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ON PETTIS INTEGRABILITY

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Abstract. Assuming that (Ω, Σ, μ) is a complete probability space and X a Banach space, in this paper we investigate the problem of the X-inheritance of certain copies of c_0 or ℓ_∞ in the linear space of all [classes of] X-valued μ-weakly measurable Pettis integrable functions equipped with the usual semivariation norm.

Keywords: Pettis integrable function space, copy of c_0, copy of ℓ_∞, countably additive vector measure, WRNP, CRP

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1. Introduction

Throughout this paper (Ω, Σ, μ) will be a complete probability space and X a real or complex Banach space. Our notation is standard [1, 2, 3]. We shall denote by $\mathscr{P}(\mu, X)$ the linear space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) of all [classes of scalarly equivalent] weakly μ-measurable X-valued Pettis integrable functions f defined on Ω, equipped with the semivariation norm

$$\|f\|_{\mathscr{P}(\mu,X)} = \sup\left\{ \int_\Omega |x^* f(\omega)| \,\mathrm{d}\mu(\omega)\colon\ x^* \in X^*, \|x^*\| \leqslant 1 \right\}.$$

The linear subspace of $\mathscr{P}(\mu, X)$ consisting of all strongly μ-measurable functions will be denoted by $P_1(\mu, X)$. As is well known, both $\mathscr{P}(\mu, X)$ and $P_1(\mu, X)$ are not in general Banach spaces, although they are barrelled normed spaces [5]. According to a result of Pettis, if $f\colon\ \Omega \to X$ is [weakly measurable and] Pettis integrable, the mapping $F\colon\ \Sigma \to X$ defined by $F(E) = (P)\int_E f \,\mathrm{d}\mu$ is a μ-continuous countably additive X-valued measure and, in addition, if f is strongly measurable, then $F(\Sigma)$

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is a relatively compact set in X. A Banach space X is said to have the weak Radon-Nikodým property (WRNP) with respect to a complete probability space (Ω, Σ, μ) if every μ-continuous measure $F\colon\ \Sigma \to X$ of σ-finite variation has a Pettis μ-integrable derivative $f\colon\ \Omega \to X$, i.e. that $F(E) = (P)\int_E f\,\mathrm{d}\mu$. If X has the WRNP with respect to every complete probability space, it is said that X has the WRNP. A Banach space X is said to have the compact range property (CRP) if any X-valued countably additive measure F of bounded variation defined on a σ-algebra of subsets has relatively compact range. These two last definitions have been taken from [9] and [10]. We shall denote by $\mathrm{ca}(\Sigma, X)$ the Banach space of all countably additive X-valued measures F on Σ equipped with the semivariation norm $\|F\|$, while $\mathrm{cca}(\Sigma, X)$ will stand for the closed subspace of $\mathrm{ca}(\Sigma, X)$ of all measures of relatively compact range. We shall denote by $\mathrm{bvca}(\Sigma, X)$ the Banach space of all X-valued countably additive measures of bounded variation F defined on Σ equipped with the variation norm $|F|$. Let us recall that the linear operator $S\colon\ \mathscr{P}(\mu, X) \to \mathrm{ca}(\Sigma, X)$ defined by $Sf(E) = (P)\int_E f(\omega)\,\mathrm{d}\mu(\omega)$ for each $E \in \Sigma$ is a linear isometry into $\mathrm{ca}(\Sigma, X)$. If X and Y are two Banach spaces over the same field $\mathbb{K}$ and $L(X, Y)$ denotes the Banach space of all bounded linear operators from X into Y equipped with the operator norm, as usual $K_{w^*}(X^*, Y)$ will denote the closed linear subspace of $L(X^*, Y)$ formed by the compact weak*-weakly continuous linear operators. Later on we shall need the following result due to Drewnowski.

Lemma 1.1. ([4]) *$K_{w^*}(X^*, Y)$ contains a copy of ℓ_∞ if and only either X contains a copy of ℓ_∞ of Y contains a copy of ℓ_∞.*

