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CONTACT ELEMENTS ON FIBERED MANIFOLDS

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Abstract. For every product preserving bundle functor T^μ on fibered manifolds, we describe the underlying functor of any order $(r,s,q), s \geqslant r \leqslant q$. We define the bundle $K^{r,s,q}_{k,l}Y$ of (k,l)-dimensional contact elements of the order (r,s,q) on a fibered manifold Y and we characterize its elements geometrically. Then we study the bundle of general contact elements of type μ. We also determine all natural transformations of $K^{r,s,q}_{k,l}Y$ into itself and of $T(K^{r,s,q}_{k,l}Y)$ into itself and we find all natural operators lifting projectable vector fields and horizontal one-forms from Y to $K^{r,s,q}_{k,l}Y$.

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It is well known that the product preserving bundle functors on the category $\mathcal{M}f$ of all manifolds coincide with the Weil functors, [7]. Recently it has been pointed out that every Weil algebra A determines an underlying Weil algebra A_k for every integer k, so that we have the underlying functors T^{A_k} of each Weil functor T^A, [5]. Moreover, the second author clarified that all product preserving bundle functors on the category $\mathcal{FM}$ of all fibered manifolds are of the form T^μ, where $\mu\colon A \to B$ is a homomorphism of Weil algebras, [10]. In the first part of the present paper we deduce there is an underlying Weil algebra homomorphism $\mu_{r,s,q}$ of μ for every integers r,s,q satisfying $s \geqslant r \leqslant q$. This defines the underlying functors $T^{\mu_{r,s,q}}$ of T^μ. In the case of a fibered velocities functor, our construction reduces to decreasing the order of fibered jets.

In the second part we start with the definition of the bundle $K^{r,s,q}_{k,l}Y$ of contact elements of dimension (k,l) and order (r,s,q), $s \geqslant r \leqslant q$, on a fibered manifold Y. Our approach is based on the classical formal construction by C. Ehresmann, [4, p. 356]. Then we clarify that the formally defined contact elements characterize properly the contact of fibered submanifolds of Y. Next we show how the recent ideas by J. Muñoz, R. J. Muriel and J. Rodríguez, [11], and the first author, [5], can be used for introducing the bundle $K^{\mu}Y \to Y$ of contact elements determined by an arbitrary Weil algebra homomorphism μ.

The last part of the present paper is devoted to some naturality problems. First we deduce that the only natural transformation of $K^{r,s,q}_{k,l}Y$ into itself is the identity. Then we prove that every natural operator transforming projectable vector fields on Y into vector fields on $K^{r,s,q}_{k,l}Y$ is a constant multiple of the flow operator. This implies that every natural transformation of the tangent bundle $TK^{r,s,q}_{k,l}Y$ into itself is a constant multiple of the identity. Finally we deduce that every natural operator transforming horizontal one-forms on Y into one-forms on $K^{r,s,q}_{k,l}Y$ is a constant multiple of the vertical lifting.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from [7].

1. The underlying functors of T^{μ}

We recall that the classical concept of r-jet can be generalized as follows. Consider a fibered manifold $p\colon Y \to M$ and a manifold Q. For two maps $f,\ g\colon Y \to Q$ we define $j^{r,s}_y f = j^{r,s}_y g$, $y \in Y$ by requiring the r-th order contact of f and g at y and the s-th order contact, $s \geqslant r$, of the restrictions to the fiber Y_x passing through $y, x = p(y)$, i.e.

$$j^r_y f = j^r_y g \quad \text{and} \quad j^s_y(f|Y_x) = j^s_y(g|Y_x). \tag{1}$$

The space of all such (r,s)-jets is denoted by $r,s(Y,Q)$.

If also Q is a fibered manifold $\pi\colon Z \to N$ and $f,\ g\colon Y \to Z$ are two $\mathscr{FM}$-morphisms, whose base maps are denoted by $\underline{f},\ \underline{g}\colon M \to N$, we can require a higher order contact of the base maps as well. Hence for every $q \geqslant r$ we define $j^{r,s,q}_y f = j^{r,s,q}_y g$ by (1) and

$$j^q_x \underline{f} = j^q_x \underline{g}. \tag{2}$$

If $h\colon Z \to W$ is another $\mathscr{FM}$-morphism, the formula

$$j^{r,s,q}_y (h \circ f) = \big(j^{r,s,q}_{f(y)} h\big) \circ \big(j^{r,s,q}_y f\big) \tag{3}$$

introduces a well defined composition of (r,s,q)-jets. The space of all (r,s,q)-jets of $\mathscr{FM}$-morphisms of Y into Z is denoted by $J^{r,s,q}(Y,Z)$.

A classical r-jet $X \in J^r_y(Y,Z)_z$ is called projectable if there is an r-jet $\underline{X} \in J^r_{p(y)}(M,N)_{\pi(z)}$ satisfying $(j^r_z\pi)\circ X = \underline{X}\circ(j^r_y p)$. One verifies easily that $J^{r,r,r}(Y,Z) \subset J^r(Y,Z)$ is the subspace of all projectable r-jets.

