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VARIETIES OF HALF LATTICE-ORDERED GROUPS OF MONOTONIC PERMUTATIONS OF CHAINS

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0. Introduction

As is well-known, the theory of lattice ordered groups (ℓ-groups) is an axiomatization of groups of automorphisms of chains (under composition) endowed with the pointwise order, because by Holland's representation theorem [Ho1], any ℓ-group is isomorphic to an ℓ-subgroup (i.e. both subgroup and sublattice) of the group of automorphisms of a chain.

For the possibility of an axiomatization of groups of all monotonic permutations (i.e. both automorphisms and anti-automorphisms) of chains, one can use a special kind of right ordered groups. By a *right ordered group* we mean a group endowed with an order relation whicg is compatible to the right with the multiplication. If $G = (G, \cdot, \leqslant)$ is a right ordered group and $c \in G$, then c is called *increasing* if it preserves order and is called *decreasing* if it reverses order under group multiplication from the left. Denote by G_1 the set of increasing elements and by G_2 the set of decreasing elements in G.

Definition 0.1. A *half ordered group* is a right ordered group G such that $G = G_1 \cup G_2$. If, moreover, G_1 is a lattice then G is called a *half lattice ordered group* (a *half ℓ-group*).

If G is a half ℓ-group and $G_2 \neq \emptyset$, then G_2 is a lattice and G_1 and G_2 are isomorphic as lattices.

Let T be a chain and $M(T)$ the group (under composition "$\circ$") of all monotonic permutations of T. If "$\leqslant$" is the pointwise order then $M(T) = \big(M(T), \circ, \leqslant\,\big)$ is a

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half ℓ-group and the increasing elements of $M(T)$ are precisely the automorphisms while the decreasing elements of $M(T)$ are exactly the anti-automorphisms of T.

Half ordered groups and especially half ℓ-group were introduced and studied by M. Giraudet and F. Lucas in [Gi-L]. They proved the following generalization of Holland's theorem.

Theorem 0.1. ([Gi-L, Theorem I.3.2]) *For any half ℓ-group $(G, \cdot, \leqslant)$ there is a chain T such that $(G, \cdot, \leqslant)$ is a substructure of $\big(M(T), \circ, \leqslant\big)$.*

Hence the theory of half ℓ-groups is an axiomatization of groups of monotonic permutations of chains, and conversely, one can use in this theory the technique of permutation groups. But in contrast to the ℓ-groups which form a variety of algebras in the language $(\cdot, e, {}^{-1}, \wedge, \vee)$ of type $\langle 2, 0, 1, 2, 2\rangle$, the class of half ℓ-groups is not a variety of algebras of any type. (For example, the product of half ℓ-groups need not be a half ℓ-group.)

Nevertheless, we can investigate the half ℓ-groups from another point of view making it possible to study varieties of related algebras.

Definition 0.2. An *m-group* is a pair (H, φ) where H is an ℓ-group and φ is a decreasing group automorphism of H of order two, i.e. for each a, $b \in H$,

$$\begin{aligned}
\varphi(ab) &= \varphi(a)\varphi(b),\\
\varphi(a \vee b) &= \varphi(a) \wedge \varphi(b),\\
\varphi^2(a) &= a.
\end{aligned}$$

It is obvious that the m-groups form a variety in the language $(\cdot, e, {}^{-1}, \vee, \wedge, \varphi)$ of type $\langle 2, 0, 1, 2, 2, 1\rangle$. It is known ([Gi-L]) that if G is a half ℓ-group and $G_2 \neq \emptyset$ then there is an element $u \in G_2$ with $u^2 = e$, and if $\varphi_u \colon G_1 \to G_1$ is the inner automorphism defined by $\varphi_u(x) = uxu^{-1}$ then (G_1, φ_u) is an m-group.

For two m-group structures (H, φ) and (H, φ') on the same ℓ-group H, $(H, \varphi)\#(H, \varphi')$ means that, for some $u \in H$, $\varphi(u) = u^{-1}$ and $\varphi\varphi' = \varphi_u$.

Theorem 0.2. ([Gi-L, Theorem I.3.2 and Lemma III.3]) *For any m-group (H, φ) there is a half ℓ-group $(G, \cdot, \leqslant)$ such that $G_1 = H$ and, for some $u \in G \setminus H$ and all $x \in H$, $\varphi(x) = uxu$ and $u^2 = e$. This establishes a 1-1 correspondence between the quotient by $\#$ of the class of m-groups and the class of half ℓ-groups which (in non-trivial cases) are not ℓ-groups.*

Therefore in the following we will study the m-groups instead of the half ℓ-groups.

Remark. Note that not every ℓ-group has decreasing group automorphisms of order two and that (by [Gi-L, Corollary III.8]) the existence of such an automorphism of an ℓ-group H is not characterizable by the theory of the first order of H.

Let (G, φ) be an m-group and M an ℓ-subgroup of G. Then M is called an *m-subgroup* of (G, φ) if it is stable under φ, and so $(M, \varphi|M)$ is an m-group. (We will often write (M, φ) instead of $(M, \varphi|M)$. A normal convex m-subgroup of (G, φ) is called an *m-ideal* of (G, φ). It is obvious that the set of convex m-subgroups of (G, φ) is a complete lattice which is a closed sublattice of the lattice of convex ℓ-subgroups of the ℓ-group G, and that the set $\mathcal{M}(G) = \mathcal{M}((G, \varphi))$ of m-ideals of (G, φ) is in the same way a complete lattice which is a closed sublattice of the lattice of ℓ-ideals of the ℓ-group G. Hence this means that $\mathcal{M}(G)$ is a Brouwerian complete lattice.

Moreover, the kernels of homomorphisms of m-groups are exactly all m-ideals. (An *m-homomorphism* is any ℓ-homomorphism that also respects φ.) If M is an m-ideal of an m-group (G, φ) and $\overline{\varphi}\colon G/M \to G/M$ is the mapping defined by $\overline{\varphi}(gM) = \varphi(g)M$ for each $g \in G$, then $(G/M, \overline{\varphi})$ is an m-group (the *factor m-group* of (G, φ) by M).

Denote by $\mathcal{M}$ the variety (in language $(\cdot, e, {}^{-1}, \vee, \wedge, \varphi)$) of all m-groups. It is clear, by the above, that $\mathcal{M}$ is an arithmetical variety (see [B-S]).

Let M be the set of all varieties of m-groups. M, ordered by inclusion, is a complete lattice in which the trivial variety $\mathscr{E}_m$ is the least element, the variety $\mathcal{M}$ is the greatest element and infima are formed by intersections. Since M is anti-isomorphic to the lattice of fully invariant congruences on the free m-group with a countable subset of generators, the lattice M is distributive and, moreover, is dually Brouwerian. We will show, in Proposition 2.1, that M is not Brouwerian.

Also, if $(\mathcal{V}_i\,;\ i \in \omega)$ is an increasing chain of varieties of m-groups and $\mathcal{W}$ is any variety of m-groups then

$$\mathcal{W} \cap \Big(\bigvee_{i\in\omega} \mathcal{V}_i\Big) = \bigvee_{i\in\omega} (\mathcal{W} \cap \mathcal{V}_i).$$

Notations. We shall write $\mathrm{Var}_\ell\, \mathcal{X}$ for the variety of ℓ-groups generated by a class $\mathcal{X}$ of ℓ-groups and $\mathrm{Var}_m\, \mathcal{Y}$ for the variety generated by a class $\mathcal{Y}$ of m-groups.

An *ℓ-equation* is an equation of the form

$$\bigvee_{i\in I} \bigwedge_{j\in J} \mathfrak{w}_{ij}(\overline{x}) = e, \quad I \text{ and } J \text{ finite},$$

where $\mathfrak{w}$ is a word of the language of groups of n variables and $\overline{x} = (x_1, x_2, \dots, x_n)$, $n \in \mathbb{N}^*$, in other words an *equation of the language* of ℓ-groups, while an equation of the language of m-groups is of the form

$$\bigvee_{i\in I}\bigwedge_{j\in J}\mathfrak{w}_{ij}\big(\overline{x},\varphi(\overline{x})\big)=e,$$

with $\mathfrak{w}$ of $2n$ variables.

In the first section of the paper the lattice and semigroup M of varieties of m-groups is studied. It is shown that multiplication distributes over some joins and meets. The fact that the complete lattice M is dually Brouwerian but not Brouwerian is proved in the second section. In that section some connections between varieties of ℓ-groups and those of m-groups are described and representations of m-groups by permutations of m-groups are used to generate varieties of m-groups. The third section is devoted to the study of some special varieties of m-groups. For example, the least non-trivial variety of m-groups (which, in contrast to the situation for ℓ-groups, differs from the variety of abelian m-groups) is found and a new idempotent in the semigroup M is described, and some classes of varieties of m-groups that are simultaneously torsion classes are shown. Free m-groups in some varieties of m-groups are described in the concluding section.

