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PRÜFER RINGS WITH INVOLUTION

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Abstract. The concept of a Prüfer ring is studied in the case of rings with involution such that it coincides with the corresponding notion in the case of commutative rings.

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1. Introduction

Rings with involution have been studied intensively, especially in some applications to Lie algebras, Jordan algebras, and rings of operators. More recently, the category of rings with involution has been taken under investigation (see [1]). The ideals of an object in this category must be closed under the involution *, and are called *-ideals.

The main purpose of this paper is to study the concept of a Prüfer ring in the case of rings with involution such that it coincides with the corresponding notion in the case of commutative rings. This is done in Section 3 for prime Goldie rings. In Section 4, we consider domains with involution. We give generalizations for some of the conditions of a commutative domain to be a Prüfer domain. We show that a domain R is a Prüfer ring if for any two *-ideals $A \subseteq B$ of R, there exists a *-ideal C of R with $A = BC$. A commutative integral domain R has a distributive lattice of ideals if and only if the localizations R_M of R at any maximal ideal M are valuation rings. These rings are called Dedekind rings if they are also noetherian. In Section 4, we extend this result to noncommutative domains with involution.

In [2], Dubrovin generalizes the concept of a Prüfer ring to orders in simple Artinian rings, but that concept does not extend the class of Prüfer domains in the commutative case. To study this concept in the case of rings with involution, it must extend the class of commutative Prüfer domains, because when we take the involution to be the identity, the ring turns out to be commutative.

2. Definitions and basic facts

In this section, we state the basic definitions and some facts that will be needed in this work. All rings considered will be noncommutative with unity and with involution * (an anti-automorphism of period 2). A subring of a ring must contain the unity. If R is a *-closed subring ($r \in R$ implies $r^* \in R$), then we simply say that R is a *-subring. For a *-closed additive subgroup I of a ring with involution Q, the order of I is defined to be the *-subring $O(I) = \{q \in Q\colon qI \subseteq I, q^*I \subseteq I\}$, or equivalently $O(I) = \{q \in Q\colon Iq \subseteq I, Iq^* \subseteq I\}$. We also define the inverse of I to be $I^{-1} = \{q \in Q\colon IqI \subseteq I,\ Iq^*I \subseteq I\}$. Clearly I^{-1} is *-closed. The quotient of a *-subring R by a *-closed subset A of R is defined as $[R : I] = \{q \in Q\colon qA \subseteq R,\ q^*A \subseteq R\}$ or equivalently $[R : I] = \{q \in Q\colon Aq \subseteq R,\ Aq^* \subseteq R\}$. Let R be a *-subring of Q. A *-closed R-submodule I of Q is called a fractional *-ideal of R if there is a regular element (an element which is not a zero-divisor) $d \in Q$ such that $dI \subseteq R,\ d^*I \subseteq R$. Since I and R are *-closed, we also have $Id \subseteq R,\ Id^* \subseteq R$, and hence we do not need to define right and left fractional *-ideals. Clearly, every fractional *-ideal is a fractional ideal. Also, each *-ideal of R is a fractional *-ideal of R. We note that, if I is a fractional *-ideal, then $[R : I]$ is a fractional *-ideal. A fractional *-ideal I is called invertible if $I^{-1}I = R$.

Lemma 1. *Let I be a fractional *-ideal of a *-subring R and assume that there exist a *-closed subset M of Q such that $MI = R$. Then $O(I) = R$.*

Proof. Since I is an R-submodule, hence $R \subseteq O(I)$. Conversely, for $q \in O(I)$ we have $MIq \subseteq MI$ so that $Rq \subseteq R$ and $q \in R$. □

Lemma 2. *Let I be a fractional *-ideal of a *-ring R. Then $[R : I]I = R$ if and only if I is invertible.*

