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# TORSIONS OF CONNECTIONS ON HIGHER ORDER COTANGENT BUNDLES

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*Abstract.* By a torsion of a general connection $\Gamma$ on a fibered manifold $Y \to M$ we understand the Frölicher-Nijenhuis bracket of $\Gamma$ and some canonical tangent valued one-form (affinor) on $Y$. Using all natural affinors on higher order cotangent bundles, we determine all torsions of general connections on such bundles. We present the geometrical interpretation and study some properties of the torsions.

*Keywords*: affinor, general connection, torsion

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## 1. Introduction

Given a linear connection $\Gamma$ on a manifold $M$, there are two classical approaches to the concept of the torsion of $\Gamma$. First, if we interpret $\Gamma$ as a linear connection on the tangent bundle $TM$, we can define the torsion of $\Gamma$ as the covariant exterior differential in the sense of Koszul of the identity tensor on $M$. This approach leads to the well known formula

$$\tau(X, Y) = \frac{1}{2}(\nabla_X Y - \nabla_Y X - [X, Y]) \tag{1}$$

for every two vector fields $X, Y$ on $M$. On the other hand, $\Gamma$ can be also interpreted as a principal connection on the frame bundle $PM$ of $M$. In this case we can introduce the torsion of $\Gamma$ as the standard covariant differential of the canonical $\mathbb{R}^m$-valued

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form $\Theta\colon TPM \to \mathbb{R}^m$ of $PM$, $m = \dim M$. Clearly, both above definitions of the torsion of $\Gamma$ give the same result. Further, if $\Gamma^*$ is the dual linear connection on the cotangent bundle $T^*M$, the torsion of $\Gamma^*$ can be introduced as the composition

$$\sigma \circ \Gamma^* \tag{2}$$

of the formal exterior differential $\sigma\colon J^1T^*M \to \wedge^2 T^*M$ with the connection $\Gamma^*$ itself, [8]. But the composition (2) can be defined for arbitrary (not only linear) connections on the cotangent bundle.

If $Y \to M$ is a fibered manifold, the (general) connection on $Y$ is a smooth section $\Gamma\colon Y \to J^1Y$ of the first jet prolongation of $Y$. Obviously, $\Gamma$ can be identified with its horizontal projection $TY \to TY$, which is a tangent valued one-form on $Y$. If $Q\colon TY \to TY$ is an arbitrary canonical tangent valued one-form on $Y$ (in other words an affinor), the Frölicher-Nijenhuis bracket $[\Gamma, Q]$ of $\Gamma$ and $Q$ is called a (general) torsion of $\Gamma$. Let $F$ be a natural bundle on the category $\mathcal{M}f_m$ of all $m$-dimensional manifolds and their local diffeomorphisms and $\Gamma\colon FM \to J^1FM$ be a connection on $FM$. If we determine all natural affinors on $FM$, then we can completely describe all general torsions of $\Gamma$. Such an approach has been used e.g. in [2], [6] and [9] for many particular cases of $F$.

In this paper we determine all general torsions of connections on $r$-th order cotangent bundles $T^{r*}M$. We also study the geometrical interpretation and properties of the torsions. In the geometrical construction of the torsions we essentially use the vertical prolongation of connections on vector bundles. We characterize linear connections and projectable connections on $T^{r*}M$ by means of certain properties of their torsions. We also show the interpretation of (2) for arbitrary (not only linear) connections. We remark that higher order cotangent bundles and their dual bundles (which are called higher order tangent bundles) are used e.g. in higher order mechanics. For example, in the study, within a continuum mechanical context, of higher order gross bodies, it is useful to study fields with values in tensor products of tangent and cotangent spaces of "higher order contact", cf. [10]. All manifolds and maps are assumed to be infinitely differentiable.

## 2. Affinors and general torsions

Denoting by $C^\infty(TM \otimes \wedge^p T^*M)$ the space of all tangent valued $p$-forms on $M$, the Frölicher-Nijenhuis bracket is a map

$$[\,,]\colon C^\infty(TM \otimes \wedge^p T^*M) \times C^\infty(TM \otimes \wedge^q T^*M) \to C^\infty(TM \otimes \wedge^{p+q} T^*M),$$

(cf. [3]). In particular, for $p = 0 = q$ the Frölicher-Nijenhuis bracket coincides with the Lie bracket of vector fields on $M$. By the Frölicher-Nijenhuis theory, tangent valued 1-forms on $M$ are exactly *affinors* on $M$, which are defined as linear morphisms $TM \to TM$ over the identity of $M$. Then the Frölicher-Nijenhuis bracket of two such affinors $K$, $L$ is a tangent valued 2-form $[K, L]$ given by

$$\begin{aligned}[K,L](X,Y) = {} & [KX,LY] + [LX,KY] + KL[X,Y] + LK[X,Y] \\ & - K[X,LY] - K[LX,Y] - L[X,KY] - L[KX,Y],\end{aligned} \tag{3}$$

$X$, $Y$ being arbitrary vector fields on $M$.