1. The ordered semigroup of varieties of m-groups

Definition 1.1. Let $\mathcal{U}$ and $\mathcal{V}$ be varieties of m-groups. Then the *product* of $\mathcal{U}$ and $\mathcal{V}$ is the variety $\mathcal{U}\mathcal{V}$ defined by: $(G,\varphi)\in\mathcal{U}\mathcal{V}$ if and only if there is an m-homomorphism of (G,φ) onto an element in $\mathcal{V}$ with the kernel in $\mathcal{U}$. (In other words: There is an m-ideal M of (G,φ) such that $M\in\mathcal{U}$ and $G/M\in\mathcal{V}$.)

It is obvious that M endowed with multiplication defined in this way is a semigroup which, with inclusion, is an ordered semigroup. For the study of questions concerning the distributivity of multiplication over the lattice operations in M, the following proposition is useful.

Proposition 1.1. *Let $\mathcal{U}$ and $\mathcal{V}$ be varieties of m-groups and let G be an m-group. Then $G\in\mathcal{U}\vee\mathcal{V}$ if and only if there exist m-ideals M and N of G such that $M\cap N=\{e\}$, $G/M\in\mathcal{U}$ and $G/N\in\mathcal{V}$.*

Proof. Thanks to distributivity of the lattice $\mathcal{M}(G)$, the proposition can be proved similarly as the analogous proposition in [Ma1] for varieties of ℓ-groups. □

Theorem 1.1. *For any varieties $\mathcal{U}$, $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{V}_i$ $(i\in I)$ of m-groups the following equalities hold:*

a) $(\mathcal{U}_1\vee\mathcal{U}_2)\mathcal{U}=\mathcal{U}_1\mathcal{U}\vee\mathcal{U}_2\mathcal{U}$;

b) $(\bigcap_{i\in I}\mathcal{V}_i)\mathcal{U}=\bigcap_{i\in I}\mathcal{V}_i\mathcal{U}$.

P r o o f. a) It is obvious that the right hand side of the equality is contained in the left hand side.

Let $H=(H,\varphi)\in(\mathcal{U}_1\vee\mathcal{U}_2)\mathcal{U}$. Then H contains an m-ideal G such that $G\in\mathcal{U}_1\vee\mathcal{U}_2$ and $H/G\in\mathcal{U}$. By Proposition 1.1, G contains m-ideals A_1 and A_2 such that $A_1\cap A_2=\{e\}$, $G/A_1\in\mathcal{U}_1$ and $G/A_2\in\mathcal{U}_2$.

For any $x\in H$, consider the subgroup $x^{-1}A_1x$ of H, a conjugate of A_1. Clearly $x^{-1}A_1x\subseteq G$. If $g\in G$ then $g^{-1}x^{-1}A_1xg=x^{-1}g_1^{-1}A_1g_1x$ where $g_1\in G$, thus $g^{-1}x^{-1}A_1xg=x^{-1}A_1x$, i.e. $x^{-1}A_1x$ is normal in G. Clearly this is an ℓ-ideal of G (and the ℓ-ideals A_1 and $x^{-1}A_1x$ are isomorphic).

Set $N_1=\bigcap_{x\in H}x^{-1}A_1x$. Then N_1 is an ℓ-ideal not only of G but also of H. Let $c\in N_1$. Then for each $x\in H$ there is $c_x\in A_1$ such that $c=x^{-1}c_xx$ and $\varphi(c)=\varphi(x)^{-1}\varphi(c_x)\varphi(x)$, where $\varphi(c_x)\in A_1$. If $y\in H$ then for $x=\varphi(y)$ we have $y=\varphi(x)$, hence for each $y\in H$, $\varphi(c)=y^{-1}d_yy$ where $d_y\in A_1$, and so $\varphi(c)\in\bigcap_{x\in H}x^{-1}A_1x=N_1$.

Therefore N_1 is an m-ideal of the m-group (G,φ), and thus $(G/N_1,\overline{\varphi})$ is an m-group.

Consider the mapping

$$\alpha\colon\ G/N_1\to\prod_{x\in H}G/x^{-1}A_1x$$

such that

$$\alpha(gN_1)=\big(\ldots,g\cdot(x^{-1}A_1x),\ldots\big)$$

for each $g\in G$. It is obvious that α is an embedding of the ℓ-group G/N_1 into the ℓ-group $\prod_{x\in H}G/D_x$, where $D_x=x^{-1}A_1x$.

Let ψ be the mapping of $\prod_{x\in H}G/D_x$ into $\prod_{x\in H}G/D_x$ such that

$$\psi\colon\ (\ldots,g\cdot D_x,\ldots)\to\big(\ldots,\varphi(g)D_x,\ldots\big).$$

We have

$$\begin{aligned}\alpha\big(\overline{\varphi}(gN_1)\big)&=\alpha\big(\varphi(g)N_1\big)=\big(\ldots,\varphi(g)D_x,\ldots\big),\\ \psi\big(\alpha(gN_1)\big)&=\psi((\ldots,gN_1,\ldots))=\big(\ldots,\varphi(g)D_x,\ldots\big).\end{aligned}$$

Therefore the embedding $\alpha\colon\ G/N_1\to\prod_{x\in H}G/D_x$ is an m-isomorphism of $(G/N_1,\overline{\varphi})$ into $\prod_{x\in H}(G/D_x,\overline{\varphi}_x)$.

Let $f_x\colon A_1 \to D_x = x^{-1}A_1x$ ($x \in H$) be the isomorphism such that $f_x(a) = x^{-1}ax$ for each $a \in A_1$. Let $\widetilde{f_x}$ denote the extension of f_x to G, where $\widetilde{f_x}(g) = x^{-1}gx$ for all $g \in G$. (Clearly $\widetilde{f_x}(A_1) = x^{-1}A_1x$.) It is obvious that $\widetilde{f_x}$ is an ℓ-automorphism of G.

Let $\varphi = \varphi\big|G$ be the restriction of φ on to G. Then φ is a decreasing involutory group automorphism of G. Let an identity

$$(*) \qquad \mathfrak{w}(\xi_1, \xi_2, \ldots, \xi_k) = e,$$

where $\mathfrak{w}(\xi_1, \xi_2, \ldots, \xi_k) = \xi_1^{i_1}\xi_2^{i_2}\ldots\xi_k^{i_k}$ and $\xi_j = x_j$ or $\xi_j = \varphi(x_j)$, be satisfied in G/A_1. That means that for each g_1A_1, g_2A_1, $\ldots, g_kA_1 \in G/A_1$ we have

$$(\gamma_1A_1)^{i_1} \cdot (\gamma_2A_1)^{i_2} \cdot \ldots \cdot (\gamma_kA_1)^{i_k} = A_1,$$

where $\gamma_j = g_j$ if $\xi_j = x_j$ and $\gamma_j = \varphi(g_j)$ if $\xi_j = \varphi(x_j)$.

Let h_1, h_2, $\ldots, h_k \in G$. Set $\eta_j = h_j$ for $\xi_j = x_j$ and $\eta_j = \varphi(h_j)$ otherwise. Then there exist g_1, $g_2, \ldots, g_k \in G$ such that $h_j = \widetilde{f_x}(g_j)$. If $\gamma_j = \varphi(g_j)$ then there are $g_j', g_j'' \in G$ such that $\varphi\bigl(\widetilde{f_x}(g_j)\bigr) = \widetilde{f_x}(g_j')$ and $g_j' = \varphi(g_j'')$.

Then in the case $\xi_j = x_j$ we have $\eta_j^{i_j} = \widetilde{f_x}(g_j)^{i_j}$ and in the case $\xi_j = \varphi(x_j)$ we have $\eta_j^{i_j} = f_x(g_j')^{i_j} = \widetilde{f_x}\bigl(\varphi(g_j'')\bigr)^{i_j}$. Moreover, $\widetilde{f_x}(A_1) = D_x$. Hence

$$(\eta_1D_x)^{i_1} \cdot (\eta_2D_x)^{i_2} \ldots (\eta_kD_x)^{i_k} = \widetilde{f_x}(\sigma_1^{i_1} \cdot \sigma_2^{i_2} \ldots \sigma_k^{i_k}),$$

where $\sigma_j = g_j$ for $\xi_j = x_j$ and $\sigma_j = \varphi(g_j'')$ for $\xi_j = \varphi(x_j)$, and since the identity $(*)$ is satisfied in G/A_1,

$$(\eta_1D_x)^{i_1} \cdot (\eta_2D_x)^{i_2} \ldots (\eta_kD_x)^{i_k} = \widetilde{f_x}(A_1) = D_x.$$

Therefore $(*)$ is also satisfied in G/D_x for each $x \in H$.

Similarly as for group identities, one can prove that if $\mathfrak{w}_{pq}$ are words in the form of the left hand side of identity $(*)$, where $p \in P$, $q \in Q$ and P, Q are finite, and if the identity

$$\mathfrak{v} = \bigvee_{p\in P} \bigwedge_{q\in Q} \mathfrak{w}_{pq} = e$$

is satisfied in G/A_1, then this identity $\mathfrak{v} = e$ is also satisfied in G/D_x for each $x \in H$. Therefore, $\mathfrak{v} = e$ is satisfied in the m-group $(G/N_1, \overline{\varphi})$.