Proof. Assume $I^{-1}I = R$. Since $O(I)R \subseteq O(I)$, Lemma (1) implies $O(I)I^{-1}I \subseteq R$ so that $O(I)I^{-1} \subseteq [R : I]$ and $O(I) \subseteq [R : I]I$, i.e., $R \subseteq [R : I]I$. Hence $R = [R : I]I$. Conversely, assume $[R : I]I = R$. Using the definitions we have $[R : I]I \subseteq I^{-1}I \subseteq O(I)$. So, $R \subseteq I^{-1}I \subseteq R$ and $R = I^{-1}I$. □

Let R be a subring of Q, R is said to be symmetric if $aR = a^*R$ for every $a \in Q$. R is called invariant in Q if $aR = Ra$ for every $a \in Q$. These definitions generalize those given in the case of a division ring D with involution (a symmetric subring of D is a subring which contains $x^{-1}x^*$ for every non-zero element x in D). Also, when the involution is the identity then Q is commutative and every subring R of Q will be symmetric and invariant.

We note that every symmetric subring R is *-closed, because for $r \in R$ we have $r^* \in r^*R = rR \subseteq R$. Then R is a symmetric subring if and only if $Ra = Ra^*$ for every $a \in Q$.

Lemma 3. *Every symmetric subring R is invariant.*

Proof. We first note that $abR = baR$ for every $a, b \in Q$, because $abR = b^*a^*R = b^*aR = a^*bR = a^*b^*R = baR$. Similarly, $Rab = Rba$ for every $a, b \in Q$. Now, let $ra \in Ra$ for $r \in R$. Then $ra \in raR = arR \subseteq aR$, so that $Ra \subseteq aR$. Similarly, $aR \subseteq Ra$ and R is invariant. □

Lemma 4. *Let R be a symmetric ring. Then each ideal of R is *-closed, two sided, and such that $xy \in I$ implies $yx \in I$.*

Proof. Let I be a left ideal of R, $x \in I$, then $x^* \in Rx^* = Rx \subseteq RI \subseteq I$ and I is *-closed. Now, let $x \in I$, $y \in R$; then $x^* \in I$ and $y^* \in R$ so that $(xy)^* = y^*x^* \in I$. Since I is *-closed, it follows that $xy \in I$ and I is a right ideal. Similarly, if I is a right ideal then it is *-closed and a left ideal. Finally, if $xy \in I$, then $x^*y^* \in Rx^*y^*R = RxyR \subseteq RIR \subseteq I$. Hence $yx = (x^*y^*)^* \in I$ as I is *-closed. □

If R is any ring with identity, then R is called a right chain ring if $aR \subseteq bR$ or $bR \subseteq aR$ for any a, b in R. Similarly, left chain rings are defined. For *-rings all chain rings are two sided because, if R is a right chain *-ring, then $a^*R \subseteq b^*R$ or $b^*R \subseteq a^*R$ for $a, b \in R$ so that $Ra \subseteq Rb$ or $Rb \subseteq Ra$ and R is also a left chain ring.

In the case of a division ring D with involution, a subring $R \subseteq D$ is called total if for every non-zero x in D, x or $x^{-1} \in R$. R is called a valuation ring if it is total and symmetric. Total rings and chain rings are the same in this case as follows from the following proposition.

Proposition 5. *Let R be a subring of a *-division ring D. Then R is a chain ring if and only if R is total in D.*

Proof. If R is total, then $a^{-1}b \in R$ or $b^{-1}a \in R$ for every non-zero a, b in D. Hence $a^{-1}bR \subseteq R$ or $b^{-1}aR \subseteq R$, so that $bR \subseteq aR$ or $aR \subseteq bR$ and R is a chain ring. Conversely, if R is a chain ring then $aR \subseteq R$ or $R \subseteq aR$. Hence $aR \subseteq R$ or $a^{-1}R \subseteq R$ so that $a \in R$ or $a^{-1} \in R$ for every non-zero a in D. □

Corollary 6. *A symmetric subring $R \subset D$ is a chain ring if and only if it is a valuation ring.*

This shows that chain rings play for symmetric rings the role valuation rings play for division rings. Finally, let us look at rings of quotients of rings with involution. The following proposition shows that the involution makes the situation symmetric.

