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ANNIHILATORS IN BCK-ALGEBRAS

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Abstract. We introduce the concepts of an annihilator and a relative annihilator of a given subset of a BCK-algebra $\mathscr{A}$. We prove that annihilators of deductive systems of BCK-algebras are again deductive systems and moreover pseudocomplements in the lattice $\mathscr{D}(A)$ of all deductive systems on $\mathscr{A}$. Moreover, relative annihilators of $C \in \mathscr{D}(A)$ with respect to $B \in \mathscr{D}(A)$ are introduced and serve as relative pseudocomplements of C w.r.t. B in $\mathscr{D}(A)$.

Keywords: BCK-algebra, deductive system, annihilator, pseudocomplement

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1. Introduction

BCK-algebras are important tools for recent investigations in algebraic logic. They are algebras arising as an algebraic counterpart of purely implicational logics (see [2]) containing only a logical connective implication $\rightarrow$ and the constant 1 considered as the value "true", in which the formulas

$$(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r)), \tag{B}$$

$$(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r)) \tag{C}$$

and

$$p \rightarrow (q \rightarrow p) \tag{K}$$

are theorems. Here (B) or (C) means transitivity or commutativity, respectively.

BCK algebras were treated from various points of view, see e.g. [7], [8] or [9].

We will start with a formal definition of a larger class of algebras called BCC-algebras ([6]):

Definition. An algebra $\mathscr{A} = (A, \cdot, 1)$ with a binary operation $\cdot$ and a nullary operation 1 is called a BCC-*algebra* if it satisfies the following axioms:
(BCC1) $(z \cdot x) \cdot [(y \cdot z) \cdot (y \cdot x)] = 1$,
(BCC2) $x \cdot x = 1$,
(BCC3) $x \cdot 1 = 1$,
(BCC4) $1 \cdot x = x$,
(BCC5) $x \cdot y = 1$ & $y \cdot x = 1 \Rightarrow x = y$.

As usual, a congruence Θ on a BCC-algebra $\mathscr{A}$ is every compatible eqivalence on A, its equivalence block $[1]_\Theta$ containing the element 1 is called the *kernel* of Θ. The set $\operatorname{Con}\mathscr{A}$ of all congruences on $\mathscr{A}$ forms a lattice with respect to set inclusion. A BCC-algebra satisfying the identity

$$x \cdot (y \cdot z) = y \cdot (x \cdot z)$$

is a BCK-*algebra*, see e.g. [7], [8]. Left distributive BCK-algebras, i.e. those in which the identity

$$x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$$

is valid, are called *Hilbert algebras* ([5]). The notion of a deductive system in a BCK-algebra was introduced in [8]:

Definition. A subset $D \subseteq A$ of a BCK-algebra $\mathscr{A} = (A, \cdot, 1)$ is called a *deductive system* of $\mathscr{A}$ if

(D1) $1 \in D$,
(D2) $x \cdot y \in D$ and $x \in D$ imply $y \in D$.

Denote $\operatorname{Ded}\mathscr{A}$ the set of all deductive systems of $\mathscr{A}$. Since $\operatorname{Ded}\mathscr{A}$ is closed under arbitrary intersections, $(\operatorname{Ded}\mathscr{A}, \subseteq)$ is a complete algebraic lattice. For $M \subseteq A$ let $D(M)$ denotes the deductive system generated by M.

In [8] it is shown that deductive systems of BCK-algebras are in a 1-1 correspondence with their congruence kernels, namely we have

Lemma 1. *Let $\mathscr{A} = (A, \cdot, 1)$ be a BCK-algebra, $\Theta, \Psi \in \operatorname{Con}\mathscr{A}$, $D \in \operatorname{Ded}\mathscr{A}$. Then*
(1) *$[1]_\Theta$ is a deductive system of $\mathscr{A}$*
(2) *the relation Θ_D on A defined by $\langle x, y\rangle \in \Theta_D$ iff $x \cdot y, y \cdot x \in D$ is a congruence on $\mathscr{A}$ with $[1]_{\Theta_D} = D$,*
(3) *each congruence is completely determined by its kernel, i.e. $[1]_\Theta = [1]_\Psi$ implies $\Theta = \Psi$.*

Moreover, assignments (1) *and* (2) *are isomorphisms between the lattices* $\operatorname{Con}\mathscr{A}$ *and* $\operatorname{Ded}\mathscr{A}$.

It can be easily checked that the relation $\leqslant$ defined on a BCK-algebra $\mathscr{A} = (A, \cdot, 1)$ by

$$x \leqslant y \text{ if and only if } x \cdot y = 1$$

is a partial order on A with 1 as the greatest element. This order relation is called a *natural ordering* on $\mathscr{A}$.