Let $p\colon Y \to M$ be a fibered manifold and denote by $VY$ the vertical tangent bundle. An affinor $\varphi \in C^\infty(TY \otimes T^*Y)$ on $Y$ is called *vertical* if $\varphi \in C^\infty(VY \otimes T^*Y)$. Consider now the inclusion $T^*M \subset T^*Y$ of cotangent bundles.

**Definition.** A vertical affinor $\varphi \in C^\infty(VY \otimes T^*M)$ is called a *soldering form.*

Clearly, $J^1Y \to Y$ is an affine bundle, the associated vector bundle of which is $VY \otimes T^*M$. In this way soldering forms on $Y$ can be considered as "deformations" of connections on $Y$ in the following sense: If $\Gamma\colon Y \to J^1Y$ is a connection and $\varphi\colon Y \to VY \otimes T^*M$ a soldering form on $Y$, then their sum $(\Gamma + \varphi)$ is a connection on $Y$. On the other hand, the difference of two connections on $Y$ is a soldering form.

If we identify a connection $\Gamma\colon Y \to J^1Y$ with its horizontal projection $TY \to TY$ (denoted by the same symbol $\Gamma$), we obtain an affinor on $Y$. In this situation the Frölicher-Nijenhuis bracket turns out to be a very powerful tool in the theory of connections. For example, by [3], the curvature of $\Gamma$ is given by

$$C\Gamma = \frac{1}{2}[\Gamma, \Gamma] \tag{4}$$

and the Bianchi identity can be simply expressed in the form $[\Gamma, C\Gamma] = 0$. Consider now a soldering form $\varphi\colon Y \to VY \otimes T^*M$ (i.e. a deformation of $\Gamma$) and modify the curvature formula (4) by

$$\tau_\varphi := [\Gamma, \varphi]. \tag{5}$$

By Mangiarotti and Modugno, [8], $\tau_\varphi\colon Y \to VY \otimes \wedge^2 T^*M$ is called a *torsion of* $\Gamma$ *with respect to the soldering form* $\varphi$. By [8], $\tau_\varphi$ generalizes the classical concept of the torsion of a connection. Finally, the most general definition of a torsion was given by Kolář and Modugno, [2]. The main idea is to replace the soldering form $\varphi$ in (5) by an arbitrary affinor on $Y$, which is canonical in the following sense:

**Definition.** A *natural affinor* on a natural bundle $F$ over manifolds is a system of affinors $Q_M\colon TFM \to TFM$ for every $m$-manifold $M$ satisfying $TFf \circ Q_M = Q_N \circ TFf$ for every local diffeomorphism $f\colon M \to N$.

Let $F$ be a natural bundle on the category $\mathscr{M}f_m$ and $\Gamma\colon FM \to J^1(FM \to M)$ be a connection on $FM$.

**Definition.** Let $Q$ be a non identical natural affinor on $F$. The Frölicher-Nijenhuis bracket $[\Gamma, Q]$ is called the (general) *torsion* of a connection $\Gamma$.

Clearly, $[\Gamma,\varphi] \in C^\infty(TY \otimes \wedge^2 T^*Y)$ and if $\mathrm{Id}\colon FM \to FM$ is a trivial identical affinor, then we have $[\Gamma, \mathrm{Id}] = 0$. For many particular cases of a natural bundle $F$ we are able to determine all natural affinors on $F$ and in this way all general torsions of a given connection $\Gamma$ on $FM$. For example, any product preserving functor $F$ on the category $\mathscr{M}f$ of all smooth manifolds and all smooth maps is a Weil functor of the form $F = T^A$, where $A = F\mathbb{R}$ is the corresponding Weil algebra, [3]. By Kolář and Modugno, [2], all natural affinors on $F$ are parametrized by $F\mathbb{R}$. An important example of a product preserving functor is the functor $T^r_k$ of $k$-dimensional velocities of order $r$, which is defined by

$$T^r_k M = J^r_0(\mathbb{R}^k, M), \quad T^r_k f(j^r_0 g) = j^r_0(f \circ g)$$

for an arbitrary smooth manifold $M$ and a smooth map $f\colon M \to N$. On the basis of the complete list of all natural affinors on product preserving bundles, Kolář and Modugno have described all general torsions of connections on the bundles $TM$, $T^1_k M$, $T^2_1 M$ and on the frame bundle $PM$, [2]. The same authors have also computed general torsions on the cotangent bundle.