Regarding the space $P_1(\mu, X)$, it can be shown that $P_1(\mu, X)$ contains a copy of c_0 if and only if X does (cf. [7, Thm. 5]) and, as far as copies of ℓ_∞ in $P_1(\mu, X)$ is concerned, due to the fact that $P_1(\mu, X)$ embeds isometrically into $\mathrm{cca}(\Sigma, X)$ and $\mathrm{cca}(\Sigma, X)$ is linearly isometric to $K_{w^*}(\mathrm{ca}(\Sigma)^*, X)$, Lemma 1.1 guarantees that ℓ_∞ embeds into $P_1(\mu, X)$ if and only if X does. In this note we investigate the presence of certain copies of c_0 or ℓ_∞ in the wider space $\mathscr{P}(\mu, X)$. As a first observation notice that if $\mathscr{P}(\mu, X^*)$ contains a copy of ℓ_∞, then either ℓ_∞ embeds into X^* or X contains a copy of ℓ_1 (if ℓ_1 does not embed into X it is well known that X^* has the CRP, consequently $\mathscr{P}(\mu, X^*)$ embeds into $\mathrm{cca}(\Sigma, X^*)$ and we are done). On the other hand, if (Ω, Σ, μ) is a perfect probability space, as a consequence of Fremlin's subsequences theorem, for each $f \in \mathscr{P}(\mu, X)$ the weak*-weakly continuous linear operator $T_f\colon\ X^* \to L_1(\mu)$ defined by $x^* \to x^* f$ is compact [6, Prop. 5.7]. Since $\|T_f\| = \|f\|_{\mathscr{P}(\mu,X)}$, the map $f \to T_f$ embeds $\mathscr{P}(\mu, X)$ isometrically into $K_{w^*}(X^*, L_1(\mu))$. Hence, if (Ω, Σ, μ) is a perfect probability space, then $\mathscr{P}(\mu, X)$ contains a copy of ℓ_∞ if and only if X does. In what follows we shall abbreviate by 'wuC' the phrase "weakly unconditionally Cauchy".

2. Embedding c_0 into $\mathscr{P}(\mu, X)$

Let us denote by $\mathscr{P}_1(\mu, X)$ the subspace of $\mathscr{P}(\mu, X)$ of all those functions $f \in \mathscr{P}(\mu, X)$ for which there exists a scalar function $g \in \mathscr{L}_1(\mu)$ such that $\|f(\omega)\| \leqslant g(\omega)$ for μ-almost all $\omega \in \Omega$.

Theorem 2.1. *Let (Ω, Σ, μ) be a perfect probability space and X a Banach space that has the WRNP with respect to (Ω, Σ, μ). If $\mathscr{P}_1(\mu, X)$ contains a copy of c_0, then X contains a copy of c_0.*

Proof. Let $\{e_n \colon n \in \mathbb{N}\}$ be the unit vector basis of c_0 and let J be a topological isomorphism from c_0 into $\mathscr{P}_1(\mu, X)$. Given $\zeta \in c_0$, select a sequence $\{x_n^*\}$ in B_{X^*} such that $\int_\Omega |x_n^* J\zeta(\omega)| \,\mathrm{d}\mu(\omega) \to \|J\zeta\|_{\mathscr{P}(\mu,X)}$ and set $\Phi_\zeta(\omega) := \sup\limits_{n\in\mathbb{N}} |x_n^* J\zeta(\omega)|$ for each $\omega \in \Omega$. Noting that $\Phi_\zeta(\omega) \leqslant \|J\zeta(\omega)\|$ for each $\omega \in \Omega$, according to the hypotheses there exists $h_\zeta \in \mathscr{L}_1(\mu)$ such that $\Phi_\zeta(\omega) \leqslant h_\zeta(\omega)$ for almost all $\omega \in \Omega$, which shows that each Φ_ζ belongs to $L_1(\mu)$. If S denotes the isometrical embedding of $\mathscr{P}(\mu, X)$ into $\mathrm{ca}(\Sigma, X)$ defined by $(Sf)(E) = (P)\int_E f \,\mathrm{d}\mu$ for each $E \in \Sigma$, the inequality $|x^* J\zeta(\omega)| \leqslant \Phi_\zeta(\omega)$ for almost all $\omega \in \Omega$ and each $x^* \in B_{X^*}$ implies that $\|SJ\zeta(E)\| \leqslant \int_E \Phi_\zeta \,\mathrm{d}\mu$, from where it follows that $SJ\zeta$ is an X-valued measure of bounded variation. Therefore SJ maps c_0 into $\mathrm{bvca}(\Sigma, X)$, and since $S|_{J(c_0)}$ has closed graph as may be easily seen, SJ happens to be a bounded linear operator when considered from c_0 into $\mathrm{bvca}(\Sigma, X)$. Moreover, since $|SJe_n| \geqslant \|SJe_n\| = \|Je_n\|_{\mathscr{P}(\mu,X)} \nrightarrow 0$, Rosenthal's c_0 theorem guarantees that there exists an infinite set M of positive integers such that $SJ|_{c_0(M)}$ is a topological isomorphism from c_0 (M) into $\mathrm{bvca}(\Sigma, X)$. In the sequel we shall identify $c_0(M)$ with c_0 and we shall denote $SJ|_{c_0(M)}$ by Q, keeping in mind that $Qe_n = SJe_n \nrightarrow 0$ in $\mathrm{bvca}(\Sigma, X)$.