If $m = \dim M$ and $m+n = \dim Y$, we introduce the principal fiber bundle of all (r,s,q)-frames on Y by

$$P^{r,s,q}Y = \mathrm{inv}J^{r,s,q}_{0,0}(\mathbb{R}^{m,n}, Y),$$

where inv indicates the invertible jets and $(0,0) \in \mathbb{R}^m \times \mathbb{R}^n$. Its structure group is

$$G^{r,s,q}_{m,n} = \mathrm{inv}J^{r,s,q}_{0,0}(\mathbb{R}^{m,n}, \mathbb{R}^{m,n})_{0,0}$$

and both multiplication in $G^{r,s,q}_{m,n}$ and its action on $P^{r,s,q}Y$ are given by the jet composition. We define a bundle functor $T^{r,s}_{k,l}$ of $(k,l;r,s)$-velocities on $\mathscr{M}f$ by $T^{r,s}_{k,l}Q = J^{r,s}_{0,0}(\mathbb{R}^{k,l}, Q)$ for every manifold Q and

$$T^{r,s}_{k,l}f(j^{r,s}_{0,0}g) = j^{r,s}_{0,0}(f\circ g)\,, \qquad j^{r,s}_{0,0}g \in T^{r,s}_{k,l}Q \tag{4}$$

for every smooth map $f\colon\ Q \to \overline{Q}$. Moreover, we introduce a bundle functor $T^{r,s,q}_{k,l}$ of $(k,l;r,s,q)$-velocities on $\mathscr{F}\mathscr{M}$ by

$$T^{r,s,q}_{k,l}Y = J^{r,s,q}_{0,0}(\mathbb{R}^{k,l}, Y)$$

for every fibered manifold Y. Then every $\mathscr{F}\mathscr{M}$-morphism $f\colon\ Y \to Z$ induces $T^{r,s,q}_{k,l}f\colon$ $T^{r,s,q}_{k,l}Y \to T^{r,s,q}_{k,l}Z$ by means of the jet composition. One finds easily

$$T^{0,s,q}_{k,l}Y = T^q_k M \times_M V^s_l Y, \tag{5}$$

where $T^q_k M$ is the bundle of all (k,q)-velocities on M and $V^s_l Y$ is the bundle of all vertical (l,s)-velocities on Y.

Remark 1. If $E \to N$ is an epimorphism of vector spaces, then $J^{r,s,q}(Y,E) \to Y$ has an induced structure of a vector bundle. So we can define, analogously to Ehresmann, [4], a vector bundle over Y

$$T^{r,s,q*}_{k,l}Y = J^{r,s,q}(Y, \mathbb{R}^{k,l})_{0,0}. \tag{6}$$

Every $\mathscr{F}\mathscr{M}$-morphism $f\colon\ Y \to Z$, $f(y) = z$, induces a linear map

$$\lambda(j^{r,s,q}_y f)\colon\ (T^{r,s,q*}_{k,l}Z)_z \to (T^{r,s,q*}_{k,l}Y)_y \tag{7}$$

by means of the jet composition

$$\lambda(j^{r,s,q}_y f)(X) = X \circ (j^{r,s,q}_y f), \qquad X \in (T^{r,s,q*}_{k,l}Z)_z.$$

Similarly to [7, p. 123], if we denote by $T^{r,s,q\square}_{k,l}Y$ the dual vector bundle of (6) and define $T^{r,s,q\square}_{k,l}f\colon\ T^{r,s,q\square}_{k,l}Y \to T^{r,s,q\square}_{k,l}Z$ by using the dual maps to (7), we obtain another bundle functor $T^{r,s,q\square}_{k,l}$ on $\mathscr{F}\mathscr{M}$.

Clearly, the functor $T^{r,s,q}_{k,l}$ preserves products. The second author showed that the product preserving bundle functors on $\mathcal{FM}$ are in bijection with the homomorphisms $\mu\colon A \to B$ of Weil algebras, [10]. The functor T^μ determined by such a homomorphism is defined by

$$T^\mu Y = T^A M \times_{T^B M} T^B Y \tag{8}$$

where we consider the map $\mu_M\colon T^A M \to T^B M$ induced by μ and the submersion $T^B p\colon T^B Y \to T^B M$. For an $\mathcal{FM}$-morphism $f\colon Y \to Z$, one defines

$$T^\mu f = T^A \underline{f} \times_{T^B \underline{f}} T^B f\colon T^\mu Y \to T^\mu Z. \tag{9}$$

In the case of $T^{r,s,q}_{k,l}$, A is the jet algebra $\mathbb{D}^q_k = \mathbb{R}(k)/\mathfrak{m}(k)^{q+1}$, where $\mathbb{R}(k)$ is the algebra of polynomials in k variables and $\mathfrak{m}(k)$ is its maximal ideal,

$$\mathbb{D}^{r,s}_{k,l} = \mathbb{R}(k+l)/\langle \mathfrak{m}(k)\mathfrak{m}(k+l)^r, \mathfrak{m}(k+l)^{s+1}\rangle \tag{10}$$

and the homomorphism

$$\delta^{r,s,q}_{k,l}\colon \mathbb{D}^q_k \to \mathbb{D}^{r,s}_{k,l} \tag{11}$$

is induced by the canonical injection $\mathbb{R}(k) \to \mathbb{R}(k+l)$, [3]. So $T^A = T^q_k$ and $T^B = T^{r,s}_{k,l}$ in this case. For every $\overline{r} \leqslant r$, $\overline{s} \leqslant s$, $\overline{q} \leqslant q$, $\overline{s} \geqslant \overline{r} \leqslant \overline{q}$, the construction of lower order jets induces a natural transformation $T^{r,s,q}_{k,l} \to T^{\overline{r},\overline{s},\overline{q}}_{k,l}$. Generalizing [5], we introduce analogous underlying bundles for every T^μ.