By assumption, G/A_1 belongs to the variety $\mathcal{U}_1$, hence G/N_1 also belongs to $\mathcal{U}_1$.

Moreover, $(H/N_1)/(G/N_1) \simeq H/G \in \mathcal{U}$, thus $H/N_1 \in \mathcal{U}_1\mathcal{U}$.

If we denote analogously $N_2 = \bigcap\limits_{x\in H} x^{-1}A_2x$, then N_2 is an m-ideal of H and $G/N_2 \in \mathcal{U}_2$, $(H/N_2)/(G/N_2) \in \mathcal{U}$, that means $H/N_2 \in \mathcal{U}_2\mathcal{U}$.

In addition, $N_1 \cap N_2 = \{e\}$, and so $H \in \mathcal{U}_1\mathcal{U} \vee \mathcal{U}_2\mathcal{U}$.

b) Obviously, the left hand side of the equality in b) is contained in the right hand side.

Let $H \in \bigcap\limits_{i\in I} \mathcal{V}_i\mathcal{U}$ and let $i \in I$. Then $h \in \mathcal{V}_i\mathcal{U}$, hence there exists an m-ideal G_i of H such that $C_i \in \mathcal{V}_i$ and $H/G_i \in \mathcal{U}$. Set $G = \bigcap\limits_{i\in I} G_i$. Evidently, G is an m-ideal of H and $G \in \bigcap\limits_{i\in I} \mathcal{V}_i$. Moreover, H/G is isomorphic to an m-subgroup of $\prod\limits_{i\in I} H/G_i$, and $H \in (\bigcap\limits_{i\in I} \mathcal{V}_i)\mathcal{U}$. □

Remark 1.1. The following questions are open.

a) Does multiplication distribute over joins from the right also for infinite cases?

b) Does multiplication also distribute over joins and meets from the left?

(For varieties of ℓ-groups see [Gℓ-Ho-Mc, Theorem 6.1].)

2. Representations and varieties of m-groups

Definition 2.1. For (G, φ) an m-group, T a chain, and α a decreasing automorphism of T, we say that (G, T, α) is a *representation* of (G, φ) if and only if $G \subseteq \operatorname{Aut} T$ and $\varphi(g) = \alpha g \alpha$ for all $g \in G$.

Definition 2.2. Let (G, φ) be an m-group and let (H, T, α) be a representation of an m-group (H, ψ). Then the *wreath product* $GWrH$ of (G, φ) and (H, ψ, T) is defined as the usual wreath product of the ℓ-groups G and (H, T) provided with the decreasing automorphism $\varphi Wr \psi$ of order two defined by:

$$\forall\big((g_t)_{t\in T}, h\big) \in GWrH\,;\ (\varphi Wr\psi)\big((g_t)_{t\in T}, h\big) = \Big(\big(\varphi(g_{\alpha(t)}\big)_{t\in T}, \psi(h)\Big).$$

It is straightforward to check that $GWrH$ defined in this way is an m-group and that, if $\mathcal{U}$ and $\mathcal{V}$ are m-varieties and $G \in \mathcal{U}$, $H \in \mathcal{V}$ then $GWrH \in \mathcal{U}\mathcal{V}$.

Example 2.1. Let $\operatorname{Inv}\colon\ \mathbb{Z} \to \mathbb{Z}$ be the automorphism of $\mathbb{Z}$ defined by $\operatorname{Inv}(a) = -a$ for each $a \in \mathbb{Z}$. Then the wreath product of $(\mathbb{Z}, \operatorname{Inv})$ by itself is the ℓ-group $\mathbb{Z}Wr\mathbb{Z}$ provided with the automorphism I_2 defined by

$$I_2\big((k_i)_{i\in\mathbb{Z}}, n\big) = \big((l_i)_{i\in\mathbb{Z}}, -n\big),$$

where $l_i = -k_{-i}$ for all $i \in \mathbb{Z}$.

The m-group $(\mathbb{Z}Wr\mathbb{Z}, I_2)$ together with the m-groups introduced in the following example will enable us to prove that the lattice M is not Brouwerian.

Example 2.2. Let $q \geqslant 1$ be an integer and $p = 2q+1$. Let G_p be an ℓ-group with generators $a_{-qp}, \ldots, a_{-1p}, a_{0p}, a_{1p}, \ldots, a_{qp}, b_p$ and defining identities $[a_{ip}, a_{jp}] = e$, $-q \leqslant i, j \leqslant q$, and $b_p^{-1} a_{ip} b_p = a_{jp}$, where $i+1 \equiv j \pmod p$, and with the ordering such that $b_p^n a_{-qp}^{k_{-q}} \ldots a_{-1p}^{k_{-1}} a_{0p}^{k_0} a_{1p}^{k_1} \ldots a_{qp}^{k_q} \geqslant e$ if and only if $n > 0$, or $n = 0$ and $k_i \geqslant 0$, for each $-q \leqslant i \leqslant q$. (That means, for p prime, that G_p is the Scrimger p-group.)

Denote by $\varphi_p \colon G_p \to G_p$ a mapping such that

$$\varphi_p\Big(b_p^n a_{-qp}^{k_{-q}} \ldots a_{-1p}^{k_{-1}} a_{0p}^{k_0} a_{1p}^{k_1} \ldots a_{qp}^{k_q}\Big) = b_p^{-n} a_{-qp}^{-k_q} \ldots a_{-1p}^{-k_1} a_{0p}^{-k_0} a_{1p}^{-k_{-1}} \ldots a_{qp}^{-k_{-q}}.$$

It is easy to verify that φ_p is an involutory decreasing group automorphism, and so (G_p, φ_p) is an m-group.

Similarly, let G_2 be an ℓ-group generated by elements a_{-12}, a_{12}, b_2 with defining identities $[a_{-12}, a_{12}] = e$, $b_2^{-1} a_{-12} b_2 = a_{12}$, and $b_2^{-1} a_{12} b_2 = a_{-12}$, lattice ordered by $b_2^n a_{-12}^{k_{-1}} a_{12}^{k_1} \geqslant e$ if and only if $n = 0$ or $n > 0$ and $k_{-1} \geqslant 0$, $k_1 \geqslant 0$, and let $\varphi_2 \colon G_2 \to G_2$ be such that $\varphi_2(b_2^n a_{-12}^{k_{-1}} a_{12}^{k_1}) = b_2^{-n} a_{-12}^{-k_1} a_{12}^{-k_{-1}}$. Then (G_2, φ_2) is also an m-group.

The following result was inspired by N. Ya. Medvedev. (For an analogous theorem concerning varieties of ℓ-groups see [K-M 1, Theorem 3].)

Proposition 2.1. *The lattice M of varieties of m-groups is not Brouwerian (and so it is not completely distributive).*

P r o o f. Let p be an arbitrary prime number. Denote by $(\mathscr{A}\mathscr{B}_p)_m$ the variety of m-groups defined by the identity $[x^p, y^p] = e$. Then $(G_p, \varphi_p) \in (\mathscr{A}\mathscr{B}_p)_m$.

Let $\mathcal{P} = \bigvee\limits_{p \neq 2} (\mathscr{A}\mathscr{B}_p)_m$ be the join of all varieties $(\mathscr{A}\mathscr{B}_p)_m$, where p is any odd prime number. Consider the m-group $(H, I_2) = (\mathbb{Z} Wr \mathbb{Z}, I_2)$ and prove that $(H, I_2) \in \mathcal{P}$. Let $\overline{G} = \overline{\prod\limits_{p \neq 2}} G_p$ be the cartesian product of the ℓ-groups G_p and $\overline{\varphi} \colon \overline{G} \to \overline{G}$ the mapping such that if $a \in \overline{G}$ then $\big(\overline{\varphi}(a)\big)(p) = \varphi_p\big(a(p)\big)$ for every p. Obviously, $(\overline{G}, \overline{\varphi})$ is an m-group. If we denote by $G = \prod\limits_{p \neq 2} G_p$ the direct product of the ℓ-groups G_p and $\varphi = \overline{\varphi}\big| G$, then (G, φ) is an m-ideal of $(\overline{G}, \overline{\varphi})$.

Consider $\overline{a}$, $\overline{b} \in \overline{G}$ such that $\overline{a}(p) = a_{1p}$ and $\overline{b}(p) = b_p$ for each p. We have $\overline{b}G \gg \overline{a}^n G$ for every $n \in \mathbb{Z}$, $\overline{a}^m G \perp \overline{a}^n G$ for every $m, n \in \mathbb{Z}$, $m \neq n$, and $(\overline{a}G^{\overline{b}^n G})^{\overline{b}G} = \overline{a}G^{b^{-n+1}G}$ for every $n \in \mathbb{Z}$.

Hence the ℓ-subgroup of $\overline{G}/G$ generated by the elements $\overline{a}G$ and $\overline{b}G$ is isomorphic to H. Moreover, it is an m-subgroup of $(\overline{G}/G, \varphi)$ isomorphic (as an m-group) to (H, I_2). Therefore $(H, I_2) \in \mathcal{P}$.