Proposition 7. *Rings with involution have only two-sided rings of quotients.*

Proof. Let R be a ring with involution, Q a right ring of quotients of R. Then every element x in Q is of the form $x = as^{-1}$, where $a \in R$, and s is a regular element in R. Since s^* is also a regular element, then there is an element $b \in R$ and a regular element t such that $a^*t = s^*b$. Taking the involution we get $t^*a = b^*s$, so that $x = as^{-1} = (t^*)^{-1}b^*$ and Q is a left ring of quotients of R. $\square$

Proposition 8 ([3]). *Let R be a ring with involution, Q a ring of quotients of R. Then the involution of R extends uniquely to Q.*

For a general noncommutative ring R the existence of a localization is a difficult problem, so we will assume in the next section that R is a prime Goldie ring with involution and so R has a ring of quotients Q with involution extending the involution in R. In this case, $Q = M_n(D)$, the ring of $n \times n$ matrices over a division ring D, or $Q = M_n(D) \oplus M_n^{\mathrm{op}}(D)$ where the latter is endowed with the exchange involution (see [3]).

Another class of rings with involution that have rings of quotients are those studied in [4]. Such rings with involution satisfy a commutative condition on the products of norms, and have rings of quotients where symmetric elements are invertible. If, further, these rings are semiprime rings, then their rings of quotients are division rings, direct sums of a division ring and its opposite, or 2×2 matrices over a field (see [4]).

3. Prüfer rings

A *-subring R is called a Prüfer ring if every finitely generated fractional *-ideal I of R is invertible. If R is a commutative Prüfer domain, then every finitely generated fractional ideal is invertible (see [5]), so every commutative Prüfer domain is a Prüfer ring, where the involution can be taken as the identity. Also, if R is a symmetric Prüfer subring of a division ring with involution, then R is a Prüfer ring (see [6]). We first show that an overring of a Prüfer ring is also a Prüfer ring.

Proposition 9. *Let R be a symmetric prime Goldie ring which is a Prüfer ring. Then any symmetric overring of R is also a Prüfer ring.*

P r o o f. Let Q be the *-ring of quotients of R. Let S be a symmetric overring of R, i.e., S is a symmetric subring of Q such that $R \subseteq S \subseteq Q$. Clearly every regular element of S is invertible in Q and hence S is a prime Goldie ring with Q as its *-ring of quotients.

Suppose that J is a finitely generated fractional *-ideal of S, say $J = a_1S + \ldots + a_nS$. Consider $I = a_1R + \ldots + a_nR$. Then $IS = J$ and I is a finitely generated fractional *-ideal of R, so that $I^{-1}I = R$. Since $I^{-1}J = I^{-1}(IS) = (I^{-1}I)S = RS = S$, we have $O(J) = S$ by Lemma 1. Also, $I^{-1} \subseteq J^{-1}$ implies that$R = I^{-1}I \subseteq J^{-1}J$ so that $1 \in J^{-1}J$. Thus, $J^{-1}J = O(J) = S$, i.e., J is invertible and S is a Prüfer ring. □

It is well known that a commutative domain R is a Prüfer domain if and only if the localizations R_P at prime ideals P of R are valuation domains (see [5]). Also, in the case of a symmetric subring R of a division ring with involution, R is a Prüfer ring if and only if the localizations R_M, at maximal *-ideals M of R are valuation rings (see [6]). For a noncommutative *-ring we can prove the following result under the assumption that localizations exist. A non-empty subset S of a *-ring R is called an Ore *-set if it is an Ore set and closed under *.

