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PROJECTABILITY AND SPLITTING PROPERTY OF LATTICE ORDERED GROUPS

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Abstract. In this paper we deal with the notions of projectability, spliting property and Dedekind completeness of lattice ordered groups, and with the relations between these notions.

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Projectable and strongly projectable lattice ordered groups were investigated in [1], [3], [5], [7], [8], [10]–[13]; for the case of vector lattices cf. [14].

The splitting property in the class $\mathscr{A}$ of all archimedean lattice ordered groups was studied in [1], [5], [6], [14].

Let $\mathscr{K}$ be a class of lattice ordered groups. In the present paper we define the splitting property in $\mathscr{K}$ in such a way that for the case when $\mathscr{K} = \mathscr{A}$ the new definition coincides with that used in the above mentioned papers.

We deal with the notions of projectability, splitting property and Dedekind completeness of lattice ordered groups, and with the relations between these notions.

The results and methods from [6] and [11] will be applied.

1. Splitting property

In this section we investigate the splitting property in the class of projectable lattice ordered groups.

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In the whole paper G denotes a lattice ordered group. Further, let $\mathscr{A}$ and $\mathscr{K}$ be as above.

1.1. Definition. G is said to be a strong subgroup of a lattice ordered group H if

(i) G is a convex ℓ-subgroup of H, and

(ii) whenever $0 < g \in G$, $h \in H$ and $ng < h$ for each positive integer n, then there exist $g_1 \in G^+$ and $h_1 \in H$ such that $h = g_1 + h_1$ and $h_1 \wedge g_2 = 0$ for each $g_2 \in G^+$.

If G is a strong subgroup of H, then we express this fact by writing $G \subseteq_s H$.

1.2. Definition (cf. [5]). G has the splitting property in $\mathscr{A}$ if it satisfies the following conditions:

(i) $G \in \mathscr{A}$;

(ii) whenever $H \in \mathscr{A}$ and G is a convex ℓ-subgroup of H, then G is a direct factor of H.

1.3. Definition. G has the splitting property in $\mathscr{K}$ if

(i) $G \in \mathscr{K}$, and

(ii) whenever $H \in \mathscr{K}$ and $G \subseteq_s H$, then G is a direct factor of H.

Since each convex ℓ-subgroup of an archimedean lattice ordered group H is a strong subgroup of H, we infer that for the class $\mathscr{A}$, Definitions 1.2 and 1.3 coincide.

Let H be a lattice ordered group such that G is a convex ℓ-subgroup of H.

For $X \subseteq G$ and $Y \subseteq H$ we denote by X^β and Y^δ the corresponding polars in G or in H, respectively; i.e.,

$$X^\beta = \{g \in G : |g| \wedge |x| = 0 \quad \text{for each } x \in X\},$$

and Y^δ is defined analogously.

If X is a one-element set, then $X^{\beta\beta}$ is said to be a principal polar of G.

G is called projectable (strongly projectable) if each principal polar of G (or each polar of G, respectively) is a direct factor of G.

We denote by $\mathscr{P}$ the class of all projectable lattice ordered groups. Then neither the relation $\mathscr{A} \subseteq \mathscr{P}$ nor the relation $\mathscr{P} \subseteq \mathscr{A}$ is valid.

If A is a direct factor of G and $g \in G$, then the component of g in A will be denoted by gA.

A nonempty indexed system $(a_i)_{i\in I}$ of elements of G is said to be disjoint if $x_i \geqslant 0$ for each $i \in I$ and $x_{i(1)} \wedge x_{i(2)} = 0$ whenever $i(1)$ and $i(2)$ are distinct elements of I.

G is called laterally complete if each disjoint indexed system of elements of G possesses the supremum in G.

We recall the following two theorems concerning the splitting property in $\mathscr{A}$ (Theorem (A) was deduced in [1] by applying results of [2] and [6]; for (B), cf. [6]).

(A) Let G belong to $\mathscr{A}$ and be laterally complete. Then G has the splitting property in $\mathscr{A}$.

(B) Let G be a complete lattice ordered group. Then G has the splitting property in $\mathscr{A}$ if and only if it is laterally complete.

For $y \in H$ we denote $[y] = \{y\}^{\delta\delta}$.