Now assume by contradiction that X contains no copy of c_0. Given $F \in \mathrm{bvca}(\Sigma, X)$, since $F \to F(E)$ is a continuous map for each $E \in \Sigma$ and X does not contain a copy of c_0, the series $\sum\limits_{n=1}^{\infty} Qe_n(E)$ converges unconditionally in X for each $E \in \Sigma$. This allows us to define the linear operator $T \colon \ell_\infty \to ba(\Sigma, X)$ by $T\xi(E) = \sum\limits_{n=1}^{\infty} \xi_n Qe_n(E)$ for each $E \in \Sigma$. If $\{E_1, \ldots, E_n\}$ is a partition of Ω by elements of Σ, setting $\xi^n := (\xi_1, \ldots, \xi_n, 0, \ldots, 0)$ we have

$$\sum_{i=1}^{n} \|T\xi(E_i)\| \leqslant \sup_{k\in\mathbb{N}} \sum_{i=1}^{n} \|Q\xi^k(E_i)\| \leqslant \sup_{k\in\mathbb{N}} |Q\xi^k| \leqslant \|Q\| \, \|\xi\|_\infty$$

showing that $T\xi$ has bounded variation and $|T| \leqslant \|Q\|$. Since $Q\xi^k \ll \mu$ for each $k \in \mathbb{N}$, according to the Vitali-Hahn-Saks theorem, $T\xi \in \mathrm{ca}(\Sigma, X)$ and $T\xi \ll \mu$ for each $\xi \in \ell_\infty$. Thus $T(\ell_\infty) \subseteq \mathrm{bvca}(\Sigma, X)$.

Given that X is assumed to have the WRNP with respect to (Ω, Σ, μ) and, as we have seen, $T\xi$ has finite variation and $T\xi \ll \mu$, there exists f_ξ in $\mathscr{P}(\mu, X)$ such that $T\xi(E) = (P)\int_E f_\xi \,\mathrm{d}\mu$ for each $\xi \in \ell_\infty$, $E \in \Sigma$ and $n \in \mathbb{N}$. But, since (Ω, Σ, μ) is a perfect finite measure space, Fremlin's subsequences theorem guarantees that $E \to (P)\int_E f_\xi \,\mathrm{d}\mu$ has relatively compact range [6], i.e. $T\xi \in \operatorname{cca}(\Sigma, X)$ for each $\xi \in \ell_\infty$. This shows that T is a bounded linear operator from ℓ_∞ into $\operatorname{cca}(\Sigma, X)$. As $Te_n = Qe_n$ for each $n \in \mathbb{N}$ and $\inf\limits_{n\in} \|Qe_n\| > 0$, Rosenthal's ℓ_∞ theorem allows us to conclude that $\operatorname{cca}(\Sigma, X)$ contains a copy of ℓ_∞. Hence Lemma 1.1 forces X to contain a copy of ℓ_∞, a contradiction. $\square$

Theorem 2.2. *If X has a weak* sequentially compact dual ball, then $\mathscr{P}(\mu, X)$ contains no copy of ℓ_∞.*

Proof. Given $f \in (\mu, X)$, the linear operator $T_f\colon X^* \to L_1(\mu)$ defined by $(T_f x^*)(\omega) = x^* f(\omega)$ for each $\omega \in \Omega$ is weak*-weakly continuous and hence $T_f \in L_{w^*}(X^*, L_1(\mu))$. Moreover the operator $\psi\colon \mathscr{P}(\mu, X) \to L_{w^*}(X^*, L_1(\mu))$ defined by $\psi(f) = T_f$ embeds $\mathscr{P}(\mu, X)$ isometrically into $L_{w^*}(X^*, L_1(\mu))$ since $\|T_f\| = \|f\|_{\mathscr{P}(\mu,X)}$. Let us see that the range of ψ is contained in $K_{w^*}(X^*, L_1(\mu))$, which amounts to each operator T_f being compact. If $\{x_n^*\}$ is a sequence in the closed unit ball B_{X^*} of X^*, since B_{X^*} is weak* sequentially compact there exists a subsequence $\{x_{n_k}^*\}$ that converges to some $x^* \in B_{X^*}$ in the weak* topology. Considering the sequence $\{T_f(x_{n_k}^* - x^*)\}$ in $L_1(\mu)$, for each $E \in \Sigma$ one has

$$\sup_{k\in\mathbb{N}} \int_E |T_f(x_{n_k}^* - x^*)| \,\mathrm{d}\mu \leqslant 2\|\chi_E f\|_{\mathscr{P}(\mu,X)}.$$