From (3) it follows that the Frölicher-Nijenhuis bracket of a connection $\Gamma\colon Y \to J^1 Y$ and a soldering form $\varphi\colon Y \to VY \otimes T^*M$ is of the form

$$[\Gamma,\varphi](U,V) = \frac{1}{2}\big([\Gamma U, V] - [\Gamma V, \varphi U] - \varphi[U,V]\big) \tag{6}$$

for every two vector fields $U$, $V$ on $M$. Denoting by $(x^i, y^p)$ the local fibered coordinates on $Y$, the equations of a connection $\Gamma\colon Y \to J^1 Y$ are

$$\mathrm{d}y^p = F^p_i(x,y)\,\mathrm{d}x^i$$

so that the corresponding horizontal projection $\Gamma\colon TY \to TY$ is of the form $(\mathrm{d}x^i, \mathrm{d}y^p) \mapsto (\mathrm{d}x^i, F^p_i \mathrm{d}x^i)$, i.e.

$$\delta^i_j \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j + F^p_i \frac{\partial}{\partial y^p} \otimes \mathrm{d}x^i.$$

Further, if

$$\varphi_i^p(x,y)\frac{\partial}{\partial y^p}\otimes \mathrm{d}x^i$$

is the coordinate expression of a soldering form $\varphi\in C^\infty(VY\otimes T^*M)$, then (6) is of the form

$$\left(\frac{\partial\varphi_j^p}{\partial x^i}+F_i^q\frac{\partial\varphi_j^p}{\partial y^q}-\varphi_j^q\frac{\partial F_i^p}{\partial y^q}\right)\frac{\partial}{\partial y^p}\otimes(\mathrm{d}x^i\wedge\mathrm{d}x^j). \tag{7}$$

## 3. Natural affinors on higher order cotangent bundles

Let $M$ be a smooth manifold. The space

$$T^{r*}M=J^r(M,\mathbb{R})_0$$

is called the $r$-th order cotangent bundle. Every local diffeomorphism $f\colon M\to N$ extends to a vector bundle morphism $T^{r*}f\colon T^{r*}M\to T^{r*}N$, $j^r_x\varphi\mapsto j^r_{f(x)}(\varphi\circ f^{-1})$, where $f^{-1}$ is constructed locally. Then $T^{r*}$ is a vector bundle functor defined on the category $\mathscr{M}f_m$. Clearly, the functor $T^{r*}$ does not preserve products and for $r=1$ we obtain the classical cotangent functor $T^*$. Denoting by $(x^i)$ some local coordinates on $M$, the induced coordinates $(u_i,u_{ij},\dots,u_{i_1\dots i_r})$ on $T^{r*}M$ (symmetric in all indices) are given by

$$u_i(j^r_xf)=\frac{\partial f}{\partial x^i}\bigg|_x,\quad u_{ij}(j^r_xf)=\frac{\partial^2 f}{\partial x^i\partial x^j}\bigg|_x,\quad u_{i_1\dots i_r}(j^r_xf)=\frac{\partial^r f}{\partial x^{i_1}\dots\partial x^{i_r}}\bigg|_x.$$

Denote by $\pi_M\colon T^{r*}M\to M$ and $p_M\colon TM\to M$ the bundle projections and $q_M\colon T^{r*}M\to T^*M$ the projection given by $q_M(j^r_xf)=j^1_xf$. Let $\lambda_M\colon TT^{r*}M\to\mathbb{R}$ be the generalized Liouville form on $T^{r*}M$ defined by

$$\lambda_M(X)=\langle q_M(p_{T^{r*}M}(X)),T\pi_M(X)\rangle$$

and $A^r_s\colon T^{r*}M\to T^{r*}M$ be the $s$-th power natural transformation of $T^{r*}M$ into itself defined by $A^r_s(j^r_xf)=j^r_x(f)^s$, where $(f)^s$ denote the $s$-th power of $f$. Since $\pi_M\colon T^{r*}M\to M$ is a vector bundle, the vertical tangent bundle $VT^{r*}M$ can be identified with the Whitney sum $T^{r*}M\oplus T^{r*}M$. Taking into account this identification, we can define natural affinors $Q^s_M\colon TT^{r*}M\to VT^{r*}M$ by

$$Q^s_M(X)=(p_{T^{r*}M}(X),\lambda_M(X)A^r_s(p_{T^{r*}M}(X))). \tag{8}$$

In coordinates,

$$Q^s_M = u_{i_1\dots i_s} u_j \frac{\partial}{\partial u_{i_1\dots i_s}} \otimes \mathrm{d}x^j + \frac{(s+1)!}{(s-1)!\,2!} u_{(i_1} \dots u_{i_{s-1}} u_{i_s i_{s+1})} u_j \frac{\partial}{\partial u_{i_1\dots i_{s+1}}} \otimes \mathrm{d}x^j + \dots + \frac{r!}{(s-1)!\,(r-s+1)!} u_{(i_1} \dots u_{i_{s-1}} u_{i_s\dots i_r)} u_j \frac{\partial}{\partial u_{i_1\dots i_r}} \otimes \mathrm{d}x^j \tag{9}$$

where $(i_1 \dots i_r)$ denote the symmetrization. The second author has proved, [4]:

**Lemma 1.** *All natural affinors on the $r$-th order cotangent bundle $T^{r*}M$ are of the form*

$$k_0 \operatorname{Id} + k_1 Q^1_M + \dots + k_r Q^r_M \tag{10}$$

*with any real parameters $k_0, k_1, \dots k_r$.*

Obviously, all nontrivial natural affinors on $T^{r*}M$ are vertical and we have $Q^i_M \in C^\infty(VT^{r*}M \otimes T^*M)$, i.e. all $Q^i_M$ are even soldering forms, $i = 1, \dots, r$. The second author has also described all natural transformations of $T^{r*}$ into itself, [5].

**Lemma 2.** *All natural transformations $T^{r*} \to T^{r*}$ are of the form*

$$k_1 A^r_1 + \dots + k_r A^r_r \tag{11}$$

*with any real parameters $k_1, \dots, k_r$.*

## 4. Vertical prolongation of connections on vector bundles

We first recall the concept of the vertical prolongation of a connection, see [3]. Let $\Gamma\colon Y \to J^1Y$ be a connection on a fibered manifold $p\colon Y \to M$. Applying the vertical tangent functor $V$, we obtain a map $V\Gamma\colon VY \to VJ^1Y$. Let

$$i_Y\colon VJ^1Y \to J^1VY$$

be the canonical exchange isomorphism, [3]. Then the composition $\mathscr{V}\Gamma := i_Y \circ V\Gamma\colon VY \to J^1VY$ is a connection on $VY \to M$. Denote by $(x^i, y^p, \eta^p = \mathrm{d}y^p)$ the canonical coordinates on $VY$. If $\mathrm{d}y^p = F^p_i(x,y)\mathrm{d}x^i$ is the coordinate form of $\Gamma$, then the vertical prolongation $\mathscr{V}\Gamma$ has equations

$$\begin{aligned}\mathrm{d}y^p &= F^p_i(x,y)\mathrm{d}x^i,\\ \mathrm{d}\eta^p &= \frac{\partial F^p_i(x,y)}{\partial y^q}\eta^q\mathrm{d}x^i.\end{aligned}$$

Now let $p\colon E\to M$ be a vector bundle. Then $J^1E\to M$ is a vector bundle too and we have the canonical identifications $VE\cong E\oplus_M E$ and $VJ^1E\cong J^1E\oplus_M J^1E$. Using the exchange isomorphism $i_E\colon VJ^1E\to J^1VE$ we obtain an identification

$$\alpha_E\colon J^1VE\cong J^1E\oplus_M J^1E.$$

Consider the classical Liouville vector field

$$L_E\colon E\to VE\subset TE$$

which is generated by the one-parameter group of homotheties of the vector bundle $E$, in coordinates $L_E = y^p\frac{\partial}{\partial y^p}$. If $\Gamma\colon E\to J^1E$ is a connection on $E$ (not necessarily linear), then the composition

$$\overline{\Gamma} := \mathrm{pr}_2\circ\alpha_E\circ\mathscr{V}\Gamma\circ L_E\colon E\to J^1E \tag{12}$$

is a section of $J^1E$. Clearly, $\overline{\Gamma}$ is a connection on $E\to M$ with the coordinate expression

$$\mathrm{d}y^p = \frac{\partial F^p_i(x,y)}{\partial y^q}y^q\mathrm{d}x^i.$$

We have

**Proposition 1.** $\Gamma=\overline{\Gamma}$ *if and only if the connection* $\Gamma$ *is linear.*

Proof. Linearity of $\Gamma$ means $F^p_i(x,y) = F^p_{ir}(x)y^r$. On the other hand, $\Gamma=\overline{\Gamma}$ iff $F^p_i(x,y) = \frac{\partial F^p_i(x,y)}{\partial y^q}y^q$. $\square$

One verifies directly

**Proposition 2.** *Let $\Gamma$ be a connection on the vector bundle $E \to M$. Then the vertical prolongation $\mathscr{V}\Gamma\colon VE \to J^1VE$ is of the form*

$$\mathscr{V}\Gamma = (\Gamma \times \overline{\Gamma})\colon\ E \oplus_M E \cong VE \to J^1VE \cong J^1E \oplus_M J^1E.$$

Suppose now that $E = FM$, where $F$ is a natural vector bundle functor. To construct the connection $\overline{\Gamma}$, we have used the Liouville vector field $L_{FM}\colon FM \to VFM$. This vector field is canonical (more precisely natural) in the following sense:

**Definition.** A *natural vector field* $\xi$ on $F$ is a system of vector fields $\xi_M\colon\ FM \to TFM$ for every $m$-manifold $M$ satisfying $TFf \circ \xi_M = \xi_N \circ Ff$ for all local diffeomorphisms $f\colon\ M \to N$.

