $(r_1,m_1)(r_2,m_2) = (r_1r_2, r_1m_2 + m_1r_2)$ for $r_1,r_2 \in R$ and $m_1,m_2 \in M$. Now we investigate GM-rings for module extensions and introduce a large class of such rings.

Theorem 11. *Let R be an introduce ring, M a R-R-bimodule. Then the following conditions are equivalent:*
(1) *R is a GM-ring.*
(2) *$R \bowtie M$ is a GM-ring.*

Proof. (1) $\Rightarrow$ (2) Given any $(r_1,m_1),(r_2,m_2) \in R \bowtie M$, we have idempotents $e,f \in R$ and units $u,v_1,v_2 \in R$ such that $r_1 - eu = v_1,\ r_2 - fu^{-1} = v_2$. One easily verifies that $(r_1,m_1) - (e,0)(u,0) = (v_1,0) \in U(R \bowtie M)$ and $(r_2,m_2) - (e,0)(u^{-1},0) = (v_2,0) \in U(R \bowtie M)$. Clearly, $(u,0)^{-1} = (u^{-1},0) \in U(R \bowtie M)$. Hence $R \bowtie M$ is a GM-ring.

(2) $\Rightarrow$ (1) Given any $r_1,r_2 \in R$, then $(r_1,0),(r_2,0) \in R \bowtie M$. Thus we have idempotents $(e,m_1),(f,m_2) \in R \bowtie M$ and a unit $(u,n) \in R \bowtie M$ such that $(r_1,0) - (e,m_1)(u,n), (r_2,0) - (f,m_2)(u,n)^{-1} \in U(R \bowtie M)$. Obviously, $e,f \in R$ are idempotents and $u \in U(R)$. Moreover, we claim that $r_1 - eu, r_2 - fu^{-1} \in U(R)$. So R is a GM-ring, as asserted. $\square$

Corollary 12. *Let R be a ring. Then R is a GM-ring if and only if so is $R \bowtie R$.*

P r o o f. It is an immediate consequence of Theorem 11. $\square$

Corollary 13. *Let R be an exchange ring with artinian primitive factors. Then $R \bowtie R$ is a GM-ring.*

P r o o f. Since R is an exchange ring with artinian primitive factors, it is a GM-ring. Thus we get the result by Corollary 12. $\square$

Theorem 14. *Let R be an exchange ring. Then the following conditions are equivalent:*

(1) *R is a GM-ring.*

(2) *$R[[x_1, \ldots, x_n]]$ is a GM-ring.*

P r o o f. (1) $\Rightarrow$ (2) It suffices to show that the result holds for $n = 1$. Given any $f(x_1), g(x_1) \in R[[x_1]]$, we have $f(0), g(0) \in R$. Since R is a GM-ring, we can find idempotents $e, f \in R$ and a unit $u \in R$ such that $f(0) - eu, g(0) - fu^{-1} \in U(R)$. It is well known that $h(x_1) \in R[[x_1]]$ is a unit if and only if $h(0) \in R$ is a unit. Therefore we can find $f'(x_1), g'(x_1) \in R[[x_1]]$ such that $f(x_1) - eu = (f(0) - eu) + f'(x_1)x_1, g(x_1) - fu^{-1} = (g(0) - fu^{-1}) + g'(x_1)x_1 \in U(R[[x_1]])$, as required.

(2) $\Rightarrow$ (1) We also prove that the result holds for $n = 1$. Given any $x, y \in R$, we have $x, y \in R[[x_1]]$ as well. Thus we can find idempotents $e(x_1), f(x_1) \in R[[x_1]]$ and a unit $u(x_1) \in R[[x_1]]$ such that $x - e(x_1)u(x_1), y - f(x_1)u(x_1)^{-1} \in U(R[[x_1]])$. Thus we know that $x - e(0)u(0), y - f(0)u(0)^{-1} \in U(R)$. One easily checks that $e(0), f(0)$ are idempotents and $u(0) \in R$ is a unit. So we complete the proof. $\square$

Corollary 15. *Let R be an exchange ring with artinian primitive factors. Then $R[[x_1, \ldots, x_n]]$ is a GM-ring.*

P r o o f. Since every exchange ring with artinian primitive factors is a GM-ring, we get the result from Theorem 14. $\square$

Know that every semiperfect ring is a GM-ring, by virtue of Theorem 14, we can derive the following corollary:

Corollary 16. *Let R be a semiperfect ring. Then $R[[x_1, \ldots, x_n]]$ is a GM-ring.*

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