Since $\lim\limits_{\mu(E)\to 0} \|\chi_E f\|_{\mathscr{P}(\mu,X)} = 0$ then $\lim\limits_{\mu(E)\to 0} \sup\limits_{k\in\mathbb{N}} \int_E |T_f(x_{n_k}^* - x^*)| \,\mathrm{d}\mu = 0$, which shows that the sequence $\{|T_f(x_{n_k}^* - x^*)|\}$ is uniformly integrable. Hence, due to the fact that

$$\lim_{k\to\infty} T_f(x_{n_k}^* - x^*)(\omega) = \lim_{k\to\infty} (x_{n_k}^* f(\omega) - x^* f(\omega)) = 0$$

for each $\omega \in \Omega$, Vitali's lemma [8, Exercise 13.38] allow us to conclude that

$$\lim_{k\to\infty} \int_\Omega |T_f(x_{n_k}^* - x^*)| \,\mathrm{d}\mu = 0$$

Therefore $T_f x_{n_k}^* \to T_f x^*$ in the norm topology of $L_1(\mu)$ and, consequently, $T_f \in K_{w^*}(X^*, L_1(\mu))$. According to Lemma 1.1, if $\mathscr{P}(\mu, X)$ contains a copy of ℓ_∞, then X must contain a copy of ℓ_∞. This is a contradiction, since X, having a weak* sequentially compact dual ball, cannot contain a copy of ℓ_∞. $\square$

Theorem 2.3. *If $\mathscr{P}(\mu,X)$ contains a copy of c_0, then either X contains a copy of c_0 or $L_{w^*}(X^*,L_1(\mu))$ contains a copy of ℓ_∞.*

P r o o f. Let J be an isomorphism from c_0 into $\mathscr{P}(\mu,X)$ and let $\{e_n\colon n\in\mathbb{N}\}$ denote the unit vector basis of c_0. Set $f_n:=Je_n$ for each $n\in\mathbb{N}$ and note that the series $\sum\limits_{n=1}^{\infty} f_n$ is wuC in $\mathscr{P}(\mu,X)$. This implies that the series $\sum\limits_{n=1}^{\infty} x^*f_n$ is wuC in $L_1(\mu)$ for each $x^*\in X^*$ and, since $L_1(\mu)$ contains no copy of c_0, that, actually, $\sum\limits_{n=1}^{\infty} x^*f_n$ is (BM)-convergent in $L_1(\mu)$. On the other hand, as the series $\sum\limits_{n=1}^{\infty}(P)\int_E f_n\,\mathrm{d}\mu$ is wuC in X for each $E\in\Sigma$, assuming that c_0 is not embedded into X, then $\sum\limits_{n=1}^{\infty}\xi_n(P)\int_E f_n\,\mathrm{d}\mu$ converges in X for each $\xi\in\ell_\infty$ and each $E\in\Sigma$. Therefore, assuming that X does not contain a copy of c_0, we may define a bounded linear operator $\varphi\colon\ \ell_\infty\to L_{w^*}(X^*,L_1(\mu))$ by $(\varphi\xi)x^*=\sum\limits_{n=1}^{\infty}\xi_n x^*f_n$ [convergence in $L_1(\mu)$] for each $x^*\in X^*$. In fact, $\varphi\xi\in L(X^*,L_1(\mu))$ for each $\xi\in\ell_\infty$ since, given $x^*\in X^*$ and $\varepsilon>0$, choosing $n\in\mathbb{N}$ with $\Big\|\sum\limits_{j>n}\xi_j x^*f_j\Big\|_{L_1(\mu)}<\varepsilon$ and noting that for some $C>0$

$$\|(\varphi\xi)x^*\|_{L_1}\leqslant\Big\|\sum_{j=1}^{n}\xi_j x^*f_j\Big\|_{L_1(\mu)}+\varepsilon\leqslant C\|x^*\|\,\|\xi\|_\infty+\varepsilon,$$