Having a Weil algebra A, we write $A = \mathbb{R} \times N_A$, where N_A is the nilpotent ideal. For every integer q, we define the induced algebra A_q to be A/N^{q+1}_A, [5]. Since the order of A is the smallest integer $h = \operatorname{ord} A$ satisfying $N^{h+1}_A = 0$, we have $A_q = A$ for $q \geqslant \operatorname{ord} A$. Consider another Weil algebra $B = \mathbb{R} \times N_B$ and a homomorphism $\mu\colon A \to B$. For $s \geqslant r$, we define

$$B^\mu_{r,s} = B/\langle \mu(N_A)N^r_B, N^{s+1}_B\rangle. \tag{12}$$

If $q \geqslant r$, we have $\mu(N^{q+1}_A) \subset \mu(N_A)N^r_B$. So there is an induced Weil algebra homomorphism

$$\mu_{r,s,q}\colon A_q \to B^\mu_{r,s}. \tag{13}$$

Definition 1. The morphism (13) is called the underlying homomorphism of μ of the order (r,s,q), $s \geqslant r \leqslant q$.

Consider another Weil algebra homomorphism $\nu\colon C \to D$. By a morphism $f\colon \mu \to \nu$ we mean a pair $f = (f_1, f_2)$ of Weil algebra homomorphisms $f_1\colon A \to C$, $f_2\colon B \to D$ such that the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\mu} & B \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\
C & \xrightarrow{\nu} & D
\end{array}
\tag{14}
$$

We say that f is an epimorphism if both f_1 and f_2 are surjective. The group of all isomorphisms $\mu \to \mu$ will be denoted by Aut μ.

Since $f_1(N_A^{q+1}) \subset N_C^{q+1}$, there is an induced homomorphism $f_{1,q}\colon A_q \to C_q$. Similarly, we have $f_2(\langle \mu(N_A)N_B^r, N_B^{s+1}\rangle) \subset \langle \nu(N_C)N_D^r, N_D^{s+1}\rangle$, so that there is an induced homomorphism

$$f_{2,r,s}\colon B_{r,s}^{\mu} \to D_{r,s}^{\nu}.$$

Using the standard algebra, we deduce

Proposition 1. *We have*

$$f_{2,r,s} \circ \mu_{r,s,q} = \nu_{r,s,q} \circ f_{1,q}.$$

So, for every $s \geqslant r \leqslant q$, there is an induced morphism

$$f_{r,s,q} = (f_{1,q}, f_{2,r,s})\colon \mu_{r,s,q} \to \nu_{r,s,q}. \tag{15}$$

Definition 2. The functor $T^{\mu_{r,s,q}}$ is called the underlying functor of the order (r,s,q) of T^{μ}, $s \geqslant r \leqslant q$.

By [10] the natural transformations $T^{\mu} \to T^{\nu}$ are in bijection with the morphisms $\mu \to \nu$. So we have

Corollary 1. *Every natural transformation $T^{\mu} \to T^{\nu}$ is projectable over a natural transformation $T^{\mu_{r,s,q}} \to T^{\nu_{r,s,q}}$ for every $s \geqslant r \leqslant q$.*

Remark 2. In [5], the first author showed that $T^{A_r}M \to T^{A_{r-1}}M$ is in affine bundle, whose associated vector bundle is the pullback of $TM \otimes (N_A^r/N_A^{r+1})$ over $T^{A_{r-1}}M$. In the fibered case, one deduces in the same way the following two results.

(i) If $s > r$, then $T^{\mu_{r,s,q}}Y \to T^{\mu_{r,s-1,q}}Y$ is an affine bundle, whose associated vector bundle is the pullback of $VY \otimes (N_B^s/N_B^{s+1})$ over $T^{\mu_{r,s-1,q}}Y$, where VY denotes the vertical tangent bundle of Y.

(ii) If $q > r$, then $T^{\mu_{r,s,q}}Y \to T^{\mu_{r,s,q-1}}Y$ is an affine bundle, whose associated vector bundle is the pullback of $TM \otimes (N_A^q/N_A^{q+1})$ over $T^{\mu_{r,s,q-1}}Y$.

We recall that $X \in T^r_k M$ is said to be regular if X is r-jet of an immersion. For $k \leqslant m$, the subset $\operatorname{reg} T^r_k M$ of all regular elements is an open dense submanifold of $T^r_k M$. The bundle $K^r_k M$ of contact (k, r)-elements on M is the factor space

$$K^r_k M := \operatorname{reg} T^r_k M / G^r_k \tag{16}$$

with respect to the right action of G^r_k defined by the jet composition.

In the fibered case, an $\mathscr{FM}$-morphism $f\colon Y \to Z$ with the base map $\underline{f}$ will be called a fibered immersion if both f and $\underline{f}$ are immersions.

Definition 3. $\operatorname{reg} T^{r,s,q}_{k,l} Y \subset T^{r,s,q}_{k,l} Y$ is the subset of all (r, s, q)-jets of fibered immersions.

By (5) we have $T^{0,1,1}_{k,l} Y = T^1_k M \times_M V^1_l Y$. One verifies easily that

$$\operatorname{reg} T^{0,1,1}_{k,l} Y = \operatorname{reg} T^1_k M \times_M \operatorname{reg} V^1_l Y. \tag{17}$$

As a direct consequence of the definition, $X \in T^{r,s,q}_{k,l} Y$ is regular if and only if its projection into $T^{0,1,1}_{k,l} Y$ is regular. So we have

$$\operatorname{reg} T^{r,s,q}_{k,l} Y = \operatorname{reg} T^q_k M \times_{T^{r,s}_{k,l} M} \operatorname{reg} T^{r,s}_{k,l} Y, \tag{18}$$

where $\operatorname{reg} T^{r,s}_{k,l} Y \subset T^{r,s}_{k,l} Y$ is the subset of all (r, s)-jets of immersions.

Analogously to the manifold case, we introduce

Definition 4. The bundle $K^{r,s,q}_{k,l} Y$ of contact $(k, l; r, s, q)$-elements of Y is the factor space $\operatorname{reg} T^{r,s,q}_{k,l} Y / G^{r,s,q}_{k,l}$.