Set $(\overline{H}, I_2) = (\mathbb{Z} Wr \mathbb{Z}, I_2)$ and show that $\mathrm{Var}_m(\overline{H}, I_2) = \mathrm{Var}_m(H, I_2)$. Let $\mathfrak{w}$ be a word which is not an identity in $(\overline{H}, I_2)$. Then there exist (finitely many) elements

$\big((k_i^1),m_1\big),\dots,\big((k_i^n),m_n\big) \in \overline{H}$ such that $\mathfrak{w}\Big(\big((k_i^1),m_1\big),\dots,\big((k_i^n),m_n\big)\Big) \neq e$. We can write $\mathfrak{w}$ in a form $\mathfrak{w} = \bigvee\limits_{\alpha\in A}\bigwedge\limits_{\beta\in B}\mathfrak{w}_{\alpha\beta}$, where A and B are finite sets and $\mathfrak{w}_{\alpha\beta} = \mathfrak{w}_{\alpha\beta}\Big(\big((k_i^1),m_1\big),\dots,\big((k_i^n),m_n\big),I_2\big(\big((k_i^1),m_1\big)\big),\dots,I_2\big(\big((k_i^n),m_n\big)\big)\Big)$ are word in the group signature.

If in $(\overline{H},I_2)$, $\mathfrak{w}\Big(\big((k_i^1),m_1\big),\dots,\big((k_i^n),m_n\big)\Big) = \big((l_i),m\big)$ and $m \neq 0$, then in (H,I_2), $\mathfrak{w}\Big(\big((0),m_1\big),\dots,\big((0),m_n\big)\Big) = \big((0),m\big) \neq e$.

Hence, let $m = 0$. Express every word $\mathfrak{w}_{\alpha\beta}\Big(\big((k_i^1),m_1\big),\dots,\big((k_i^n),m_n\big),I_2\big(\big((k_i^1),m_1\big)\big),\dots,I_2\big(\big((k_i^n),m_n\big)\big)\Big)$ in the form $\Big(\big(l_i^{p_{1\alpha\beta}}\big)^{r_{1\alpha\beta}}\dots\big(l_i^{p_{s\alpha\beta}}\big)^{r_{s\alpha\beta}},\widetilde{m}\Big)$, where each of $\big(l_i^{p_{1\alpha\beta}}\big),\dots,\big(l_i^{p_{s\alpha\beta}}\big)$ is equal to either some of (k_i^j) or of $I_2(k_i^j)$, $j = 1,\dots,n$. Let r_0 be the maximum of the absolute values of all $r_{t\alpha\beta}$. Choose $t_o \in \mathbb{Z}$ such that $\mathfrak{w}\Big(\big((k_i^1),m_1\big),\dots,\big((k_i^n),m_n\big)\Big)(t_0) \neq e$. Define elements $\big((\widetilde{k}_i^1),m_1\big),\dots,\big((\widetilde{k}_i^n),m_n\big) \in H$ such that $\widetilde{k}_i^j = k_i^j$ if $|i-t_0| \leqslant r_0$, and $\widetilde{k}_i^j = 0$ for $|i = t_0| > r_0$. Then $\mathfrak{w}\Big(\big((\widetilde{k}_i^1),m_1\big),\dots,\big((\widetilde{k}_i^n),m_n\big)\Big)(t_0) = \mathfrak{w}\Big(\big((k_i^1),m_1\big),\dots,\big((k_i^n),m_n\big)\Big)(t_0) \neq e$. Hence $(\overline{H},I_2) \in \mathrm{Var}_m(H,I_2)$, which means $\mathrm{Var}_m(\overline{H},I_2) = \mathrm{Var}_m(H,I_2) \subseteq \mathcal{P}$.

Denote by H_2 the subgroup of $\overline{H}$ consisting of all elements $\big((k_i),m\big)$ such that $k_{i+2} = k_i$ for each $i \in \mathbb{Z}$. Then (H_2,I_2) is an m-subgroup of $(\overline{H},I_2)$ and $(H_2,I_2) \cong (G_2,\varphi_2)$. Hence $(G_2,\varphi_2) \in \mathcal{P}$.

Denote for any prime number p by $(\mathscr{A}\mathscr{B}_p)_\ell$ the variety of ℓ-groups defined by the identity $[x^p,y^p] = e$, i.e. by the same identity as the variety of m-groups $(\mathscr{A}\mathscr{B}_p)_m$. Similarly, denote by $\mathscr{A}_m$ and $\mathscr{A}_\ell$ the variety of abelian m-groups and ℓ-groups, respectively. As is shown in [K-M 1], for the varieties of ℓ-groups $(\mathscr{A}\mathscr{B}_2)_\ell$, $(\mathscr{A}\mathscr{B}_p)_\ell$, where $p \neq 2$, and $\mathscr{A}_\ell$, $(\mathscr{A}\mathscr{B}_2)_\ell \cap (\mathscr{A}\mathscr{B}_p)_\ell \subseteq \mathscr{A}_\ell$. Therefore also $(\mathscr{A}\mathscr{B}_2)_m \cap (\mathscr{A}\mathscr{B}_p)_m \subseteq \mathscr{A}_m$. As $(G_2,\varphi_2) \notin \mathscr{A}_m$, we get

$$\bigvee_{p\neq 2}\big((\mathscr{A}\mathscr{B}_2)_m \cap (\mathscr{A}\mathscr{B}_p)_m\big) \neq (\mathscr{A}\mathscr{B}_2)_m \cap \bigvee_{p\neq 2}(\mathscr{A}\mathscr{B}_p)_m,$$

therefore the lattice M of varieties of m-groups is not Brouwerian. $\square$

Notation 2.1. Let H be an ℓ-group. Then the ℓ-groups H^*, obtained from H by reversing the order, and H_*, obtained by reversing the group operation, are isomorphic. For a variety $\mathcal{V}$ of ℓ-groups, $\mathcal{V}^*$ will denote the variety of those H^* with H in $\mathcal{V}$, in other words, the variety whose defining set of equations is obtained from that of $\mathcal{V}$ either by exchanging $\wedge$ and $\vee$ or by reading the operations from right to left.

Let $\mathscr{E}$ denote the trivial variety of ℓ-groups (defined by $x = e$), and $\mathscr{L}\mathscr{G}$ the universal one (defined by $x = x$).

A *reversible variety* is a variety of ℓ-groups such that $\mathcal{V} = \mathcal{V}^*$. The set of reversible varieties of ℓ-groups was introduced and studied in [Hu-Re], where it was proved that it is an uncountable proper subsemigroup and sublattice of the set of varieties of ℓ-groups, and that the following ℓ-group varieties belong to it:

– All varieties defined by group identities, in particular the Abelian variety $\mathscr{A}$, and hence also all $\mathscr{A}^n$ for each positive integer n.
– The variety $\mathscr{N}$ of normal valued ℓ-groups defined by the identity

$$(x \vee e)(y \vee e) \leqslant (y \vee e)^2(x \vee e)^2.$$

– The variety $\mathcal{R}$ of representable ℓ-groups defined by the identity

$$(x \vee e)^2 \wedge (y \vee e)^2 = \big((x \vee e) \wedge (y \vee e)\big)^2.$$

The following facts are well known (see [Ho2] or [Re]):

– $\mathscr{A}$ is the smallest and $\mathscr{N}$ is the largest proper variety of ℓ-groups.
– $\mathcal{R}$ is generated by the class of totally ordered groups, $\mathscr{A}$ is generated by the totally ordered group $\mathbb{Z}$ of integers, and $\mathscr{L}\mathscr{G}$ is generated by $A\mathbb{Q}$, the ℓ-group of all automorphisms of the chain $\mathbb{Q}$ of rational numbers.

Also recall from [Gi-L] that the totally ordered m-groups are just the abelian ones provided with the map $\varphi(x) = x^{-1}$.

Notation 2.2. For any ℓ-group H, $\mathrm{Exch} = \mathrm{Exch}\, H$ will denote the permutation of $H \times H^*$ defined, for any $a, b \in H$, by $\mathrm{Exch}(a, b) = (b, a)$. It is clear that $(H \times H^*, \mathrm{Exch})$ is an m-group.

Theorem 2.1. *Each set of identities defining a reversible variety of ℓ-groups defines a variety of m-groups.*

Proof. Let $\mathcal{V} = \mathcal{V}^*$ be a reversible variety of ℓ-groups and let $H \in \mathcal{V}$. Then H^*, and hence also $H \times H^*$, satisfy the same ℓ-group identities as H. Therefore the m-groups in the form $(H \times H^*, \mathrm{Exch})$, where H is an arbitrary ℓ-group in $\mathcal{V}$, generate a variety of m-groups with the same ℓ-identities as $\mathcal{V}$. □

Corollary 2.1. *The ordered semigroup M of varieties of m-groups contains a copy of the set of reversible varieties of all ℓ-groups as a $\wedge$-subsemilattice.*

Notation 2.3. If $\mathcal{V}$ is a reversible variety of ℓ-groups, then the variety of m-groups defined by the same ℓ-group identities as $\mathcal{V}$ will be denoted by $\mathcal{V}_m$. A variety $\mathcal{U}$ of m-groups will be called an *ℓ-variety* if $\mathcal{U} = \mathcal{V}_m$ for some variety of ℓ-groups $\mathcal{V}$.