1.4. Lemma. *Suppose that $G, H \in \mathscr{P}$ and that G is a convex ℓ-subgroup of H. Let $0 < k \in H$. Then there exists $0 \leqslant g_0 \in G$ such that $g_0 \leqslant k$ and either*

(i) *$g \wedge (k - g_0) = 0$ for each $g \in G^+$, or*

(ii) *$i(k - g_0) \leqslant k$ for each positive integer i.*

P r o o f. It suffices to apply the same method as in the proof of Lemma 2 in [6] (pp. 259–263) with the following distinctions:

a) we take H instead of $H^\wedge$ (p. 260^1);

b) the relation $e_1 \in G$ (p. 260^2) is a consequence of $e_1 \in [0, e]$ and of the fact that G is a convex ℓ-subgroup of H;

c) the last two lines of the proof under consideration are now to be omitted. □

1.5. Lemma. *Let $G, H \in \mathscr{P}$, $G \subseteq_s H$, $0 < k \in H$. Then there are $g_1 \in G^+$ and $h_1 \in H$ such that $h = g_1 + h_1$ and $h_1 \wedge g_2 = 0$ for each $g_2 \in G^+$.*

P r o o f. Either (i) or (ii) from 1.4 is valid. If (i) holds, then we put $g_1 = g_0$ and $h_1 = -g_0 + k$; the desired relations are satisfied.

If (ii) holds, then the validity of the assertion of the lemma is a consequence of the relation $G \subseteq_s H$. □

1.6. Theorem. *Suppose that G is a projectable lattice ordered group and that it is laterally complete. Then G has the splitting property in the class $\mathscr{P}$.*

P r o o f. We apply 1.5 and the same method as in the proof of [6, Theorem 1]. □

1.7. Theorem. *Let G be a complete lattice ordered group. Then the following conditions are equivalent:*

(i) *G has the splitting property in the class $\mathscr{P}$;*

(ii) *G is laterally complete.*

P r o o f. In view of the Riesz Theorem, G belongs to $\mathscr{P}$. Next, G is archimedean. Since 1.2 and 1.3 are equivalent for archimedean lattice ordered groups, our assertion is a consequence of [6, Theorem 2]. □

Theorems 1.6 and 1.7 are analogous to the results (A) and (B) which were mentioned above.

2. The relation $G^{DL} = G^{LD}$

Again, let G be a lattice ordered group. We denote by G^L and G^D the lateral completion or the Dedekind completion of G, respectively (for definitions, cf. e.g., [11]). The notion of D-completeness is used in the same sense as in [11].

In [2], the following result was proved:

($*$) (Bernau) Let G be an archimedean lattice ordered group. Then the relation

$$G^{DL} = G^{LD} \tag{1}$$

is valid.

In [11] it was shown that the relation (1) holds also for the case when G is strongly projectable.

In the present section we prove that (1) is valid for projectable lattice ordered groups, generalizing the result of [11].

Since the lateral completion and the Dedekind completion are defined uniquely up to isomorphism, the relation (1) is also to be considered up to isomorphism (leaving all elements of G fixed).

For $X \subseteq G$ we now denote the polar of X in G by X^{δ}; this differs from the notation in Section 1 above (the reason for this change is that we are now to be compatible with the notation in [11]).

If $X_1 = \{x_i\}_{i\in I} \subseteq X \subset G$ is such that

(i) the set X_1 is disjoint,

(ii) $x_i > 0$ for each $i \in I$,

(iii) for each $0 < x \in X$ there exists $i \in I$ with $x_i \wedge x > 0$,

then X_1 is said to be a maximal disjoint subset of X.

2.1. Lemma. *Assume that G is projectable and laterally complete. Then it is strongly projectable.*

Proof. Let $X \subseteq G$. Put $B = X^{\delta}$. We have to verify that B is a direct factor of G. The cases $X = \emptyset$, $X = \{0\}$ and $B = \{0\}$ being trivial we can suppose that $\emptyset \neq X \neq \{0\} \neq B$.

By applying Axiom of Choice we conclude that there exist

(i) a maximal disjoint subset $\{x_i\}_{i\in I_1}$ in X, and

(ii) a maximal disjoint subset $\{x_i\}_{i\in I_2}$ in B.

We have $I_1 \cap I_2 = \emptyset$. Put $I = I_1 \cup I_2$. Then $\{x_i\}_{i\in I}$ is a maximal disjoint subset in G. Next, the relation

$$B = (\{x_i\}_{i\in I_1})^{\delta} \tag{2}$$

is valid.

For each $i \in I$ we put $G_i = \{x_i\}^{\delta\delta}$. Since G is projectable, G_i is a direct factor of G. If $g \in G$, then we denote by g_i the component of g in G_i.

Consider the mapping $\varphi : G \to \prod\limits_{i\in I} G_i$ defined by

$$\varphi(g) = (g_i)_{i\in I}$$

for each $g \in G$. Then φ is a homomorphism of G into $\prod\limits_{i\in I} G_i$.