Roughly speaking, natural vector fields on $F$ can be interpreted as absolute (or constant) natural operators transforming vector fields on $M$ into vector fields on $FM$, [3]. By [3], every natural vector field on $FM$ is vertical.

**Example.** Let $\Phi(t)$ be a smooth one-parameter family of natural transformations $F \to F$, where smoothness means that the map $(\Phi(t))_M\colon\ \mathbb{R} \times FM \to FM$ is smooth for every manifold $M$. Then the formula $\xi_M = \frac{\partial}{\partial t}\big|_0 (\Phi(t))_M$ defines a natural vector field $\xi_M\colon\ FM \to VFM$.

If we replace the Liouville vector field $L_{FM}$ in (12) by an arbitrary natural vector field $L\colon\ FM \to VFM$, we obtain

**Proposition 3.** *Let $\Gamma\colon\ FM \to J^1FM$ be a connection on a natural vector bundle $F$. Then every natural vector field $L\colon\ FM \to VFM$ induces a connection $\Gamma_L\colon\ FM \to J^1FM$ by*

$$\Gamma_L = \mathrm{pr}_2 \circ \alpha_{FM} \circ \mathscr{V}\Gamma \circ L. \tag{13}$$

If $L$ is the Liouville vector field $L_{FM}$, then $\Gamma_L = \overline{\Gamma}$.

Let $\Gamma\colon\ E \to J^1E$ be a connection on the vector bundle $E \to M$ and $f\colon\ E \to E$ be a vector bundle morphism over the identity of $M$. In the rest of this section we describe the construction of a connection on $E$ by means of the vector bundle morphism $f$. Denoting by $\mathrm{Id}_E\colon\ E \to E$ the identity morphism, the difference $(\mathrm{Id}_E - f)\colon\ E \to E$ is also a vector bundle morphism over $\mathrm{id}_M$. Using the inclusion $E \subset J^1E$ we obtain a vector bundle morphism

$$\widetilde{f} := (\mathrm{Id}_E - f)\colon\ E \to J^1E$$

over the identity of $M$. On the other hand, the composition

$$g := J^1 f \circ \Gamma\colon\ E \to J^1 E$$

is another vector bundle morphism over the identity of $M$. Then the sum of $\widetilde{f}$ and $g$ on the vector bundle structure $J^1 E$ is a section of $J^1 E$,

$$\Gamma_f := (\widetilde{f} + g)\colon\ E \to J^1 E. \tag{14}$$

Clearly, $\Gamma_f$ is a connection on $E$ and for $f = \mathrm{Id}_E$ we have $\Gamma_{\mathrm{Id}_E} = \Gamma$.

## 5. The torsion on the cotangent bundle

By Lemma 1, we have one nontrivial affinor $Q^1_M\colon\ TT^*M \to VT^*M$ on the cotangent bundle, $Q^1_M(\mathrm{d}x^i, \mathrm{d}u_i) = (0, u_i u_k \mathrm{d}x^k)$. If

$$\mathrm{d}u_i = \Gamma_{i,l}(x,u)\mathrm{d}x^l$$

are equations of a connection $\Gamma$ on $T^*M$, then by (7) the torsion $\tau := [\Gamma, Q^1_M]\colon$ $T^*M \to VT^*M \otimes \wedge^2 T^*M$ is of the form

$$\tau = \left(\Gamma_{k,i}u_j + \Gamma_{j,i}u_k - \frac{\partial \Gamma_{k,i}}{\partial u_m} u_m u_j\right) \frac{\partial}{\partial u_k} \otimes (\mathrm{d}x^i \wedge \mathrm{d}x^j).$$

Write

$$\begin{aligned} \tau^L &= \Gamma_{j,i}(\mathrm{d}x^i \wedge \mathrm{d}x^j) \otimes \left(u_k \frac{\partial}{\partial u_k}\right), \\ \tau^* &= \left(\Gamma_{k,i} - \frac{\partial \Gamma_{k,i}}{\partial u_m} u_m\right) u_j \frac{\partial}{\partial u_k} \otimes (\mathrm{d}x^i \wedge \mathrm{d}x^j). \end{aligned}$$

Then we have

**Proposition 4.** *The torsion on the cotangent bundle $T^*M$ is of the form*

$$\tau = \tau^L + \tau^*.$$

*Further, $\tau^* = 0$ if and only if the connection $\Gamma$ is linear.*

Proof. If $\Gamma$ is a linear connection, then $\Gamma_{k,i}(x,u) = \Gamma^r_{k,i}(x)u_r$ which yields $\tau^* = 0$. On the other hand, if $\tau^* = 0$, then we obtain the differential equation $\Gamma_{k,i} = \frac{\partial \Gamma_{k,i}}{\partial u_m} u_m$. Clearly, the differential equation $y = y' \cdot x$ has the general solution $y = cx$, i.e. $\Gamma_{k,i}(x,u) = \Gamma^r_{k,i}(x)u_r$. $\square$

In other words, a connection $\Gamma$ on $T^*M$ is linear if and only if $\tau = \tau^L$. In the rest of this section we describe the geometrical construction of $\tau^L$ and $\tau^*$. We remark that the construction of the "linear" term $\tau^L$ was given also by Kolář and Modugno in [2].