Remark 2.1. For any m-group (G, φ), the map i defined by $i(g) = \bigl(g, \varphi(g)\bigr)$ is an embedding of (G, φ) into $(G \times G^*, \mathrm{Exch})$.

Definition 2.3. Let $\mathcal{V}$ be a variety of m-groups. We will write

$$\mathcal{V}_\ell = \mathrm{Var}_m \bigl\{(G \times G^*, \mathrm{Exch});\ (G \times G^*, \mathrm{Exch}) \in \mathcal{V}\bigr\};$$
$$\mathcal{V}^\ell = \mathrm{Var}_m \bigl\{(G \times G^*, \mathrm{Exch});\ \text{for some } \varphi,\ (G, \varphi) \text{ is an } m\text{-group in } \mathcal{V}\bigr\}.$$

Lemma 2.1. *$\mathcal{V}_\ell$ is the ℓ-variety of m-groups axiomatized by the set of equalities*

$$\Bigl\{\bigvee_i \bigwedge_j \mathfrak{w}_{ij}(\overline{x}, \overline{y}) = e;\ \bigvee_i \bigwedge_j \mathfrak{w}_{ij}(\overline{x}, \varphi(\overline{x})) = e \textit{ is an axiom of } \mathcal{V}\Bigr\}.$$

Proof. Take $\bigvee_i \bigwedge_j \mathfrak{w}_{ij}(\overline{x}, \varphi(\overline{x})) = e$, an axiom of $\mathcal{V}$, and $(G \times G^*, \mathrm{Exch}) \in \mathcal{V}_\ell$. Then for all $(\overline{g}, \overline{g}') \in G \times G^*$ we have

$$\bigvee_i \bigwedge_j \mathfrak{w}_{ij}\bigl((\overline{g}, \overline{g}') \cdot \mathrm{Exch}(\overline{g}, \overline{g}')\bigr) = \Bigl(\bigvee_i \bigwedge_j \mathfrak{w}_{ij}(\overline{g} \cdot \overline{g}'), \bigwedge_i \bigvee_j \mathfrak{w}_{ij}(\overline{g}' \cdot \overline{g})\Bigr) = (e, e),$$

hence G and so $G \times G^*$ and $(G \times G^*, \mathrm{Exch})$ satisfy $\bigvee_i \bigwedge_j \mathfrak{w}_{ij}(\overline{x}, \overline{y}) = e$, and so does any m-group in $\mathcal{V}_\ell$.

Conversely, take an m-group (G, φ) satisfying the set of axioms, then clearly $(G \times G^*, \mathrm{Exch})$ satisfies all axioms for $\mathcal{V}$, hence, by Remark 2.1, so does (G, φ). □

Lemma 2.2. *$\mathcal{V}^\ell$ is the ℓ-variety of m-groups axiomatized by the set of all ℓ-equations true in all $(G, \varphi) \in \mathcal{V}$.*

Proof. Clear by Remark 2.1 and the fact that (reversible) ℓ-equations are preserved under direct products and reversibility of order. □

Lemma 2.3. *For any variety $\mathcal{V}$ of m-groups the following conditions are equivalent:*

a) *$\mathcal{V}$ is an ℓ-variety of m-groups.*
b) $\mathcal{V}^\ell = \mathcal{V}$.
c) $\mathcal{V}_\ell = \mathcal{V}$.
d) $\mathcal{V}^\ell = \mathcal{V}_\ell$.

Proof. Clear from Lemmas 2.1 and 2.2. □

Lemma 2.4. $\mathcal{V}_\ell \subseteq \mathcal{V} \subseteq \mathcal{V}^\ell$.

Theorem 2.2. a) *$\mathcal{V}_\ell$ is the largest ℓ-variety contained in $\mathcal{V}$.*
b) *$\mathcal{V}^\ell$ is the smallest ℓ-variety containing $\mathcal{V}$.*

Proof. Clear from Lemmas 2.3 and 2.4. $\square$

Theorem 2.3. *If $\mathcal{U}$ is a reversible variety of ℓ-groups generated by a family $\{G_i;\ i \in I\}$ of ℓ-groups then $\mathcal{U}_m$ is generated by $\{(G_i \times G_i^*, \mathrm{Exch});\ i \in I\}$.*

Proof. Clear from Lemma 2.1 as well as from Lemma 2.2. $\square$

Example 2.2. (From well known facts, see [Re].)
a) $(\mathbb{Z} \times \mathbb{Z}^*, \mathrm{Exch})$ generates $\mathscr{A}_m$.
b) $\{(T_i \times T_i^*, \mathrm{Exch});\ T_i$ totally ordered group$\}$ generates $\mathcal{R}_m$.
c) $(\mathrm{Aut}\,\mathbb{Q} \times (\mathrm{Aut}\,\mathbb{Q})^*, \varphi)$, where we saw that φ was unique up to m-isomorphism, generates $\mathcal{M}$.

Mimicking, as introduced in [Gℓ-Ho-Mc], proved a very powerful tool in the study of ℓ-groups. (The reader can also refer to [Re], §10.3 for the ℓ-group version.)

Definition 2.4. a) We say that a representation (G, Ω, a) of an m-group (G, φ) *mimics a representation* (H, Λ, b) of an m-group (H, ψ) if and only if, whenever
(i) $\lambda \in \Lambda$,
(ii) $\{\mathfrak{w}_r(\overline{x}, \overline{\varphi(x)})\}$ is a finite set of words of the language of m-groups in variables $\overline{x} = (x_1, \ldots, x_n)$,
(iii) $\overline{h} = (h_1, \ldots, h_n) \in H^n$,
then
(iv) there exist $\alpha \in \Omega$ and $\overline{g} = (g_1, \ldots, g_n) \in G^n$ such that
(v) $(\mathfrak{w}_i(\overline{h}))(\lambda) < (\mathfrak{w}_j(\overline{h}))(\alpha)$ if and only if $(\mathfrak{w}_i(\overline{g}))(\alpha) < (\mathfrak{w}_j(\overline{g}))(\alpha)$.

b) We say that (G, Ω, a) *mimics an m-group* (H, ψ) if and only if it mimics all representations of this m-group, and that (G, Ω, a) *mimics a variety* $\mathcal{V}$ of m-groups if and only if $(G, \varphi) \in \mathcal{V}$ and (G, Ω, a) mimics all m-groups in $\mathcal{V}$.

Lemma 2.5. *If a representation (G, Ω, a) of an m-group (G, φ) mimics a variety $\mathcal{V}$, then (G, φ) generates $\mathcal{V}$.*

Proof. Let $\mathfrak{w}(\overline{x}) = e$ not hold in $\mathcal{V}$. Then there exist $(H, \psi) \in \mathcal{V}$ and $\overline{h} = (h_1, \ldots, h_n) \in H^n$ such that $\mathfrak{w}(\overline{h}) \neq e$ in (H, ψ). Let (H, Λ, b) be a representation of (H, ψ). Then for some $\lambda \in \Lambda$, $\mathfrak{w}(\overline{h}, \overline{bhb})(\lambda) \neq \lambda$. Since (G, Ω, a) mimics (H, ψ), there are $\alpha \in \Omega$ and $\overline{g} \in G^n$ such that $\mathfrak{w}(\overline{g}, \overline{aga})(\alpha) \neq \alpha$, hence $\mathfrak{w}(\overline{g}, \overline{\varphi(g)}) = e$ does not hold in (G, φ). $\square$

Notation 2.4. It was established in [Gi-L, Corollaire III.7] that if T is an o-2-homogeneous chain without outer automorphisms, the representation of the m-group $(\operatorname{Aut} T, \varphi)$ is unique up to half ℓ-group isomorphisms. This holds in particular for $T = \mathbb{Q}$, the rational line, and $T = \mathbb{R}$, the real line. We shall denote by Inv the "unique" decreasing automorphism of order two defined on $\operatorname{Aut}\mathbb{Q}$ as well as on $\operatorname{Aut}\mathbb{R}$ by $\big(\operatorname{Inv}(f)\big)(x) = -f(-x)$.

Theorem 2.4. *If (G, Ω, a) is a representation for an m-group (G, φ) such that G is 2-transitive on Ω, then (G, Ω, a) mimics the variety of all m-groups.*

Proof. The proof is just an adaptation of that given in [Re] in Example 10.3.4.

Let (H, Λ, b), λ, $\big\{\mathfrak{w}_r\big(\overline{x}, \overline{\varphi(x)}\big)\big\}$ and $h_1, \ldots, h_n$ be as in the definition of mimicking. Every word $\mathfrak{w}_r$ can be written in the form $\mathfrak{w}_r = \bigvee_i \bigwedge_j \mathfrak{w}_{rij}$, $i \in I$, $j \in J$, where I and J are finite and $\mathfrak{w}_{rij}$ is a group word. Now, in the subgroup $\overline{G \cup Ga}$ of $\mathcal{M}(\Omega)$, each $\mathfrak{w}_{rij}\big(\overline{h}, \overline{\varphi(h)}\big)$ is either $\mathfrak{w}_{rij}\big(\overline{h}, \overline{\varphi(h)}\big) = \mathfrak{w}'_{rij}(\overline{h})$ of $\mathfrak{w}_{rij}\big(\overline{h}, \overline{\varphi(h)}\big) = \mathfrak{w}'_{rij}(\overline{ah})$, where $\mathfrak{w}'_{rij}$ is a group word.