Let $g \in G$ and suppose that $\varphi(g) = 0$. Then $\varphi(|g|) = 0$. Hence

$$|g|_i \wedge x_i = 0 \quad \text{for each} \quad i \in I.$$

Thus in view of the maximality of $\{x_i\}_{i\in I}$ in G we obtain that $|x| = 0$. Therefore $x = 0$ and so φ is an isomorphism of G into $\prod\limits_{i\in I} G_i$.

Choose $x^i \in G_i$, $x^i \geqslant 0$ for each $i \in I$. Then $(x^i)_{i\in I}$ is a disjoint indexed system in G. Since G is laterally complete there exists $x^0 \in G$ with

$$x^0 = \bigvee_{i\in I} x^i.$$

It is easy to verify that $(x^0)_i = x^i$ for each $i \in I$. Hence $\Big(\prod\limits_{i\in I} G_i\Big)^+ \subseteq \varphi(G)$. This yields that $\prod\limits_{i\in I} G_i \subseteq \varphi(G)$. Hence φ is an isomorphism of G onto $\prod\limits_{i\in I} G_i$.

Put

$$G^1 = \prod_{i\in I_1} G_i, \quad G^2 = \prod_{i\in I_2} G_i.$$

Then $G = G^1 \times G^2$. From the definition of G^2 and from (2) we obtain $B = G^2$, completing the proof.

□

2.2. Proposition (cf. [13]). *Let G be projectable and let H be a lateral completion of G, $0 < h \in H$. Then there exists a disjoint indexed system $(x_i)_{i\in I}$ of elements of G such that the relation $h = \bigvee\limits_{i\in I} x_i$ is valid in H.*

2.3. Lemma. *Let A and B be laterally complete lattice ordered groups. Then $A \times B$ is laterally complete as well.*

Proof. Let $(x_i)_{i \in I}$ be a disjoint indexed system of elements of $A \times B$. Hence $(x_i A)_{i \in I}$ is a disjoint indexed system of elements of A and $(x_i B)_{i \in I}$ is a disjoint indexed system of elements of B. Then there are $a \in A$ and $b \in B$ such that

$$a = \bigvee_{i \in I} x_i A \quad \text{in} \quad A,$$

$$b = \bigvee_{i \in I} x_i B \quad \text{in} \quad B.$$

Hence these relations are valid also in $A \times B$. For each $i \in I$ we have

$$x_i = (x_i A) \vee (x_i B).$$

Thus

$$a \vee b = \Bigl(\bigvee_{i \in I} x_i A\Bigr) \vee \Bigl(\bigvee_{i \in I} x_i B\Bigr) = \bigvee_{i \in I} (x_i A \vee x_i B) = \bigvee_{i \in I} x_i = \bigvee_{i \in I} x_i (A \times B).$$

$\square$

2.4. Lemma. *Suppose that G is projectable and $G = A \times B$. Let H be a lateral completion of G. Then H can be written in the form $H = A^1 \times B^1$, where A^1 and B^1 are lateral completions of A and B, respectively.*

Proof. Let A^1 and B^1 be lateral completions of A and B, respectively. According to 2.3, $A^1 \times B^1$ is laterally complete. In view of 2.2 we have $A \subseteq_s A^1$ and $B \subseteq_s B^1$. This yields that $G \subseteq_s (A^1 \times B^1)$. Then there exists a lateral completion H of G such that $H \subseteq_s (A^1 \times B^1)$ (cf. [4] and [11], 1.5).

Let $0 < x \in A^1 \times B^1$. Thus $x = a^1 \vee b^1$ for $a^1 = xA^1$, $b^1 = xB^1$. There exist disjoint indexed systems $(a_i)_{i \in I}$ of elements of A and $(b_j)_{j \in J}$ of elements of B such that

$$a^1 = \bigvee_{i \in I} a_i \quad \text{is valid in } A^1,$$

$$b^1 = \bigvee_{j \in J} b_j \quad \text{is valid in } B^1.$$

Hence the indexed system S consisting of all a_i $(i \in I)$ and all b_j $(j \in J)$ is a disjoint indexed system of elements of G and the join of this system in H is the element x. Therefore $A^1 \times B^1$ is a lateral completion of G. $\square$

2.5. Lemma. *Under the assumptions and notation as in* 2.4, A^1 *is a double polar of* A *in* H, *and* B^1 *is a double polar of* B *in* H.