We show later in a more general setting that there is a canonical manifold structure on $K^{r,s,q}_{k,l} Y$. We shall need the following assertion.

Proposition 2. *The group* $G^{r,s,q}_{k,l}$ *coincides with* $\operatorname{Aut} \delta^{r,s,q}_{k,l}$.

P r o o f. Write $\delta = \delta^{r,s,q}_{k,l}$ for short. Let x_i or x_i, y_p be the generating elements of $\mathbb{R}(k)$ or $\mathbb{R}(k + l)$, respectively. The elements of $\mathbb{D}^q_k$ are polynomials in x_i of degree at most q. Each element of $\mathbb{D}^{r,s}_{k,l}$ is a polynomial of degree at most s in y_p and of degree at most r in the monomials that contain at least one x_i.

Consider a morphism $f\colon \delta \to \delta$. It is determined by the values $f_1(x_i) \in \mathbb{D}^q_k$ and $f_2(y_p) \in \mathbb{D}^{r,s}_{k,l}$. These data define an element of $J^{r,s,q}_{0,0}(\mathbb{R}^{k,l}, \mathbb{R}^{k,l})_{0,0}$. Consider $X_1 = j^q_0 \varphi_1 \in \mathbb{D}^q_k = J^r_0(\mathbb{R}^k, \mathbb{R})$ and $X_2 = j^{r,s}_{0,0} \varphi_2 \in \mathbb{D}^{r,s}_{k,l} = J^{r,s}_{0,0}(\mathbb{R}^{k,l}, \mathbb{R})$. Construct an $\mathscr{FM}$-morphism $\varphi\colon \mathbb{R}^{k,l} \to \mathbb{R}^{1,1}$, $\varphi(t, \tau) = (\varphi_1(t), \varphi_2(t, \tau))$, $t \in \mathbb{R}^k$, $\tau \in \mathbb{R}^l$. This identifies (X_1, X_2) with an element of $J^{r,s,q}_{0,0}(\mathbb{R}^{k,l}, \mathbb{R}^{1,1})$. In the covariant approach to natural transformations of Weil functors, [6], the action of the semigroup $\mathrm{Mor}(\delta, \delta)$ of all morphisms $\delta \to \delta$ corresponds to the composition of (r, s, q)-jets. This identifies $J^{r,s,q}_{0,0}(\mathbb{R}^{k,l}, \mathbb{R}^{k,l})_{0,0}$ with $\mathrm{Mor}(\delta, \delta)$. Clearly, the invertible (r, s, q)-jets correspond to the isomorphisms. □

We are going to describe how the contact $(k, l; r, s, q)$-elements on Y characterize the contact of fibered submanifolds of Y. We say that a submanifold $Z \subset Y$ is a fibered submanifold of Y if $N = p(Z) \subset M$ is a submanifold and the restricted and corestricted map $Z \to N$ is a fibered manifold. The fibered dimension of Z is the pair (k, l), $k = \dim N$, $k + l = \dim Z$. A local parametrization of Z is a fibered immersion $\varphi\colon \mathbb{R}^{k,l} \to Z$. Hence $j^{r,s,q}_{0,0}\varphi \in \operatorname{reg} T^{r,s,q}_{k,l} Y$ and the change of fibered parametrization at $(0, 0)$ corresponds to the jet composition of $j^{r,s,q}_{0,0}\varphi$ with an element $g \in G^{r,s,q}_{k,l}$.

We recall that, in the manifold case, every n-dimensional submanifold $N \subset M$ determines canonically a contact (n, r)-element $k^r_x N \subset K^r_n M$ for every $x \in N$, and two n-dimensional submanifolds $N, \overline{N} \subset M$ have r-th order contact at $x \in N \cap \overline{N}$ if $k^r_x N = k^r_x \overline{N}$, [4], [7].

Definition 5. We say that fibered submanifolds $Z, \overline{Z} \subset Y$ of the same fibered dimension (k, l) have a contact of order (r, s, q) at $y \in Z \cap \overline{Z}$, $s \geqslant r \leqslant q$, if

$$k^r_y Z = k^r_y \overline{Z} \;\; k^s_y Z_x = k^s_y \overline{Z}_x \quad \text{and} \quad k^q_x N = k^q_x \overline{N}, \tag{19}$$

where Z_x or $\overline{Z}_x$ is the fiber over $x = p(y)$.

We write $(k^r_y Z, k^s_y Z_x, k^q_x N) = k^{r,s,q}_y Z$ and say this is the contact (r, s, q)-element of Z at y. The identification of $k^{r,s,q}_y Z$ with an element of $K^{r,s,q}_{k,l} Y$ is based on the following assertion.

Proposition 3. *We have $k^{r,s,q}_y Z = k^{r,s,q}_y \overline{Z}$ iff there exist fibered parametrizations φ of Z and $\overline{\varphi}$ of $\overline{Z}$, $\varphi(0, 0) = y = \overline{\varphi}(0, 0)$, satisfying*

$$j^{r,s,q}_{0,0}\varphi = j^{r,s,q}_{0,0}\overline{\varphi}. \tag{20}$$

P r o o f. By the definition of composition of (r, s, q)-jets, (20) implies (19) directly. Conversely, assume (19). From the manifold case we know there is a local coordinate system $x^1, \ldots, x^m$ on M such that N or $\overline{N}$ can be parametrized in the form

$$x^a = t^a, x^b = \varphi^b(t^a) \quad \text{or} \quad x^b = \overline{\varphi}^b(t^a), \tag{21}$$

respectively, $a = 1, \ldots, k, b = k+1, \ldots, m$. Then $k^q_x N = k^q_x \overline{N}$ is equivalent to $j^q_0 \varphi^b = j^q_0 \overline{\varphi}^b$. Next we can add such fiber coordinates $x^{m+1}, \ldots, x^{m+n}$ on Y that the fibered parametrization of Z or $\overline{Z}$ is (21) and