Let $\{u_t;\ t = 1, \ldots, m\}$ be the set of all initial segments of the words $\mathfrak{w}'_{rij}$ when written in reduced form. Let

$$\lambda_0^1 = \lambda, \lambda_t^1 = \big(u_t(\overline{h})\big)(\lambda),$$
$$\lambda_0^2 = a(\lambda), \lambda_t^2 = \big(u_t(\overline{h})\big)\big(a(\lambda)\big).$$

Take $\{\alpha_t^\varepsilon;\ t = 1, \ldots, m, \varepsilon = 1, 2\} \subseteq \Omega$ such that $\lambda_t^\varepsilon \to \alpha_t^\varepsilon$ is an order isomorphism. For each $i = 1, \ldots, n$ take $g_i \in G$ such that, for all t, $s = 0, \ldots, m$ and ε, $\varepsilon' = 1, 2$,

$$g_i(\alpha_s^\varepsilon) \leqslant \alpha_t^{\varepsilon'} \iff h_i(\lambda_s^\varepsilon) \leqslant \lambda_t^{\varepsilon'}.$$

Then g_i, where act on $\{\lambda_t^\varepsilon;\ t = 0, \ldots, m, \varepsilon = 1, 2\}$, mimics acting of h_i on $\{\lambda_t^\varepsilon;\ t = 0, \ldots, m, \varepsilon = 1, 2\}$, and hence (G, Ω, a) mimics (H, Λ, b). $\square$

The following lemma could be also proved using [K-M 2, Proposition 4.7.1, Lemma 7.3.1, Theorem 7.3.1, and Lemma 7.4.1]. Here we present instead its direct and short proof.

Lemma 2.6. *Any ℓ-group G can be ℓ-embedded in some ℓ-permutation group $\operatorname{Aut} T$ where T is a 2-transitive chain which is isomorphic to any of its bounded intervals and anti-isomorphic to itself (in particular, the order type of $T + 1 + T$ is T) and $\operatorname{card} T \leqslant \operatorname{card} G$.*

Proof. Let $\mathbb{Q}$ be the rational line, σ an order reversing permutation of $\mathbb{Q}$, and, for each a, $b \in \mathbb{Q}$ with $a < b$, take an order isomorphism x_{ab} from the interval (a, b) to $\mathbb{Q}$. Let M be a model of $\mathbb{Q} \cup X \cup \{\sigma\}$ in a first order language including

– a unary predicate for $\mathbb{Q}$,
– a binary predicate for the natural order on $\mathbb{Q}$,
– a ternary predicate for the action of $X \cup \{\sigma\}$ on $\mathbb{Q}$.

In particular, M satisfies the first order formulas

$$\forall a, b, \in \mathbb{Q} \ \Big(a < b \Rightarrow \exists x \in X\big(x(a,b)\big) = \mathbb{Q}\Big)$$

and

$$\forall a, b \in \mathbb{Q} \ \big(a < b \Rightarrow \sigma(b) < \sigma(a)\big)$$

where σ is definable in M.

It follows that, for any model $M' = Y \cup T \cup \{\tau\}$ of the theory M, T is a 2-homogeneous chain which is isomorphic to any of its bounded intervals and anti-isomorphic to itself.

Now, by Ehrenfeucht-Mostowski construction (see [Ra]), for any chain S, $\operatorname{Aut} S$ can be embedded in to the group of automorphisms of some $M' = y \cup T \cup \{\tau\}$ of the theory of M with $S \subseteq T$ and $\operatorname{card} S = \operatorname{card} G \leqslant \operatorname{card} \operatorname{Aut} S$. This embedding induces the usual embedding of $\operatorname{Aut} S$ in $\operatorname{Aut} T$, hence any ℓ-group can be embedded in such an $\operatorname{Aut} T$.

Clearly, if T has the required properties, for $a < b < c$ in T, $T + 1 + T^* \cong (a,b) \cup \{b\} \cup (b,c) \cong T^* + 1 + T$. $\square$

Theorem 2.5. *Let (G, φ) be an m-group on a chain S. Then there exist a 2-homogeneous chain T and a decreasing automorphism $\widetilde{\varphi}$ on T such that $(G, \varphi) \subseteq (\operatorname{Aut} T, \widetilde{\varphi})$ and $\operatorname{card} T \leqslant \operatorname{card} G$.*

P r o o f. Take an m-group (G, φ). We know that (G, φ) can be embedded into $(G \times G^*, \mathrm{Exch})$, hence into some $\big(\operatorname{Aut} T \times (\operatorname{Aut} T)^*, \mathrm{Exch}\big)$ where T satisfies the requirements of Lemma 2.6. Let T_1, α, T_2 be chains such that

– α has one element,
– there is an o -isomorphism i_1 from T onto T_1,
– there is an o-isomorphism i_2 from T onto T_2.

For any $(g, h) \in \operatorname{Aut} T \times (\operatorname{Aut} T)^*$ set

$$F((g,h))i_1(t) = g(t),$$
$$F((g,h))(\alpha) = \alpha,$$
$$F((g,h,))i_2(t) = i_2 h(t).$$

Let u be the anti-isomorphism of $S = T_1 + \alpha + T_2^*$ defined by

$$u\big(i_1(t)\big) = i_2(t), \quad u(\alpha) = \alpha, \quad u\big(i_2(t)\big) = i_1(t),$$

and let ψ be defined on $\operatorname{Aut} S$ by

$$\psi(g) = ugu^{-1} \text{ for all } g.$$

Clearly $(\operatorname{Aut} S, \psi)$ is an m-group and F is an m-embedding of $\bigl(\operatorname{Aut} T \times (\operatorname{Aut} T)^*, \operatorname{Exch}\bigr)$ into it.

By Lemma 2.6., S is isomorphic to T, hence with the same properties. $\square$

3. Other varieties of m-groups

On any abelian ℓ-group H, one can define a mapping Inv such that for each $a \in H$, $\operatorname{Inv}(a) = a^{-1}$, and that (H, Inv) is an m-group.

Definition 3.1. By the variety $\mathcal{I}$ we will understand the variety of m-groups defined by the identity $\varphi(x) = \operatorname{Inv}(x) = x^{-1}$.

Proposition 3.1. *The variety $\mathcal{I}$ is generated by the m-group $(\mathbb{Z}, \operatorname{Inv})$.*

Proof. Let $(G, \operatorname{Inv}) \in \mathcal{I}$. Clearly, G is abelian, hence it lies in the variety of ℓ-groups $\mathscr{A}$ generated by $(\mathbb{Z}, \leqslant)$. The rest follows from the fact that Inv is definable in the language of groups. $\square$

As a corollary, we get the following theorem.

Theorem 3.1. *The variety $\mathcal{I}$ is the smallest proper variety of m-groups and it is not idempotent.*

Proof. Let $\mathcal{V}$ be a non-trivial variety of m-groups and let $\{e\} \neq (G, \varphi) \in \mathcal{V}$. Take $e < x \in G$ and set $y = x\varphi(x)^{-1}$. Then $\varphi(y) = y^{-1}$, hence the m-subgroup generated by y in (G, φ) is a copy of $(\mathbb{Z}, \operatorname{Inv})$, a generating structure for $\mathcal{I}$. Therefore $\mathcal{I} \subseteq \mathcal{V}$.

The variety $\mathcal{I}^2$ is generated by $(\mathbb{Z}, \operatorname{Inv}) Wr (\mathbb{Z}, \operatorname{Inv}) \notin \mathscr{A}_m$, hence $\mathcal{I}^2 \neq I$. $\square$

Definition 3.2. By the variety $\mathcal{C}$ we will understand the variety defined by the identity $\bigl[x, \varphi(x)\bigr] = e$.

Example 3.1. Consider the m-group (G_2, φ_2) from Example 2.2. Obviously, $g\varphi(g) = \varphi(g)g$ for each $g \in G$, hence $(G, \varphi) \in \mathcal{C}$. That means $\mathscr{A}_m$ is a proper subvariety of $\mathcal{C}$.

Further, set $A = \{a_{-12}^p \cdot a_{12}^q;\ p, q \in \mathbb{Z}\}$. Then A is a commutative m-ideal of (G, φ) and $(G/A, \overline{\varphi}) \in \mathscr{A}_m$, hence $(G, \varphi) \in \mathscr{A}_m^2$.

From this we get the following theorem.

Theorem 3.2. a) *$\mathscr{A}_m$ is strictly included in $\mathcal{C}$.*

b) $\mathcal{C} \cap \mathscr{A}_m^2 \neq \mathscr{A}_m$.