**I. The construction of $\tau^L$.** By the theory of sesquiholonomic 2-jets, there is a natural map

$$\sigma\colon\ J^1T^*M \to \wedge^2 T^*M$$

which is called the formal exterior differential, [7]. Denoting by $(x^i, u_i, u_{i,j})$ the local coordinates on $J^1T^*M$, the coordinate form of $\sigma$ is $u_{i,j}(\mathrm{d}x^i \wedge \mathrm{d}x^j)$. Then the composition $\sigma \circ \Gamma\colon\ T^*M \to \wedge^2 T^*M$ has the coordinate form

$$\Gamma_{i,j}(\mathrm{d}x^i \wedge \mathrm{d}x^j)$$

and $\tau^L\colon\ T^*M \to VT^*M \otimes \wedge^2 T^*M$ is of the form

$$\tau^L(u) = (\sigma \circ \Gamma)(u) \otimes L(u)$$

where $L\colon\ T^*M \to VT^*M$ is the classical Liouville vector field on $T^*M$.

**Remark.** By Proposition 4, the torsion of every connection on the cotangent bundle can be written in the form $\tau = \tau^L + \tau^*$. Using such a point of view, the "linear" part $\tau^L$ can be considered as the interpretation of the classical formula (2) for arbitrary connections.

**II. The construction of $\tau^*$.** Let $L$ be the classical Liouville vector field on $T^*M$ and denote by $\Gamma_L$ the induced connection (13) from Proposition 3, where we put $F = T^*$. Then the equations of $\Gamma_L\colon\ T^*M \to J^1T^*M$ are

$$\mathrm{d}u_i = \frac{\partial \Gamma_{i,k}}{\partial u_m} u_m \mathrm{d}x^k.$$

Since $J^1T^*M \to T^*M$ is an affine bundle with the associated vector bundle $VT^*M \otimes T^*M$, the difference of two connections $\Gamma$ and $\Gamma_L$ on $T^*M$ is a section of $VT^*M \otimes T^*M$, in coordinates

$$\left(\Gamma_{k,i} - \frac{\partial \Gamma_{k,i}}{\partial u_m} u_m\right) \frac{\partial}{\partial u_k} \otimes \mathrm{d}x^i.$$

Multiplying by the Liouville one-form $(u_j \mathrm{d}x^j)$ and then using the antisymmetrization $T^*M \otimes T^*M \to \wedge^2 T^*M$, we obtain $\tau^*$.

## 6. The first torsion on the $r$-th order cotangent bundle

A connection $\Gamma$ on $T^{r*}M$ has equations

$$\mathrm{d}u_i = \Gamma_{i,l}(x,u)\mathrm{d}x^l,\ \mathrm{d}u_{ij} = \Gamma_{ij,l}(x,u)\mathrm{d}x^l, \dots,\ du_{i_1\dots i_r} = \Gamma_{i_1\dots i_r,l}(x,u)\mathrm{d}x^l \tag{15}$$

and the first natural affinor $Q^1_M\colon TT^{r*}M \to VT^{r*}M$ is of the form

$$Q^1_M(\mathrm{d}x^i, \mathrm{d}u_i, \mathrm{d}u_{ij}, \dots, \mathrm{d}u_{i_1\dots i_r}) = (0, u_i u_p \mathrm{d}x^p, u_{ij}u_p\mathrm{d}x^p, \dots u_{i_1\dots i_r}u_p\mathrm{d}x^p),$$

see (8) and (9). In this section we show that the first torsion

$$\tau_1 := [\Gamma, Q^1_M]\colon\ T^{r*}M \to VT^{r*}M \otimes \wedge^2 T^*M$$

has quite similar properties and interpretation as the torsion $\tau$ on the cotangent bundle. Write

$$\tau_1^L = \Gamma_{j,i}(\mathrm{d}x^i\wedge\mathrm{d}x^j)\otimes\Big(u_k\frac{\partial}{\partial u_k} + u_{kl}\frac{\partial}{\partial u_{kl}} + \dots + u_{k_1\dots k_r}\frac{\partial}{\partial u_{k_1\dots k_r}}\Big), \tag{16}$$