Example. It is known that every partially ordered set $(P, \leqslant, 1)$ with the greatest element 1 can be regarded as a BCK-algebra if one defines the operation $\cdot$ on P as follows:

$$x \cdot y = 1 \text{ for } x \leqslant y \text{ and } x \cdot y = y \quad \text{otherwise.}$$

In fact such an algebra is a Hilbert one and its natural ordering coincides with the given order $\leqslant$.

Moreover, the operation $\cdot$ is compatible with the natural ordering in the following sense:

Lemma 2. *Let $\mathscr{A} = (A, \cdot, 1)$ be a* BCK*-algebra, $\leqslant$ its natural ordering and $x, y, z \in A$. Then $x \leqslant y$ implies $z \cdot x \leqslant z \cdot y$ and $y \cdot z \leqslant x \cdot z$.*

P r o o f. Follows easily from (BCC1). $\square$

Hilbert algebras were introduced in the 50-ties by L. Henkin and T. Skolem for investigations in intuitionistic and other non-classical logics. In [3] it has been shown that for a Hilbert algebra $\mathscr{A}$ the lattice $\operatorname{Ded}\mathscr{A}$ is distributive and algebraic, hence also pseudocomplemented and relatively pseudocomplemented (in spite of Lemma 1 the same holds also for the lattice $\operatorname{Con}\mathscr{A}$). In [4] the description of pseudocomplements or relative pseudocomplements, respectively, is given by means of the so-called annihilators or relative annihilators.

The aim of this paper is to find a similar description for a larger class of all BCK-algebras.

2. Annihilators and relative annihilators in BCK-algebras

In what follows suppose that $\mathscr{A} = (A, \cdot, 1)$ is a BCK-algebra. First we will focus on properties of the lattice $\operatorname{Ded}\mathscr{A}$.

Lemma 3. *Let $\mathscr{A} = (A, \cdot, 1)$ be a* BCK*-algebra. The lattice* $\operatorname{Ded}\mathscr{A}$ *is a distributive algebraic lattice, hence pseudocomplemented and relatively pseudocomplemented.*

Proof. By Lemma 1 we know that lattices $\operatorname{Ded}\mathscr{A}$ and $\operatorname{Con}\mathscr{A}$ are isomorphic and, moreover, each congruence is completely determined by its kernel. Hence to prove distributivity of $\operatorname{Con}\mathscr{A}$ it is enough to prove that for any triple $\Theta, \Psi, \varphi \in \operatorname{Con}\mathscr{A}$ the inclusion

$$[1]_{\Theta\cap(\Psi\vee\varphi)} \subseteq [1]_{(\Theta\cap\Psi)\vee(\Theta\cap\varphi)}$$

holds (the converse inclusion is valid trivially). For this suppose $x \in [1]_{\Theta\cap(\Psi\vee\varphi)}$, hence there exist $c_1, \ldots, c_n \in A$ such that $1\Theta x$ and $1 = c_1\Psi c_2\varphi c_3 \ldots c_{n-1}\Psi c_n = x$. Applying the substitution property we get

$$x = (1 \cdot x)\Psi(c_2 \cdot x)\varphi(c_3 \cdot x) \ldots (c_{n-1} \cdot x)\Psi(x \cdot x) = 1,$$

$1 = (c_i \ \cdot 1)\Theta(c_i \ \cdot x)$ and $(c_{i-1} \cdot x)\Theta(c_i \cdot x)$ for all $i \in 1, \ldots, n$. Altogether we have

$$x = (1 \cdot x)(\Psi \cap \Theta)(c_2 \cdot x)(\varphi \cap \Theta)(c_3 \cdot x) \ldots (c_{n-1} \cdot x)(\Psi \cap \Theta)(x \cdot x) = 1$$

proving $x \in [1]_{(\Theta\cap\Psi)\vee(\Theta\cap\varphi)}$. Algebraicity of $\operatorname{Ded}\mathscr{A}$ simply follows from algebraicity of $\operatorname{Con}\mathscr{A}$. The fact that every distributive algebraic lattice is pseudocomplemented is well-known. $\square$

Now we are ready to describe pseudocomplements in $\operatorname{Ded}\mathscr{A}$. For the case of commutative BCK-algebras, i.e. those which are join semilattices with respect to a natural order, this was already done in [1]. In the general case we need to know which pairs of deductive systems have trivial intersection.

Lemma 4. *Let $\mathscr{A} = (A, \cdot, 1)$ be a* BCK*-algebra and $A, B \in \operatorname{Ded}\mathscr{A}$. Then*

(1) $A \cap B = \{(b \cdot a) \cdot a;\ a \in A,\ b \in B\}$,

(2) $A \cap B = \{1\}$ *iff* $b \cdot a = a$ *for each* $a \in A$ *and* $b \in B$.