P r o o f. For $X \subseteq H$ we denote the polar of X in H by $X^{\perp}$. Since $A \subseteq A^1$, for each $0 \leqslant a \in A$ and each $0 \leqslant b^1 \in B^1$ we have $a \wedge b^1 = 0$. Hence $A^{\perp} \supseteq B^1$ and thus $A^{\perp\perp} \subseteq (B^1)^{\perp}$. Clearly $(B^1)^{\perp} = A^1$, whence $A^{\perp\perp} \subseteq A^1$. Let $0 < a^1 \in A^1$. There exists an indexed system $(a_i)_{i\in I}$ of elements of A such that the relation

$$a^1 = \bigvee_{i\in I} a_i$$

is valid in A^1, and hence this relation is valid also in H. Since $A^{\perp\perp}$ is a closed ℓ-subgroup of H we conclude that $a^1 \in A^{\perp\perp}$. Thus $A^1 = A^{\perp\perp}$. Analogously, $B^1 = B^{\perp\perp}$. □

2.6. Lemma. *Let* G *be projectable and let* H *be a lateral completion of* G. *Then* H *is projectable.*

P r o o f. Let $0 < h \in H$. If $X \subseteq H$, then let $X^{\perp}$ be as in the proof of 2.5. Put $Y = \{h\}^{\perp\perp}$. We have to verify that Y is a direct factor of H.

In view of 2.2, there exists a disjoint indexed system $(x_i)_{i\in I}$ of elements of G such that the relation

$$h = \bigvee_{i\in I} x_i$$

is valid in H and $0 < x_i$ for each $i \in I$. Then $\{x_i\}_{i\in I}$ is a maximal disjoint set in Y. Next, Axiom of Choice and 2.2 yield that there exists a maximal disjoint indexed system $(x_i)_{i\in I'}$ in H such that $I \subseteq I'$ and all the elements x_i $(i \in I')$ belong to G.

Let $i \in I'$. Put $G_i = \{x_i\}^{\delta\delta}$. Since G is projectable, there is a convex ℓ-subgroup G_i' of G such that

$$G = G_i \times G_i'.$$

Then according to 2.4 and 2.5 we have

$$H = G_i^1 \times (G_i')^1,$$

where G_i^1 is a lateral completion of G_i and, moreover, $G_i^1 = (G_i)^{\perp\perp}$.

Now by the same method as in the proof of 2.1 we obtain

$$H = \prod_{i\in I'} G_i^1,$$

$$Y = \prod_{i\in I} G_i^1.$$

In view of $I \subseteq I'$ we conclude that Y is a direct factor of G, completing the proof. □

2.7. Lemma. *Let G be projectable and let H be a lateral completion of G. Then H is strongly projectable.*

P r o o f. This is a consequence of 2.6 and 2.1. $\square$

2.8. Lemma. *Let G be projectable and let H be a Dedekind completion of G. Then H is projectable.*

P r o o f. Let $0 < h \in H$. For $X \subseteq H$ let $X^{\perp}$ be as in the proof of 2.6. We have to verify that $\{h\}^{\perp\perp}$ is a direct factor of H.

There exists a subset X of G such that the relation

$$\sup X = h$$

is valid in H. By Axiom of Choice, there exists a maximal disjoint set $\{x_i\}_{i\in I}$ of the ideal of the lattice G^+ which is generated by the set X.

The remaining steps are analogous to those which were applied in the proof of 2.6. $\square$

2.9. Lemma. *Let G be projectable and D-complete. Let H be the lateral completion of G. Assume that $0 < h \in H$, $b \in G$, $h \leqslant b$. Then $h \in G$.*

P r o o f. We apply the same method as in the proof of 3.5, [11] with the distinction that instead of Lemma 3.3 in [11] we now use Proposition 2.2. $\square$

2.10. Lemma. *Let G be projectable and D-complete. Suppose that H is a lateral completion of G. Then H is D-complete.*

P r o o f. The method is as in the proof of [11], 3.6 (instead of 3.4.1, 3.5 and 3.3 from [11] we now apply 2.7, 2.9 or 2.2, respectively). $\square$

2.11. Theorem. *Let G be a projectable lattice ordered group. Next, let H and K be a lateral completion or a Dedekind completion of G, respectively, H_1 a Dedekind completion of H and K_1 a lateral completion of K. Then there exists an isomorphism φ of K_1 onto H_1 such that $\varphi(g) = g$ for each $g \in G$.*

P r o o f. We proceed as in the proof of 4.1 in [11] with the distinction that

(i) 3.4.1, 3.4.2 and 3.6 from [11] are now replaced by 2.7, 2.8 or 2.10, respectively;

(ii) instead of 3.2 from [11] we have now to apply 3.2 from [11] and 2.1 above. $\square$

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