$$x^{m+c} = \tau^c,\ x^{m+d} = \varphi^{m+d}(t^a, \tau^c) \quad \text{or} \quad x^{m+d} = \overline{\varphi}^{m+d}(t^a, \tau^c), \tag{22}$$

respectively, $c = 1, \ldots, l$, $d = l+1, \ldots, n$. In this situation, $k^r_y Z = k^r_y \overline{Z}$ implies $j^r_{0,0} \varphi^{m+d} = j^r_{0,0} \overline{\varphi}^{m+d}$. Finally, $k^s_y Z_x = k^s_y \overline{Z}_x$ is equivalent to $j^s_0 \varphi^{m+d}(0, \tau) = j^r_0 \overline{\varphi}^{m+d}(0, \tau)$. Thus, (19) is equivalent to $j^{r,s,q}_{0,0} \varphi = j^{r,s,q}_{0,0} \overline{\varphi}$. □

Every Weil algebra A induces a vector space $\tilde{A} = N_A / N_A^2$ and every homomorphism $\mu\colon A \to B$ induces a linear map $\tilde{\mu}\colon \tilde{A} \to \tilde{B}$. Write $\overline{B}^{\mu} = \tilde{B}/\tilde{\mu}(\tilde{A})$ for the factor vector space. In the manifold case, the underlying bundle $T^{A_1} M$ of $T^A M$ is isomorphic to $T^1_k M$, $k = \dim \tilde{A}$, and $\operatorname{reg} T^1_k M \subset M$ characterizes $\operatorname{reg} T^{A_1} M \subset T^{A_1} M$. In [5], $\operatorname{reg} T^A M \subset T^A M$ is defined as the inverse image of $\operatorname{reg} T^{A_1} M$ with respect to the canonical projection $T^A M \to T^{A_1} M$, see also [11]. In the fibered case, if $l = \dim \overline{B}^{\mu}$, then (12) implies that the underlying bundle $T^{\mu_{0,1,1}} Y$ is isomorphic to

$$T^1_k M \times_M V^1_l Y.$$

Then we define

$$\operatorname{reg} T^{\mu_{0,1,1}} Y = \operatorname{reg} T^1_k M \times_M \operatorname{reg} V^1_l Y \tag{23}$$

and $\operatorname{reg} T^{\mu} Y$ is the inverse image of $\operatorname{reg} T^{\mu_{0,1,1}} Y$ with respect to the canonical projection. Thus, analogously to (18) we have

$$\operatorname{reg} T^{\mu} Y = \operatorname{reg} T^A M \times_{T^B M} \operatorname{reg} T^B Y. \tag{24}$$

In the manifold case, the following concept was introduced in [5], [11].

Definition 6. The bundle of contact elements of type μ on Y is the factor space $K^{\mu} Y = \operatorname{reg} T^{\mu} Y / \operatorname{Aut} \mu$.

We shall write $\kappa\colon \operatorname{reg} T^{\mu} Y \to K^{\mu} Y$ for the factor projection.

We introduce the manifold structure on $K^{\mu} Y$ by using the ideas by Alonso [1]. First we have to generalize his algebraic lemma. Denote by $\tilde{a}$ the image of $a \in N_A$ in $\tilde{A}$ and by $\overline{b}$ the image of $b \in N_B$ in $\overline{B}^{\mu}$.

Lemma 1. *Let $f\colon \delta^{r,s,q}_{m,n} \to \mu$ be an epimorphism. Let $a_1,\dots,a_k \in N_A$, $b_1,\dots,b_l \in N_B$ have the property that $\tilde a_1,\dots,\tilde a_k$ is a basis in $\tilde A$ and $\overline{b}_1,\dots,\overline{b}_l$ is a basis in $\overline{B}^{\mu}$. Then there exist generators $x_1,\dots,x_m$ of $\mathbb{D}^q_m$ and additional generators $y_1,\dots,y_n$ of $\mathbb{D}^{r,s}_{m,n}$ satisfying $f_1(x_1)=a_1,\dots,f_1(x_k)=a_k, f_1(x_{k+1})=0,\dots,f_1(x_m)=0$, $f_2(y_1)=b_1,\dots,f_2(y_l)=b_l, f_2(y_{l+1})=0,\dots,f_2(y_n)=0$.*

P r o o f. By the surjectivity of f_1, there exist $x_1,\dots,x_k\in\mathbb{D}^q_m$ satisfying $f_1(x_1)=a_1,\dots,f_1(x_k)=a_k$. Complete them by some $x'_{k+1},\dots,x'_m$ to a system of generators of $\mathbb{D}^q_m$. Hence we have

$$f_1(x'_u)=P_u(a_1,\dots,a_k),\qquad u=k+1,\dots,m$$

for some polynomials P_u. Then we define $x_u=x'_u-P_u(x_1,\dots,x_k)$. Further, by the surjectivity of f_2, there exist $y_1,\dots,y_l\in\mathbb{D}^{r,s}_{m,n}$ satisfying $f_2(y_1)=b_1,\dots,f_2(y_l)=b_l$. Complete them by some $y'_{l+1},\dots,y'_n$ to a system of additional generators of $\mathbb{D}^{r,s}_{m,n}$. Hence we have

$$f_2(y'_v)=P_v(\mu(a_1),\dots\mu(a_m),b_1,\dots,b_l),\qquad v=l+1,\dots,n$$

for some polynomials P_v. Then we define $y_v=y'_v-P_v(\delta^{r,s,q}_{m,n}(x_1),\dots,\delta^{r,s,q}_{m,n}(x_m), y_1,\dots,y_l)$. □