Theorem 3.3. *$\mathscr{A}_m$ is the smallest m-variety between $\mathcal{I}$ and $\mathcal{C}$. (Hence $\mathscr{A}_m$ covers $\mathcal{I}$.)*

Proof. Let $\mathcal{V}$ be a variety of m-groups such that $\mathcal{I} \subset \mathcal{V} \subseteq \mathcal{C}$. Let $(G, \varphi) \in \mathcal{V}$ be such an m-group that $\varphi \neq \mathrm{Inv}$, i.e., that there exists $a \in G$ with $\varphi(a) \neq a^{-1}$. Let us show that then there exists an element $e < b \in G$ for which $\varphi(b) \neq b^{-1}$.

Let $\varphi(b) = b^{-1}$ for each $e < b \in G$. Then $a = a^+ \cdot (a^-)^{-1}$ implies $\varphi(a^+) = \varphi(a) \cdot \varphi(a^-)$ and thus $(a^+)^{-1} = \varphi(a) \cdot (a^-)^{-1}$, that means $\varphi(a) = \big((a^-)^{-1}a^+\big)^{-1} = \big(a^+(a^-)^{-1}\big)^{-1} = a^{-1}$, contradiction.

Hence, consider an element $e < b \in G$ for which $\varphi(b) \neq b^{-1}$. Let $\langle b\rangle \cap \langle \varphi(b)\rangle \neq \{e\}$ and let $k, p \in \mathbb{Z}$, $\varphi(b)^k = b^p$, $0 < p$. Then $k < 0$ and $\varphi(b)^k = b^p$, $b^k = \varphi(b)^p$, therefore $\varphi(b)^{kp} = b^{p^2} = b^{k^2}$. Consequently $p^2 = k^2$ and so $k = -p$, that means $\varphi(b)^k = \varphi(b)^{-p}$. Hence $\varphi(b)^p \cdot b^p = e$, and since $G \in \mathcal{C}$, $\big(\varphi(b) \cdot b\big)^p = e$. This implies $\varphi(b) \cdot b = e$, thus $\varphi(b) = b^{-1}$, a contradiction.

Since $\big(\langle b\rangle, \leqslant\ \big) \simeq (\mathbb{Z}, \leqslant)$ and $\big(\langle\varphi(b)\rangle, \leqslant\ \big) \simeq (\mathbb{Z}^*, \leqslant)$, the subgroup of G generated by $\big\{b, \varphi(b)\big\}$ is an m-subgroup of (G, φ) isomorphic to $(\mathbb{Z} \times \mathbb{Z}^*, \mathrm{Exch})$. Therefore we have $\mathscr{A}_m \subseteq \mathcal{V}$. □

Theorem 3.4. $\mathcal{C} \cap \mathcal{R}_m = \mathscr{A}_m$.

Proof. Clearly $\mathscr{A}_m \subseteq \mathcal{C} \cap \mathcal{R}_m$. Take $(G, \varphi) \in \mathcal{C} \cap \mathcal{R}_m$. G is a subdirect product of m-groups $(G_i \times G_i^*, \mathrm{Exch})$, where each G_i is a totally ordered group, hence for all $g = \big((a_i), (b_i)\big) \in G \cap \Pi(G_i \times G_i)$ we have

$$g \cdot \varphi(g) = \big((a_i), (b_i)\big) \cdot \big(\mathrm{Exch}(a_i), (b_i)\big) = (a_i b_i, b_i a_i).$$

Since $G \in \mathcal{C}$, $a_i b_i = b_i a_i$.

So all the o-groups G_i, and hence the ℓ-group G, are in the ℓ-group variety $\mathscr{A}$, therefore the m-group (G, φ) belongs to $\mathscr{A}_m$. □

Definition 3.3. By the variety $\mathcal{J}$ we will mean the variety $\mathcal{J} = \bigcup\limits_{n \in \omega} \mathcal{I}^n$, the smallest variety of m-groups containing the powers of $\mathcal{I}$. (Note that $\mathcal{J}$ is the smallest non-trivial idempotent in the semigroup of m-varieties.)

Theorem 3.5. $\mathcal{C} \cap \mathcal{J} = \mathcal{I}$.

P r o o f. First we prove that $\mathcal{C} \cap \mathcal{I}^2 = \mathcal{I}$. Take $G \in \mathcal{I}^2$. There is an m-ideal M of G such that $M \in \mathcal{I}$ and $G/M \in \mathcal{I}$. Since $G/M \in \mathcal{I}$, the following identity holds in G/M:

$$(gM) \cdot \varphi(gM) = M.$$

In other words, $g \cdot \varphi(g) \in M$ for all $g \in G$, and since $M \in \mathcal{I}$,

$$e = g \cdot \varphi(g) \cdot \varphi\big(g \cdot \varphi(g)\big) = g \cdot \varphi(g) \cdot \varphi(g) \cdot g.$$

If, moreover, $G \in \mathcal{C}$, this yields $\big(g \cdot \varphi(g)\big)^2 = e$, hence $g \cdot \varphi(g) = e$.

Now assume $\mathcal{C} \cap \mathcal{I}^{n-1} = \mathcal{I}$ for some $n \geqslant 2$. Then $\mathcal{I}\mathcal{C} \cap \mathcal{I}^n = \mathcal{I}^2$, and hence

$$\mathcal{C} \cap \mathcal{I}^n = \mathcal{C} \cap \mathcal{I}\mathcal{C} \cap \mathcal{I}^n = \mathcal{C} \cap \mathcal{I}^2 = \mathcal{I}.$$

This yields

$$\mathcal{C} \cap \mathcal{J} = \mathcal{C} \cap \left(\bigcup_{n \in \omega} \mathcal{I}^n \right) = \bigcup_{n \in \omega} (\mathcal{C} \cap \mathcal{I}^n) = \mathcal{I}.$$

$\square$

Corollary 3.1. a) $\mathscr{A}_m \cap \mathcal{J} = \mathcal{I}$.

b) *$\mathcal{J}$ is strictly contained in $\mathcal{N}_m$.*

P r o o f. a) $\mathcal{I} \subseteq \mathscr{A}_m \cap \mathcal{J} \subseteq \mathcal{C} \cap \mathcal{J} = \mathcal{I}$.

Since $\mathcal{I} \subset \mathscr{A}_m$, we have $\mathcal{N}_m \supseteq \cup \mathcal{I}^n = \mathcal{J}$. At the same time, $\mathcal{N}_m \cap \mathscr{A}_m = \mathscr{A}_m$, hence $\mathcal{J} \neq \mathcal{N}_m$. $\square$

Question 3.1. It is well known that the variety $\mathcal{N}$ of normal valued ℓ-groups is the greatest proper variety of ℓ-groups. Does there exist also a greatest proper variety of m-groups?

To study some properties of varieties of m-groups, we will use methods of torsion classes and torsion radicals. These notions for ℓ-groups were introduced by J. Martinez in [Ma2]. W. C. Holland in [Ho3] proved that every variety of ℓ-groups is a torsion class of ℓ-groups. Similarly we can also define torsion classes and torsion radicals for m-groups.

Definition 3.4. A class of m-groups $\mathcal{T}$ is called a *torsion class* of m-groups if $\mathcal{T}$ is closed under

1. convex m-subgroups,
2. m-homomorphic images,
3. joins of convex m-subgroups in $\mathcal{T}$.

It is obvious that for any m-group (G,φ) and convex m-subgroups A_i of G, $i\in I$, the join $A=\bigvee_{i\in I} A_i$ in the lattice of convex m-subgroups of (G,φ) equals the subgroup of G generated by the subgroups A_i, $i\in I$.

Definition 3.5. If $\mathcal{T}$ is a torsion class of m-groups and $G=(G,\varphi)$ is an m-group then $\mathcal{T}(G)$, the join of all convex m-subgroups of G belonging to $\mathcal{T}$, is called the *$\mathcal{T}$-torsion radical* of (G,φ). (Clearly, $\mathcal{T}(G)$ is an m-ideal of (G,φ).)

The proofs of the following two propositions are analogous to those of Propositions 1.1 and 1.2 in [Ma2], and hence they are omitted.

Proposition 3.2. *Let $\mathcal{T}$ be a torsion class of m-groups and let $G=(G,\varphi)$ be an m-group.*

a) *If A is a convex m-subgroup of G, then $\mathcal{T}(A)=A\cap\mathcal{T}(G)$.*
b) *If $\Phi\colon G\to H$ is a surjective m-homomorphism, then $\Phi\big(\mathcal{T}(G)\big)\subseteq\mathcal{T}(H)$.*
c) $\mathcal{T}\big(\mathcal{T}(G)\big)=\mathcal{T}(G)$.
d) *If $\{A_i;\ i\in I\}$ is a family of convex m-subgroups of G, then $\mathcal{T}\Big(\bigvee_{i\in I} A_i\Big)=\bigvee_{i\in I}\mathcal{T}(A_i)$.*

Proposition 3.3. *Suppose we assign to each m-group $G=(G,\varphi)$ an m-ideal $\overline{\mathcal{T}}(G)$ subject to conditions* a) *and* b) *(and so also* c) *and* d)*) in Proposition* 3.2. *Let $\mathcal{T}=\big\{G;\ \overline{\mathcal{T}}(G)=G\big\}$. Then $\mathcal{T}$ is a torsion class of m-groups and $\overline{\mathcal{T}}(G)$ is a $\mathcal{T}$-torsion radical of G, for each m-group (G,φ).*

Functions satisfying conditions a) and b) in Proposition 3.2 are called *torsion radicals.* Thus there is a one-to-one correspondence between torsion classes and torsion radicals of m-groups.