Proof. (1) Denote $M = \{(b\cdot a)\cdot a;\ a \in A,\ b \in B\}$ and suppose $y = (b\cdot a)\cdot a \in M$. We have $a \cdot [(b \cdot a) \cdot a] = (b \cdot a) \cdot (a \cdot a) = (b \cdot a) \cdot 1 = 1 \in A$ and since $a \in A$, applying (D2) we get $(b \cdot a) \cdot a \in A$. Analogously,

$$b \cdot [(b \cdot a) \cdot a] = (b \cdot a) \cdot (b \cdot a) = 1 \in B.$$

Using the same argument we obtain $(b\cdot a)\cdot a \in B$ and altogether $y = (b\cdot a)\cdot a \in A\cap B$. Conversely, let $z \in A \cap B$. Then setting $a = b = z$ yields

$$z = 1 \cdot z = (z \cdot z) \cdot z \in M$$

and proves the converse inclusion.

(2) easily follows from (1). $\square$

The foregoing result motivates us to introduce the following concepts.

Definition. Let B, C be subsets of a BCK-algebra $\mathscr{A} = (A, \cdot, 1)$. The subset

$$\langle C\rangle = \{x \in A;\ x \cdot c = c \text{ for each } c \in C\}$$

is called an *annihilator* of C. The subset

$$\langle C, B\rangle = \{x \in A;\ (x \cdot c) \cdot c \in B \ \text{ for each } c \in C\}$$

is called a *relative annihilator* of C *with respect to* B. If $C = \{c\}$ is a singleton, we will write briefly $\langle c\rangle$ instead of $\langle\{c\}\rangle$.

One can easily prove the following properties of annihilators.

Lemma 5. *Let $\mathscr{A} = (A, \cdot, 1)$ be a* BCK-*algebra, $B, C \subseteq A$.*
(1) *If $B \subseteq C$ then $\langle C\rangle \subseteq \langle B\rangle$,*
(2) *$C \subseteq \langle\langle C\rangle\rangle$,*
(3) *$\langle 1\rangle = A$ and $\langle A\rangle = \{1\}$,*
(4) *$\langle C\rangle = \bigcap\{\langle x\rangle;\ x \in C\}$.*

Theorem 1. *Let $\mathscr{A} = (A, \cdot, 1)$ be a* BCK-*algebra. Then for each $c \in A$ the annihilator $\langle c\rangle$ is a deductive system of $\mathscr{A}$ and hence if $C \in \operatorname{Ded}\mathscr{A}$, the annihilator $\langle C\rangle$ is a pseudocomplement of C in $\operatorname{Ded}\mathscr{A}$.*

P r o o f. Let us prove that $\langle c\rangle$ is a deductive system of $\mathscr{A}$. Evidently, $1 \in \langle c\rangle$. Suppose further $x \cdot y,\ x \in \langle c\rangle$ for some $x, y \in A$, i.e. $x \cdot c = c$ and $(x \cdot y) \cdot c = c$. Applying (BCC1) we obtain

$$1 = (y \cdot c) \cdot [(x \cdot y) \cdot (x \cdot c)] = (y \cdot c) \cdot [(x \cdot y) \cdot c] = (y \cdot c) \cdot c.$$

This means $(y \cdot c) \leqslant c$ and since the converse inequality is valid trivially, we get the desired equality $y \cdot c = c$.

Lemma 5(4) then yields that $\langle C\rangle$ is also a deductive system for each $C \subseteq A$. It is an easy exercise to verify that $\langle C\rangle$ is a pseudocomplement of $C \in \operatorname{Ded}\mathscr{A}$. $\square$

Now, we are interested in determining conditions under which a set and the deductive system generated by this set have the same annihilators.

Theorem 2. *Let $\mathscr{A} = (A, \cdot, 1)$ be a* BCK-*algebra. The following conditions are equivalent:*
(1) $\langle M \rangle = \langle D(M) \rangle$ *for each* $M \subseteq A$,
(2) *for each* $b, c \in A$, $b \cdot c = c$ *if and only if* $c \cdot b = b$.

Proof. (1) $\Rightarrow$ (2) Let $b, c \in A$ be such that $b \cdot c = c$, i.e. $b \in \langle c \rangle$. Then we have $b \in \langle D(c) \rangle$ by (1). Since $(c \cdot b) \cdot b \in D(c)$, we have $b \in \langle (c \cdot b) \cdot b \rangle$ and $1 = (c \cdot b) \cdot (b \cdot b) = b \cdot [(c \cdot b) \cdot b] = (c \cdot b) \cdot b$, and finally, $c \cdot b = b$.

(2) $\Rightarrow$ (1) Let $b, c \in A$. By the definition of an annihilator,

$$b \in \langle c \rangle \text{ if and only if } c \in \langle b \rangle$$

for every $b, c \in A$.