Proposition 4. *Let $f,g\colon \delta^{r,s,q}_{m,n}\to\mu$ be two epimorphisms. Then there exists an isomorphism $h\colon \delta^{r,s,q}_{m,n}\to\delta^{r,s,q}_{m,n}$ satisfying $f=g\circ h$.*

P r o o f. For given $a_1,\dots,a_k,b_1,\dots,b_l$, Lemma 1 yields some $x'_1,\dots,y'_n$ for f and some $x''_1,\dots,y''_n$ for g. Define h by setting $h_1(x'_1)=x''_1,\dots,h_1(x'_m)=x''_m$, $h_2(y'_1)=y''_1,\dots,h_2(y'_n)=y''_n$. □

Consider a fixed epimorphism $f\colon \delta^{r,s,q}_{m,n}\to\mu$.

Lemma 2. *For every isomorphism $g\colon \mu\to\mu$, there exists an isomorphism $h\colon \delta^{r,s,q}_{m,n}\to\delta^{r,s,q}_{m,n}$ satisfying $g\circ f=f\circ h$.*

P r o o f. We apply Proposition 4 to f and $g\circ f$. □

By this lemma, for every $g\in\operatorname{Aut}\mu$ there is an element $h\in G^{r,s,q}_{m,n}$ that is f-projectable over g. The subgroup G of all such elements is a closed subgroup, so a Lie group, and the induced map $\tilde f\colon G\to\operatorname{Aut}\mu$ is surjective. The kernel $\overline{G}\subset G$ is a closed subgroup and the factor group $G/\overline{G}$ is isomorphic to $\operatorname{Aut}\mu$.

The epimorphism f induces a natural transformation $f_Y\colon T^{r,s,q}_{m,n}Y\to T^{\mu}Y$, which maps $\operatorname{reg}T^{r,s,q}_{m,n}Y=P^{s,r,q}Y$ onto $\operatorname{reg}T^{\mu}Y$. We start with the case $Y=\mathbb{R}^{m,n}$. We have

$$P^{r,s,q}\mathbb{R}^{m,n}=\mathbb{R}^{m,n}\times G^{r,s,q}_{m,n}. \tag{25}$$

The natural transformation $f_{\mathbb{R}^{m,n}}$ coincides with the factor projection of (25) into

$$\mathbb{R}^{m,n} \times (G^{r,s,q}_{m,n}/\overline{G}) = \operatorname{reg} T^{\mu}\mathbb{R}^{m,n}. \tag{26}$$

Then the group identification $(G^{r,s,q}_{m,n}/\overline{G})/(G/\overline{G}) = G^{r,s,q}_{m,n}/G$ implies

$$K^{\mu}\mathbb{R}^{m,n} = \mathbb{R}^{m,n} \times (G^{r,s,q}_{m,n}/G). \tag{27}$$

This decomposition introduces the manifold structure on $K^{\mu}\mathbb{R}^{m,n}$ that is independent of the choice of f. Indeed, if we replace f by another epimorphism $\delta^{r,s,q}_{m,n} \to \mu$, we find the effect of an inner automorphism of $G^{r,s,q}_{m,n}$. Globalizing this result to an arbitrary Y, we obtain

Proposition 5. *There is a unique manifold structure on $K^{\mu}Y$ such that the factor projection $\kappa\colon \operatorname{reg} T^{\mu}Y \to K^{\mu}Y$ is a submersion.*

We have also proved the following assertion.

Corollary 2. *$\operatorname{reg} T^{\mu}Y \to K^{\mu}Y$ is a principal fiber bundle with structure group* $\operatorname{Aut}\mu$.

3. Some natural properties of $K^{r,s,q}_{k,l}$

First of all we show that the functor $K^{r,s,q}_{k,l}$ is rigid from the naturality point of view.

Proposition 6. *The only natural transformation $\mathscr{C}\colon K^{r,s,q}_{k,l}Y \to K^{r,s,q}_{k,l}Y$ is the identity.*

Proof. By locality, we may assume $Y = \mathbb{R}^{m,n}$. Let $i\colon \mathbb{R}^{k,l} \to \mathbb{R}^{m,n}$ be the injection

$$\bar{x}^a = x^a,\ \bar{x}^b = 0,\ \bar{y}^c = y^c,\ \bar{y}^d = 0. \tag{28}$$

Write $\varrho = \kappa(j^{r,s,q}_{0,0}i)$. Since $K^{r,s,q}_{k,l}\mathbb{R}^{m,n}$ is the orbit of ϱ with respect to fibered isomorphisms of $\mathbb{R}^{m,n}$, it suffices to prove $\mathscr{C}(\varrho) = \varrho$. Let $\mathscr{C}(\varrho) = \kappa(j^{r,s,q}_{0,0}\eta)$. Since η is a fibered immersion, there exist integers $i_1, \dots, i_k, j_1, \dots, j_l$ such that the map $\varphi\colon \mathbb{R}^{k,l} \to \mathbb{R}^{k,l}$,

$$\bar{x}_1 = x^{i_1} \circ \eta,\ \dots,\ \bar{x}^k = x^{i_k} \circ \eta,\ \bar{y}^1 = y^{j_1} \circ \eta,\ \dots,\ \bar{y}^l = y^{j_l} \circ \eta \tag{29}$$

is a local fibered isomorphism of $\mathbb{R}^{m,n}$. Consider a fibered isomorphism e_t, $0 \neq t \in \mathbb{R}$, on $\mathbb{R}^{m,n}$ of the form

$$\bar{x}^a = tx^a + x^{i_a},\ \bar{x}^b = x^b,\ \bar{y}^c = ty^c + y^{j_c},\ \bar{y}^d = y^d. \tag{30}$$