$$\begin{aligned}\tau_1^* = {} & \Big(\Gamma_{k,i} - \frac{\partial\Gamma_{k,i}}{\partial u_m}u_m - \dots - \frac{\partial\Gamma_{k,i}}{\partial u_{m_1\dots m_r}}u_{m_1\dots m_r}\Big)u_j \\ & \times \frac{\partial}{\partial u_k}\otimes(\mathrm{d}x^i\wedge\mathrm{d}x^j) \\ & + \Big(\Gamma_{kl,i} - \frac{\partial\Gamma_{kl,i}}{\partial u_m}u_m - \dots - \frac{\partial\Gamma_{kl,i}}{\partial u_{m_1\dots m_r}}u_{m_1\dots m_r}\Big)u_j\frac{\partial}{\partial u_{kl}}\otimes(\mathrm{d}x^i\wedge\mathrm{d}x^j) \\ & \vdots \\ & + \Big(\Gamma_{k_1\dots k_r,i} - \frac{\partial\Gamma_{k_1\dots k_r,i}}{\partial u_m}u_m - \dots - \frac{\partial\Gamma_{k_1\dots k_r,i}}{\partial u_{m_1\dots m_r}}u_{m_1\dots m_r}\Big)u_j \\ & \times \frac{\partial}{\partial u_{k_1\dots k_r}}\otimes(\mathrm{d}x^i\wedge\mathrm{d}x^j).\end{aligned}$$

We prove

**Proposition 5.** *The first torsion $\tau_1$ on $T^{r*}M$ is of the form*

$$\tau_1 = \tau_1^L + \tau_1^*. \tag{17}$$

*Further, $\tau_1^* = 0$ if and only if the connection $\Gamma$ is linear.*

P r o o f. Formula (17) follows from (7) by a direct computation. If $\Gamma$ is linear, then

$$\Gamma_{k,i} = \Gamma^m_{k,i}u_m + \Gamma^{mn}_{k,i}u_{mn} + \dots + \Gamma^{m_1\dots m_r}_{k,i}u_{m_1\dots m_r}$$

so that the first term of $\tau_1^*$ vanishes. Quite analogously we show that all the remaining terms of $\tau_1^*$ are zero. On the other hand, if $\tau_1^* = 0$, then all the terms of $\tau_1^*$ vanish. In this case we obtain a partial differential equation $f(x, u_1, \dots, u_r) = \frac{\partial f}{\partial u_1} u_1 + \dots + \frac{\partial f}{\partial u_r} u_r$, whose solution is of the form $f = \widetilde{f}_1(x) u_1 + \dots + \widetilde{f}_r(x) u_r$. $\square$

**I. The geometrical construction of $\tau_1^L$.** Denote by $L\colon T^{r*}M \to VT^{r*}M$ the Liouville vector field. Using the formal exterior differential $\sigma\colon J^1T^*M \to \wedge^2 T^*M$ and the projection $q_M\colon T^{r*}M \to T^*M$ we obtain a map $\widetilde{\sigma} := \sigma \circ J^1 q_M\colon J^1T^{r*}M \to \wedge^2 T^*M$ with the coordinate form $u_{i,j}(\mathrm{d}x^i \wedge \mathrm{d}x^j)$. Then $\tau_1^L$ is of the form

$$\tau_1^L(u) = (\widetilde{\sigma} \circ \Gamma)(u) \otimes L(u). \tag{18}$$

**II. The construction of $\tau_1^*$.** Let $L$ be the Liouville vector field on $T^{r*}M$, $F = T^{r*}$ and $\Gamma_L$ be the connection (13) from Proposition 3. Taking into account that $J^1T^{r*}M \to T^{r*}M$ is an affine bundle with the corresponding associated vector bundle $VT^{r*}M \otimes T^*M$, the difference of connections $\Gamma$ and $\Gamma_L$ on $T^{r*}M$ is the section of the associated vector bundle. Using multiplication by the Liouville form $(u_j \mathrm{d}x^j)$ and then the antisymmetrization $\otimes^2 T^*M \to \wedge^2 T^*M$, we obtain $\tau_1^*$.

## 7. Torsions on the $r$-th order cotangent bundle

Let $\Gamma$ be a connection (15) on $T^{r*}M$. By Lemma 1, all natural affinors on $T^{r*}M$ are linearly generated by the identity affinor and by nontrivial affinors $Q_M^1, \dots, Q_M^r$. In this way we have the list of $r$ torsions on $T^{r*}M$,

$$\tau_s := [\Gamma, Q_M^s] \in C^\infty(VT^{r*}M \otimes \wedge^2 T^*M), \quad s = 1, \dots, r.$$

We show that analogously to Proposition 5, each such torsion can be expressed in the form $\tau_s = \tau_s^{L_s} + \tau_s^*$.