First, we prove the required equality for every singleton $M = \{c\}$. By Lemma 5(4), $\langle D(c) \rangle = \bigcap \{\langle x \rangle;\ x \in D(c)\}$. We need only to show that $\langle c \rangle \subseteq \langle D(c) \rangle$ since the opposite inclusion follows from Lemma 5 (1). Consider $z \in \langle c \rangle$. Then $c \in \langle z \rangle$ and, by Theorem 1, $\langle z \rangle$ is a deductive system of $\mathscr{A}$, whence $D(c) \subseteq \langle z \rangle$. Suppose now $x \in D(c)$. Then $x \in \langle z \rangle$ and again $z \in \langle x \rangle$, i.e.

$$z \in \bigcap \{\langle x \rangle;\ x \in D(c)\} = \langle D(c) \rangle.$$

Now let $M \subseteq A$. As was already proved, we have

$$\langle M \rangle = \bigcap \{\langle m \rangle;\ m \in M\} = \{\langle D(m) \rangle;\ m \in M\}.$$

If $y \in \langle m \rangle$ for each $m \in M$, then (2) implies $m \in \langle y \rangle$ which gives $D(M) \subseteq \langle y \rangle$. By Lemma 5 we have $y \in \langle \langle y \rangle \rangle \subseteq \langle D(M) \rangle$ finishing the proof. $\square$

Theorem 3. *Let B, C be deductive systems of a* BCK-*algebra $\mathscr{A} = (A, \cdot, 1)$. Then the relative annihilator $\langle C, B \rangle$ is a deductive system of $\mathscr{A}$ and it is a relative pseudocomplement of C with respect to B in the lattice* $\operatorname{Ded} \mathscr{A}$.

Proof. First, let us prove that for $B, C \in \operatorname{Ded} \mathscr{A}$, $\langle C, B \rangle$ is a deductive system of $\mathscr{A}$. It is immediate that $1 \in \langle C, B \rangle$. To prove (D2) suppose $x \cdot y, x \in \langle C, B \rangle$ for some $x, y \in A$. This means

$$(x \cdot c) \cdot c \in B \text{ and } ((x \cdot y) \cdot c) \cdot c \in B$$

for each $c \in C$. We already know that $x \cdot c \in C$, hence also

$$[(x \cdot y) \cdot (x \cdot c)] \cdot (x \cdot c) \in B$$

for each $c \in C$. Set $u = (y \cdot c) \cdot (x \cdot c)$. According to (BCC1), $(y \cdot c) \cdot ((x \cdot y) \cdot (x \cdot c)) = 1$ which is equivalent to $(y \cdot c) \leqslant (x \cdot y) \cdot (x \cdot c)$. Applying Lemma 2 to the last inequality we get

$$u = (y \cdot c) \cdot (x \cdot c) \geqslant [(x \cdot y) \cdot (x \cdot c)] \cdot (x \cdot c) \in B,$$

hence $u \in B$. Let us denote further $v = (y \cdot c) \cdot c$ and prove that $x \cdot c = ((x \cdot c) \cdot c) \cdot c$. The equality

$$(x \cdot c)[((x \cdot c) \cdot c) \cdot c] = [(x \cdot c) \cdot c)] \cdot [(x \cdot c) \cdot c] = 1$$

yields $x \cdot c \leqslant ((x \cdot c) \cdot c) \cdot c$. Substituting $y = (x \cdot c) \cdot c$ into the inequality $(y \cdot c) \leqslant (x \cdot y) \cdot (x \cdot c)$ we obtain

$$((x \cdot c) \cdot c) \cdot c \leqslant [x \cdot ((x \cdot c) \cdot c)] \cdot (x \cdot c) = [(x \cdot c) \cdot (x \cdot c)] \cdot (x \cdot c) = x \cdot c,$$

proving the converse inequality. Finally we compute

$$[(x \cdot c) \cdot c] \cdot [(y \cdot c) \cdot c] = (y \cdot c) \cdot [((x \cdot c) \cdot c) \cdot c] = (y \cdot c) \cdot (x \cdot c) \in B.$$

However, by the assumption also $(x \cdot c) \cdot c \in B$ and since B is a deductive system of $\mathscr{A}$, also $(y \cdot c) \cdot c \in B$ completing the proof of $\langle C, B\rangle \in \operatorname{Ded} \mathscr{A}$. An easy computation shows that $C \cap \langle C, B\rangle \subseteq B$. Let us prove that $\langle C, B\rangle$ is the greatest deductive system with the above property. Indeed, let $F \in \operatorname{Ded} \mathscr{A}$ be such that $C \cap F \subseteq B$. For each $c \in C$ and $f \in F$ the element $(f \cdot c) \cdot c \in C \cap F \subseteq B$, hence $f \in \langle C, B\rangle$ proving $F \subseteq \langle C, B\rangle$. $\square$

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