We have $K^{r,s,q}_{k,l} e_t(\varrho) = \varrho$ for all t. By naturality, e_t preserves $\mathscr{C}(\varrho)$ as well. For $t \to 0$ we obtain $\mathscr{C}(\varrho) = \kappa(j^{r,s,q}_{0,0}\bar{\eta})$, where $\bar{\eta}$ is expressed by (29) and

$$\bar{x}^a = u^a,\ \bar{x}^b = f^b(u),\ \bar{y}^c = v^c,\ \bar{y}^d = f^d(u,v). \tag{32}$$

Consider a fibered isomorphism d_t, $0 \neq t \in \mathbb{R}$, on $\mathbb{R}^{m,n}$ of the form

$$\bar{x}^a = x^a,\ \bar{x}^b = tx^b,\ \bar{y}^c = y^c,\ \bar{y}^d = ty^d. \tag{33}$$

Since d_t preserves ϱ, it preserves $\mathscr{C}(\varrho)$ as well. For $t \to 0$ we obtain $\mathscr{C}(\varrho) = \varrho$. $\square$

In the case of $n = 0$, the fibered manifold $\mathrm{id}_M \colon M \to M$ is identified with M. Then the only non-trivial situation is $l = 0$, $r = s = q$. In this case we obtain the classical bundle $K^r_k M$ of all contact (k,r)-elements on M, [8]. The following corollary represents a new result in the manifold case.

Corollary 3. *The only natural transformation $K^r_k M \to K^r_k M$ is the identity.*

Proposition 44.4 from [8] reads that every natural operator transforming vector fields on a manifold M into vector fields on $K^r_k M$ is a constant multiple of the flow operator. We generalize this result to the fibered manifold case.

Proposition 7. *For $m > k$, every natural operator $\mathscr{A}$ transforming projectable vector fields on a fibered manifold Y into vector fields on $K^{r,s,q}_{k,l} Y$ is a constant multiple of the flow operator $\mathscr{K}^{r,s,q}_{k,l}$.*

P r o o f. Consider $Y = \mathbb{R}^{m,n}$ and ϱ from the proof of Proposition 6. First we deduce that $\mathscr{A}$ is uniquely determined by $\mathscr{A}\,(\partial/\partial x^m)_\varrho$. Write $\pi \colon K^{r,s,q}_{k,l}\mathbb{R}^{m,n} \to \mathbb{R}^{m,n}$ for the bundle projection. Let X be a projectable vector field on $\mathbb{R}^{m,n}$ over a vector field X_1 on $\mathbb{R}^m$. Consider $\tau \in K^{r,s,q}_{k,l}\mathbb{R}^{m,n}$ over $\tau_0 \in K^r_k\mathbb{R}^m$, $\pi(\tau) = (x,y)$, with the property that $X_1(x)$ is transversal to τ_0. In this situation, there exists a fibered isomorphism of $\mathbb{R}^{m,n}$ transforming τ into ϱ and the germ of X at (x,y) into the germ of $\partial/\partial x^m$ at $(0,0)$. For $m > k$, all τ with this property form a dense subset in $K^{r,s,q}_{k,l}\mathbb{R}^{m,n}$.

Next we prove $\mathscr{A} = a\mathscr{K}^{r,s,q}_{k,l} + \mathscr{V}$, $a \in \mathbb{R}$, where $\mathscr{V}$ is a π-vertical operator, i.e. every $\mathscr{V}(X)$ is a π-vertical vector field. Write

$$T\pi\left(\mathscr{A}\left(\frac{\partial}{\partial x^m}\right)_\varrho\right) = \sum_{i=1}^{m} a_i \frac{\partial}{\partial x^i}\bigg|_{0,0} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial y^j}\bigg|_{0,0}, \qquad a_i, b_j \in \mathbb{R}. \tag{34}$$

Consider the fibered isomorphisms $c_t = (tx^1, \ldots, tx^{m-1}, x^m, ty^1, \ldots, ty^n)$, $t \neq 0$, on $\mathbb{R}^{m,n}$. They preserve $\partial/\partial x^m$ and ϱ, so they preserve $T\pi(\mathscr{A}(\partial/\partial x^m)_\varrho)$ as well. On the other hand, c_t transforms (34) into

$$\sum_{i=1}^{m-1} ta_i \frac{\partial}{\partial x^i}\bigg|_{0,0} + a_m \frac{\partial}{\partial x^m}\bigg|_{0,0} + \sum_{j=1}^{n} tb_j \frac{\partial}{\partial y^j}\bigg|_{0,0}.$$

This implies $a_1 = \ldots = a_{m-1} = b_1 = \ldots = b_n = 0$. Hence $\mathscr{V} = \mathscr{A} - a_m \mathscr{K}^{r,s,q}_{k,l}$ is a π-vertical operator.