it follows that $\|(\varphi\xi)x^*\|_{L_1(\mu)}\leqslant C\|x^*\|\,\|\xi\|_\infty$ for each $\xi\in\ell_\infty$ and $x^*\in X^*$, which shows that $\varphi\xi\in L(X^*,L_1(\mu))$ for each $\xi\in\ell_\infty$ and, besides, that φ is bounded. Given some fixed $\xi\in\ell_\infty$, let us show that $\varphi\xi\in L_{w^*}(X^*,L_1(\mu))$. In fact, let $\{x_d^*\}_{d\in D}$ be a net in X^* such that $x_d^*\to x^*$ under the weak* topology of X^*. Choosing some $E\in\Sigma$, we have in particular

$$\Big\langle x_d^*-x^*,\sum_{n=1}^{\infty}\xi_n(P)\int_E f_n\,\mathrm{d}\mu\Big\rangle\to 0 \tag{2.1}$$

and hence there is $k\in D$ such that $\Big|\Big\langle x_d^*-x^*,\sum\limits_{n=1}^{\infty}\xi_n(P)\int_E f_n\,\mathrm{d}\mu\Big\rangle\Big|<\varepsilon$ for each $d>k$. Bearing in mind that $\sum\limits_{n=1}^{m}\xi_n(P)\int_E f_n\,\mathrm{d}\mu\to\sum\limits_{n=1}^{\infty}\xi_n(P)\int_E f_n\,\mathrm{d}\mu$ in X in the norm topology, it follows that

$$\lim_{m\to\infty}\int_E\sum_{n=1}^{m}\xi_n(x_d^*-x^*)f_n\,\mathrm{d}\mu=\Big\langle x_d^*-x^*,\sum_{n=1}^{\infty}\xi_n(P)\int_E f_n\,\mathrm{d}\mu\Big\rangle \tag{2.2}$$

for each $d\in D$. On the other hand, since for each fixed $d\in D$ the sequence $\Big\{\sum\limits_{n=1}^{m}\xi_n(x_d^*-x^*f_n\Big\}_{m=1}^{\infty}$ converges in $L_1(\mu)$ in norm, and hence weakly, to the function

$\sum_{n=1}^{\infty} \xi_n(x_d^* - x^*)f_n$, then

$$\lim_{m\to\infty} \int_E \sum_{n=1}^{m} \xi_n(x_d^* - x^*)f_n \,\mathrm{d}\mu = \int_E \sum_{n=1}^{\infty} \xi_n(x_d^* - x^*)f_n \,\mathrm{d}\mu. \tag{2.3}$$

So, using (2.2) and (2.3), we have

$$\int_E \sum_{n=1}^{\infty} \xi_n(x_d^* - x^*)f_n \,\mathrm{d}\mu = \left\langle x_d^* - x^*, \sum_{n=1}^{\infty} \xi_n(P) \int_E f_n \,\mathrm{d}\mu \right\rangle$$

for each $d \in D$. Hence equation (2.1) leads to $\left|\int_E \sum_{n=1}^{\infty} \xi_n(x_d^* - x^*)f_n \,\mathrm{d}\mu\right| < \varepsilon$ for each $d > k$. This implies that $\int_E (\varphi\xi)x_d^* \,\mathrm{d}\mu \to \int_E (\varphi\xi)x^* \,\mathrm{d}\mu$. Since this is true for every $E \in \Sigma$, it follows that $(\varphi\xi)x_d^* \to (\varphi\xi)x^*$ in the weak topology of $L_1(\mu)$. Hence we have shown that $\varphi(\ell_\infty) \subseteq L_{w^*}(X^*, L_1(\mu))$. Finally, since $\|\varphi e_n\| = \|f_n\|_{\mathscr{P}(\mu,X)}$ for each $n \in \mathbb{N}$, then $\inf_{n\in\mathbb{N}} \|\varphi e_n\| > 0$ and Rosenthal's ℓ_∞ theorem guarantees that $L_{w^*}(X^*, L_1(\mu))$ contains a copy of ℓ_∞. □

Corollary 2.4. *If X has the Schur property, $\mathscr{P}(\mu, X)$ contains no copy of c_0.*

P r o o f. This is a straightforward consequence of Theorems 2.3 and 1.1 since, if X has the Schur property, then $K_{w^*}(X^*, L_1(\mu)) = L_{w^*}(X^*, L_1(\mu))$. □

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