Theorem 10. *Let R be a symmetric prime Goldie ring with involution such that for every maximal *-ideal M of R, the set $S(M) = \{r \in R\colon r + M$ is a regular element in $R/M\}$ is an Ore *-set of elements regular in R, and the localization R_M with respect to that set is a valuation ring in Q, the ring of quotients of R. Then R is a Prüfer ring.*

P r o o f. Let $I = b_1R + \ldots + b_nR$ be a finitely generated fractional *-ideal of R. Let M be a maximal *-ideal of R. Then there exists a regular element a_M of R such that $IR_M = a_MR_M$ and hence $a_M^{-1}I \subseteq R_M$. Then $a_M^{-1}b_i \in a_M^{-1}I \subseteq R_M$ and $a_M^{-1}b_i = c_i^{-1}d_i$ where $c_i \in S(M)$ and $d_i \in R$ $(i = 1, \ldots, n)$. We can assume that $a_M^{-1}b_i = c^{-1}r_i$ where $c \in S(M)$ and $r_i \in R$ $(i = 1, \ldots, n)$. So

$$ca_M^{-1}I = ca_M^{-1}b_1R + \ldots + ca_M^{-1}b_nR = r_1R + \ldots + r_nR \subseteq R.$$

Since R is symmetric, hence $b_i^*R = b_iR$ $(i = 1, \ldots, n)$ and

$$\begin{aligned} Ica_M^{-1} &= b_1Rca_M^{-1} + \ldots + b_nRca_M^{-1} = b_1^*Rca_M^{-1} + \ldots + b_n^*Rca_M^{-1} \\ &= b_1^*(ca_M^{-1})^*R + \ldots + b_n^*(ca_M^{-1})^*R = r_1^*R + \ldots + r_n^*R \subseteq R. \end{aligned}$$

Hence $(ca_M^{-1})^* I \subseteq R$, and $ca_M^{-1} \in [R : I]$ follows. If $ca_M^{-1} I \subseteq M$, then $R_M = ca_M^{-1} I R_M \subseteq R_M \subseteq M R_M$, so that $R_M = M R_M$, i.e., R_M is the Jacobson radical of R_M, which is a contradiction (since R_M is a valuation ring). Therefore $ca_M^{-1} I \not\subseteq M$. So, $[R : I]I \not\subseteq M$ for every maximal *-ideal M of R. Hence $[R : I]I = R$, and by Lemma 2, I is invertible. □

For a converse of Theorem 10, one can adopt the proof of Theorem 3 in [2] to get the following theorem.

Theorem 11. *Suppose M is a *-ideal of a symmetric Prüfer ring R with involution such that R/M is an Artinian ring. Then $S(M) = \{r \in R\colon r + M$ is a regular element in $R/M\}$ is an Ore *-set of regular elements of R. Moreover, if R/M is a simple Artinian ring, then R_M is a valuation ring.*

For a prime *-ring R, the center Z is an integral domain. Consequently, all of the localizations R_M, for maximal ideals M of Z, exist and lie in a common ring of quotients Q of R. In this case we have the following result.

Proposition 12. *Let Z be the center of a prime *-ring R. If for any maximal ideal M of Z the localization R_M is a Prüfer ring, then R is also a Prüfer ring.*

P r o o f. Let I be a finitely generated fractional *-ideal of R. Then $(I^{-1})_M = (I_M)^{-1}$ for any maximal ideal M of Z. So,

$$I^{-1} I = \bigcap (I^{-1} I)_M = \bigcap (I^{-1})_M I_M = \bigcap R_M = R,$$

where all of these intersections are over the set of maximal ideals M of Z. □

4. Prüfer domains

For equivalent conditions for a commutative domain to be a Prüfer domain one refers to [5, Theorem 6.6]. We consider in this section symmetric *-domains.