We first construct the terms $\tau_s^{L_s}$. Let $A_s^r\colon T^{r*}M \to T^{r*}M$ be the $s$-th power natural transformation from Lemma 2. Multiplying by a real number $t$, we obtain a smooth one-parameter family of natural transformations. Denote by

$$L_s\colon T^{r*}M \to VT^{r*}M$$

the natural vector field generated by this smooth one-parameter family, see Example. Then $L_1$ is the classical Liouville vector field. The vector field $L_s$ can be also defined as a map $T^{r*}M \to T^{r*}M \oplus T^{r*}M \cong VT^{r*}M$ of the form $u \mapsto (u, A_s^r(u))$. In coordinates,

$$L_s = u_{i_1} \dots u_{i_s} \frac{\partial}{\partial u_{i_1 \dots i_s}} + \dots + \frac{r!}{(s-1)!\,(r-s+1)!} u_{(i_1} \dots u_{i_{s-1}} u_{i_s \dots i_r)} \frac{\partial}{\partial u_{i_1 \dots i_r}}.$$

Now it suffices to modify (18) by

$$\tau_s^{L_s}(u) = (\widetilde{\sigma} \circ \Gamma)(u) \otimes L_s(u), \quad s = 1, \dots, r. \tag{19}$$

In coordinates, $\tau_s^{L_s} = \Gamma_{i,j}(\mathrm{d}x^i \wedge \mathrm{d}x^j) \otimes L_s$.

It remains to construct $\tau_s^*$. Write $f_s := A_s^r \colon T^{r*}M \to T^{r*}M$ for the $s$-th power natural transformation. Let $\Gamma_{f_s}$ be the connection on $T^{r*}M$ defined by (14) by means of the vector bundle morphism $f = f_s$. Since $f_1 = \mathrm{id}_{T^{r*}M}$, we have $\Gamma_{f_1} = \Gamma$. Further, let $\Gamma_{L_s}$ be the connection (13) from Proposition 3 corresponding to the natural vector field $L_s$ on $T^{r*}M$. Then the difference of connections $\Gamma_{f_s}$ and $\Gamma_{L_s}$ is a section $t\colon T^{r*}M \to VT^{r*}M \otimes T^*M$ of the associated vector bundle. Multiplication by the Liouville one-form $(u_j\,\mathrm{d}x^j)$ yields an element of $(VT^{r*}M \otimes T^*M) \otimes T^*M$. Finally, using antisymmetrization $\otimes^2 T^*M \to \wedge^2 T^*M$ we obtain $\tau_s\colon T^{r*}M \to VT^{r*}M \otimes \wedge^2 T^*M$. In what follows we shall need only the coordinate form of $\tau_r^*$,

$$\begin{aligned}\tau_r^* = {} & \Big(-\frac{\partial \Gamma_{k,i}}{\partial u_{m_1\dots m_r}}\, u_{m_1}\dots u_{m_r}\Big) u_j \frac{\partial}{\partial u_k} \otimes (\mathrm{d}x^i \wedge \mathrm{d}x^j) + \dots \\ & + \Big(-\frac{\partial \Gamma_{k_1\dots k_{r-1},i}}{\partial u_{m_1\dots m_r}} u_{m_1}\dots u_{m_r}\Big) u_j \frac{\partial}{\partial u_{k_1\dots k_{r-1}}} \otimes (\mathrm{d}x^i \wedge \mathrm{d}x^j) \\ & + \Big(\Gamma_{l,i}\delta^l_{(k_1} u_{k_2}\dots u_{k_r)} - \frac{\partial \Gamma_{k_1\dots k_r,i}}{\partial u_{m_1\dots m_r}}\, u_{m_1}\dots u_{m_r}\Big) u_j \\ & \times \frac{\partial}{\partial u_{k_1\dots k_r}} \otimes (\mathrm{d}x^i \wedge \mathrm{d}x^j).\end{aligned} \tag{20}$$

Using (7), (9) and (15) we prove by a direct evaluation

**Proposition 6.** *All torsions of a connection on $T^{r*}M$ are of the form*

$$\tau_s = \tau_s^{L_s} + \tau_s^*, \quad s = 1, \dots, r.$$

By Proposition 5, $\tau_1^* = 0$ if and only if the connection $\Gamma$ is linear. We show that the vanishing of $\tau_r^*$ expresses the projectability of $\Gamma$ in the following sense:

**Definition.** We say that a connection $\Gamma$ on $T^{r*}M$ is *projectable* if there is a connection $\Delta$ on $T^{(r-1)*}M$ such that

$$J^1\pi_M^{r,r-1} \circ \Gamma = \Delta \circ \pi_M^{r,r-1},$$

where $\pi_M^{r,r-1}\colon T^{r*}M \to T^{(r-1)*}M$ is the bundle projection.

**Proposition 7.** *If $\tau_r^* = 0$, then the connection $\Gamma$ is projectable.*

Proof. By (20), if $\tau_r^* = 0$, then the first $(r-1)$ components of $\Gamma$ in the coordinate expression (15) are independent of $u_{m_1 \dots m_r}$. □

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