It remains to show $\mathscr{V}(\partial/\partial x^m)_\varrho = 0$, which is equivalent to $\mathscr{V} = 0$. Let ψ_τ be the flow of $\mathscr{V}(\partial/\partial x^m)$. By π-verticality,

$$\psi_\tau(\varrho) = \kappa(j^{r,s,q}_{0,0}\eta_\tau), \tag{35}$$

where η_τ is a smoothly parametrized family of fibered immersions $\mathbb{R}^{k,l} \to \mathbb{R}^{m,n}$ sending $(0,0)$ into $(0,0)$. By continuity of ψ, we may assume $i_a = a$ and $j_c = c$ in (29) with η replaced by η_τ for τ sufficiently small. Thus, every η_τ can be chosen in the form (32) with $f^b(0) = 0$ and $f^d(0,0) = 0$. Consider the fibered isomorphism k_t

$$\bar{x}^a = \frac{1}{t}x^a,\ \bar{x}^b = x^b,\ \bar{y}^c = \frac{1}{t}y^c,\ \bar{y}^d = y^d, \qquad 0 \neq t \in \mathbb{R}. \tag{36}$$

Since k_t preserves $\partial/\partial x^m$, $K^{r,s,q}_{k,l}k_t$ commutes with ψ_τ. Clearly, $K^{r,s,q}_{k,l}k_t(\varrho) = \varrho$, so that $K^{r,s,q}_{k,l}k_t(\psi_\tau\varrho) = \psi_\tau(\varrho)$. Then (36) implies

$$\psi_\tau\varrho = \kappa(j^{r,s,q}_{0,0}(k_t \circ \eta_\tau)) = \kappa(j^{r,s,q}_{0,0}(k_t \circ \eta_\tau \circ t\,\mathrm{id}^{k,l}_{\mathbb{R}})).$$

For $t \to 0$, we obtain $\psi_\tau(\varrho) = \varrho$. Hence $\mathscr{V}(\partial/\partial x^m)_\varrho = 0$. □

Now it is easy to determine all natural transformations of $TK^{r,s,q}_{k,l}Y$ into itself.

Proposition 8. *For $m > k$, every natural transformation $\mathscr{B}\colon TK^{r,s,q}_{k,l}Y \to TK^{r,s,q}_{k,l}Y$ is a constant multiple of the identity.*

Proof. Let $p\colon TK^{r,s,q}_{k,l}Y \to K^{r,s,q}_{k,l}Y$ be the bundle projection, $\mathscr{O}$ the zero section and I the identity of $TK^{r,s,q}_{k,l}Y$. Then $p \circ \mathscr{B} \circ \mathscr{O}\colon K^{r,s,q}_{k,l}Y \to K^{r,s,q}_{k,l}Y$ is a natural transformation, so the identity of $K^{r,s,q}_{k,l}Y$ by Proposition 6.

First we show that $p \circ \mathscr{B} = p$. Write $\sigma = \mathscr{K}^{r,s,q}_{k,l}(\partial/\partial x^m)_\varrho$, where ϱ is from the proof of Proposition 6. Since the orbit of σ is dense, it suffices to verify $p(\mathscr{B}(\sigma)) = \varrho$. We have $p(\mathscr{B}(\tau\sigma)) = j^{r,s,q}_{0,0}\eta_\tau$, $\tau \in \mathbb{R}$. Analogously to the proof of Proposition 7, we may assume η_τ is of the form (32) with $f^b(0) = 0$ and $f^d(0,0) = 0$ for τ sufficiently small. The fibered isomorphisms (36) preserve $p(\mathscr{B}(\tau\sigma))$. For $t \to 0$, we obtain $p(\mathscr{B}(\tau\sigma)) = \varrho$. Using the homotheties on $\mathbb{R}^{m,n}$, we find $p(\mathscr{B}(\sigma)) = \varrho$.

This implies that $\mathscr{B} \circ \mathscr{K}^{r,s,q}_{k,l}$ is a natural operator transforming projectable vector fields from Y to $K^{r,s,q}_{k,l}Y$, so a constant multiple of $\mathscr{K}^{r,s,q}_{k,l}$ by Proposition 7. Hence $\mathscr{B}(\sigma) = c\sigma$ for some $c \in \mathbb{R}$. Using the fact the orbit of σ is dense, we obtain $\mathscr{B} = cI$. $\square$

A one-form $\omega\colon TY \to \mathbb{R}$ is called horizontal if $\omega(X) = 0$ for every vertical tangent vector X of Y. In general, given a fibered manifold $q\colon Z \to N$, the vertical lift of a one-form $\omega\colon TN \to \mathbb{R}$ is the one-form $\omega \circ Tq\colon TZ \to \mathbb{R}$.

Proposition 9. *For $m > k$, every natural operator $\mathscr{E}$ transforming horizontal one-forms on Y into one-forms on $K^{r,s,q}_{k,l}Y$ is a constant multiple of the vertical lifting.*

Proof. Consider σ from the proof of Proposition 8. Since the orbit of σ is dense, $\mathscr{E}$ is uniquely determined by the evaluations $\langle \mathscr{E}(\omega), \sigma\rangle$ for all horizontal one-forms ω on $\mathbb{R}^{m,n}$. The homotheties h_t on $\mathbb{R}^{m,n}$, $t \neq 0$, preserve ϱ and map $\partial/\partial x^m$ into $t\partial/\partial x^m$, so they send σ into $t\sigma$. Using the naturality of $\mathscr{E}$ with respect to h_t, we obtain a homogenity condition $\langle \mathscr{E}(h_t^*\omega), \sigma\rangle = t\langle \mathscr{E}(\omega), \sigma\rangle$. By the nonlinear Peetre theorem and the homogeneous function theorem, [7], we deduce that $\langle \mathscr{E}(\omega), \sigma\rangle$ is linear in $\omega_{0,0} \in T^*_{0,0}\mathbb{R}^{m,n}$. Using the naturality of $\mathscr{E}$ with respect to the transformations $(tx^1, \dots, tx^{m-1}, x^m, y^1, \dots, y^n)$, $t \neq 0$, we obtain $\langle \mathscr{E}(dx^i), \sigma\rangle = 0$ for $i = 1, \dots, m-1$. Hence $\mathscr{E}$ is determined by $\langle \mathscr{E}(dx^m), \sigma\rangle$. This proves our claim. $\square$

For the manifold case, we obtain

Corollary 4. *Every natural operator transforming one-forms on a manifold M into one-forms on K^r_kM is a constant multiple of the vertical lifting.*

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