Proposition 13. *Let R be a symmetric domain with involution and for any two *-ideals $A \subseteq B$ of R let there exist a *-ideal C of R with $A = BC$. Then the localization R_M is a chain ring for every maximal *-ideal M of R.*

P r o o f. Let M be a maximal *-ideal of R, let R_M be the corresponding local ring. For $a, b \in R$, aR and $aR + bR$ are two *-ideals of R and $aR \subseteq aR + bR$, hence there exists a *-ideal C of R such that $aR = (aR + bR)C$. So, $a = ax + by$ for some $x, y \in C$. If $x \in M$, then $1 - x$ is a unit in R_M and $a(1 - x) = by \in bR_M$ implies that $aR_M \subseteq bR_M$. If $x \notin M$, then $byx^{-1} = a(x^{-1} - 1)$. But $yx^{-1} \in CR_M = R_M$, and so $bR_M \subseteq aR_M$ follows. Thus R_M is a chain ring. □

Corollary 14. *Let R be a symmetric Ore domain with involution such that for any two *-ideals $A \subseteq B$ of R there exists a *-ideal C of R with $A = BC$. Then R is a Prüfer ring.*

P r o o f. R is now a symmetric subring of a division ring, the division ring of quotients of R. Thus, Corollary follows from Corollary 6 and [6, Theorem 11]. $\square$

Proposition 13 generalizes Theorem 6.6 (part 7) in [5] to the case of a noncommutative *-domain. To give a generalization of part 10 of the same theorem we start by giving first the meaning of a D-*-ring. A *-ring R is called a D-*-ring if for every three *-ideals A, B, C of R,

$$A \cap (B + C) = A \cap B + A \cap C,$$

i.e., the lattice of *-ideals is distributive.

Lemma 15. *Let R be a symmetric D-*-ring, M a maximal *-ideal. Then $S = R - M$ is an Ore *-set.*

P r o o f. Clearly S is *-closed since M is *-closed. To prove the right Ore condition, we note that for $r \in R$ the right ideals rR are two-sided *-ideals. Then for $r, s \in R$,

$$rR = rR \cap (sR + (r - s)R) = (rR \cap sR) + (rR \cap (r - s)R).$$

Hence $r = (r - s)t + x$ for $t \in R$, $x \in rR \cap sR$, so that

$$st = (r(t - 1) + a) \in rR \cap sR,$$

and

$$r(1 - t) = (s(-t) + a) \in rR \cap sR.$$

Now, let $r \in R$, $s \in S$. If $t \in M$, it follows that $1 - t \in S$ (otherwise $1 \in M$), and the right Ore condition is satisfied. If $t \notin M$, then $st = ru$ for some u in R and again the right Ore condition is satisfied. Since R and S are *-closed, the left Ore condition is also satisfied. $\square$

Lemma 16. *Let R and M be as in Lemma 15 and let R have no zero divisors, or be a noetherian ring. Then the ring of quotients R_M with respect to S exists.*

Proof. This is obvious if R has no zero divisors. Now, let R be a noetherian ring. Define $I = \{r \in R\colon rs = 0 \text{ for some } s \in S\}$, then I is a *-ideal of R. The image $\overline{S}$ of S in $\overline{R} = R/I$ is an Ore *-set and consists of regular elements, as $\overline{R}$ satisfies the maximum condition for annihilators in $\overline{R}$ (see [7]). Then one can form the ring $R_M = \{\overline{rs}^{-1}\colon \overline{r} \in \overline{R},\ \overline{s} \in \overline{S}\}$, the ring of quotients of $\overline{R}$ with respect to $\overline{S}$. □

Proposition 17. *Let R be a symmetric D-*-ring which has no zero divisors, or a noetherian ring. Then $S = R - M$ is an Ore *-set and $R_M = RS^{-1}$ is a chain ring for every maximal *-ideal M of R.*

Proof. By virtue of Lemma 16, it remains to show that the ring of quotients R_M is a chain ring. It is clear that R_M has a unique maximal ideal. Also, every principal ideal in R_M has the form $\overline{a}R_M$ for $\overline{a} \in \overline{R}$. Since R is a symmetric ring, $\overline{R}$ is also symmetric. Also, the lattice of *-ideals of R_M is distributive. Then as in the proof of Lemma 15, for any two elements $\overline{a}, \overline{b}$ in $\overline{R}$ there exists $\overline{t}$ in $\overline{R}$ with $\overline{a}(1 - \overline{t})$, $\overline{bt} \in \overline{a}R_M \cap \overline{b}R_M$. Either $\overline{t}$ or $1 - \overline{t}$ is in $\overline{S}$, i.e., a unit in R_M, hence $\overline{a} \in \overline{a}R_M \cap \overline{b}R_M$ or $\overline{b} \in \overline{a}R_M \cap \overline{b}R_M$. So, $\overline{a}R_M \subseteq \overline{b}R_M$ or $\overline{b}R_M \subseteq \overline{a}R_M$ and R_M is a chain ring. □

Theorem 18. *Let R be a symmetric ring which has no zero divisors or a noetherian ring. Then R is a D-*-ring if and only if $S = R - M$ is an Ore *-set and $R_M = RS^{-1}$ is a chain ring for every maximal *-ideal M of R.*

Proof. Due to Proposition 17, it remains to prove that R is a D-*-ring under the assumption that R_M as defined in the proof of Lemma 13 exists, and a chain ring for all maximal *-ideals M of R. Let φ be the canonical homomorphism from R onto $\overline{R} = R/I$. For a *-ideal L of R, we have

$$\varphi^{-1}(\overline{R} \cap \overline{L}R_M) = \{r \in R\colon rs \in L \text{ for some } s \in S\}.$$

So, $L = \bigcap \varphi^{-1}(\overline{R} \cap \overline{L}R_M)$. Then, for *-ideals L, J of R we have $L = J$ if and only if $\overline{L}R_M = \overline{J}R_M$ for all maximal *-ideals M. But for any three *-ideals A, B, C of R we have

$$(\overline{A} \cap (\overline{B} + \overline{C}))R_M = ((\overline{A} \cap \overline{B}) + (\overline{A} \cap \overline{C}))R_M,$$

as R_M is a chain ring. This proves that R is a D-*-ring. □

Corollary 19. *Let R be a symmetric Ore domain with involution. Then R is a D-*-ring if and only if R is a Prüfer domain.*

If R is a symmetric subring of a division ring with involution, then R is a Prüfer ring if and only if R_M is a valuation ring for every maximal *-ideal M of R (see [6]). Thus, in this case we have

Theorem 20. *Let R be a symmetric subring of a division ring with involution. The following conditions are equivalent:*
(1) *R is a Prüfer ring,*
(2) *R_M is a chain ring for every maximal *-ideal M of R,*
(3) *R_M is a valuation ring for every maximal *-ideal M of R,*
(4) *R_P is a valuation ring for every prime *-ideal P of R,*
(5) *R is D-*-ring.*

Let D be a division ring with involution. For a given preordering T of D, one can construct a subring consisting of elements of D, which are bounded by some rational number with respect to every ordering containing T. This subring $V = \{d \in D\colon\ r - dd^* \in T \text{ for some positive rational } r\}$ is called the bounded subring. It is shown in [6] that V is a noncommutative Prüfer domain. In fact, it is shown that this subring is the intersection of all valuation subrings of D which are compatible with the preordering T. This generalizes the commutative case (where $* =$ identity).

Let R be a *-ring, P an ordering of R (for the definitions of orderings and preorderings on *-rings one refers to [4]). The ideal $p = P \cap -P$ is a prime ideal, and is referred to as the support of P. Support p orderings on R are in correspondence with support zero orderings on the domain $\overline{R} = R/p$. So we can assume that R is an integral domain and P has support zero. If R is an Ore domain, then every support zero ordering on R extends uniquely to an ordering on the division ring of quotients of R.

Proposition 21. *For any preordering of an Ore *-domain R, the bounded subring V_o is a Prüfer ring.*

Proof. This follows from Proposition (16) in [6] and the fact that $V_o = V \cap R$, where V is the bounded subring of the extended preordering on the division ring of quotients of R. $\square$

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