Products of torsion classes of m-groups can be defined likewise as for varieties of m-groups: If $\mathcal{U}$ and $\mathcal{T}$ are torsion classes of m-groups, then an m-group (G,φ) belongs to $\mathcal{U}\cdot\mathcal{T}$ if and only if there is an m-ideal M of (G,φ) with $(M,\varphi)\in\mathcal{U}$ and $(G/M,\overline{\varphi})\in\mathcal{T}$. To verify that $\mathcal{U}\cdot\mathcal{T}$ is a torsion class of m-groups we can use, similarly as for ℓ-groups (see [Ma2, p. 287]), the corresponding torsion radicals:

If (G,φ) is an m-group, let $\mathcal{X}(G)$ be the unique m-ideal of (G,φ) such that $\mathcal{X}(G)/\mathcal{U}(G)=\mathcal{T}\big(G/\mathcal{U}(G)\big)$. Then $(G,\varphi)\in\mathcal{U}\cdot\mathcal{T}$ if and only if $\mathcal{X}(G)=G$. Since the lattice of convex m-subgroups of G is distributive, condition a) in the definition of a torsion radical can be verified in the same way as in [Ma2] for ℓ-groups. Condition b) is satisfied trivially. Hence $\mathcal{X}$ is a torsion radical and thus $\mathcal{U}\cdot\mathcal{T}$ is a torsion class of m-groups.

Moreover, the operation "$\cdot$" on torsion classes of m-groups is associative. Let σ be an ordinal number. If σ is not a limit ordinal, we define $\mathcal{T}^{\sigma}=\mathcal{T}\cdot\mathcal{T}^{\sigma-1}$, if σ is a limit ordinal, we set $\mathcal{T}^\sigma = \{G;\ \bigcup_{\alpha<\sigma} \mathcal{T}^\alpha(G) = G\}$. We have, similarly as for ℓ-groups (see [Ma2, p. 287]), that $\mathcal{T}^\sigma$ is a torsion class of m-groups.

Theorem 3.6. *If $\mathcal{V}$ is a reversible variety of ℓ-groups, then $\mathcal{V}_m$ is a torsion class of m-groups.*

P r o o f. Let (G,φ) be an m-group and C_i, $i \in I$, a family of its convex m-subgroups such that $(C_i,\varphi) \in \mathcal{V}_m$ for each $i \in I$. Since by [Ho2] $\mathcal{V}$ is a torsion class of ℓ-groups, $C = \bigvee_{i\in I} C_i$, the convex ℓ-subgroup of G generated by $C_i's$, belongs to $\mathcal{V}$ too, and hence $(C,\varphi) \in \mathcal{V}_m$. □

Corollary 3.2. a) *For each $n \geqslant 1$, $\mathcal{I}^n$ is a torsion class.*
b) *$\mathcal{J}$ is a torsion class.*

P r o o f. a) For $n = 1$, it is enough to prove that $\mathcal{I}$ is closed under the join of two convex m-subgroups.

Take an m-group (G,φ) and convex m-subgroups (A,φ) and (B,φ) of (G,φ) belonging to $\mathcal{I}$. Since $\mathcal{I} \subseteq \mathscr{A}_m$, we have for $(C,\varphi) = (A,\varphi) \vee (B,\varphi)$ that $(C,\varphi) \in \mathscr{A}_m$ and $C = AB = BA$. Hence, for any $c = ab \in C$,

$$\varphi(c) = \varphi(a)\cdot\varphi(b) = a^{-1}b^{-1} = b^{-1}a^{-1} = c^{-1},$$

thus $(C,\varphi) \in \mathcal{I}$.

We know that products of torsion classes are torsion classes, too, hence $\mathcal{I}^n$ is torsion class of m-groups for each $n \geqslant 1$.

b) Follows from the fact that $\mathcal{J} = \bigcup_{u\in\omega} \mathcal{I}^n$. □

Question 3.2. Many of properties of varieties of ℓ-groups were proved using the fact that by [Ho3] all varieties of ℓ-groups are torsion classes. Is also every variety of m-groups a torsion class of m-groups?

For varieties of m-groups that are simultaneously torsion classes of m-groups, we will prove a generalization of a result concerning varieties of ℓ-groups due to Bernau [B] (see [K-M 2, Theorem 4.3.2]). Recall that an ℓ-subgroup H of an ℓ-group G is called *closed* if for each $a_i \in H$, $i \in I$, such that $a = \bigvee_{i\in I} a_i$ in G exists, we have $a \in H$. The *closure* $\overline{H}$ of an ℓ-subgroup H of G is the intersection of all closed ℓ-subgroups of G containing H. Now, if (G,φ) is an m-group then $H \subseteq G$ is called a *closed m-subgroup* if it is both an m-subgroup and closed. The *closure* of an m-subgroup H of (G,φ) is then the intersection of all closed m-subgroups of (G,φ) containing H.

Proposition 3.3. *If $\mathcal{X}$ is a variety of m-groups, (G,φ) an m-group, (H,φ) an m-subgroup of (G,φ) and $(H,\varphi)\in\mathcal{X}$, then also $(\overline{H},\varphi)\in\mathcal{X}$.*

Proof. Let (G,φ) be an m-group and H an m-subgroup of G. Denote by K the m-subgroup of G generated by the suprema of subsets of H (for which they exist). Let $\mathcal{X}$ be a variety of m-groups and $(H,\varphi)\in\mathcal{X}$. Every element $a\in K$ can be written in the form

$$a=\bigvee_{i\in I}\bigwedge_{j\in J}\varphi^{s_{ij}}(h_{ij}) \tag{+}$$

where I and J are finite sets, $s_{ij}=0$ or 1,

$$h_{ij}=\Big(\bigvee_G M_{ij1}\Big)^{\varepsilon_{ij1}}\dots\Big(\bigvee_G M_{ijn(ij)}\Big)^{\varepsilon_{ijn(ij)}},$$

$\varepsilon_{ijk}=\pm1$, $M_{ijk}\subseteq H^+$, $\bigvee\limits_G M_{ijk}$ exists.

Let an identity $\mathfrak{w}\big(x_1,\dots,x_n,\varphi(x_1),\dots,\varphi(x_n)\big)=e$ be satisfied in $\mathcal{X}$. Let $a_1=a\in K$, $a_2,\dots,a_n\in H$, and a be in the form (+). Let m_{ijk} be the supremum of some finite subset of M_{ijk} and let

$$\overline{a}_1=\bigvee_{i\in I}\bigwedge_{j\in J}\varphi^{s_{ij1}}(m_{ij1}^{\varepsilon_{ij1}})\dots\varphi^{s_{ijn(ij)}}\big(m_{ijn(ij)}^{\varepsilon_{ijn(ij)}}\big).$$

Then $\overline{a}_1\in H$, and so $\mathfrak{w}\big(\overline{a}_1,a_2,\dots,a_n,\varphi(\overline{a}_1),\varphi(a_2),\dots,\varphi(a_n)\big)=e$. Write $\mathfrak{w}$ in the form

$$\begin{aligned}&\mathfrak{w}\big(x_1,\dots,x_n,\varphi(x_1),\dots,\varphi(x_n)\big)\\&=\bigvee_{\alpha\in A}\bigwedge_{\beta\in B}\mathfrak{w}_{\alpha\beta}\big(x_1,\dots,x_n,\varphi(x_1),\dots,\varphi(x_n)\big),\end{aligned}$$

where $\mathfrak{w}_{\alpha\beta}=\varphi^{s_1}\big(x_{\alpha\beta1}^{\varepsilon_1}\big)\dots\varphi^{s_k}\big(x_{\alpha\beta k}^{\varepsilon_k}\big)$, $x_{\alpha\beta i}\in\{x_1,\dots,x_n\}$, $\varepsilon_i=\pm1$, $s_i=0$ or 1.

Set $b_{\alpha\beta j}=a_{\alpha\beta j}$ for $x_{\alpha\beta j}\neq x_1$, and $b_{\alpha\beta j}=\overline{a}_1$ for $x_{\alpha\beta j}=x_1$. Now, if $\overline{\mathfrak{w}}_{\alpha\beta}=\varphi^{s_1}(b_{\alpha\beta1})\dots\varphi^{s_k}(b_{\alpha\beta k})$, then $\mathfrak{w}\big(\overline{a}_1,a_2,\dots,a_n,\varphi(\overline{a}_1),\varphi(a_2),\dots,\varphi(a_n)\big)=\bigvee\limits_{\alpha\in A}\bigwedge\limits_{\beta\in B}\overline{\mathfrak{w}}_{\alpha\beta}\big(\overline{a}_1,a_2,\dots,a_n,\varphi(\overline{a}_1),\varphi(a_2),\dots,\varphi(a_n)\big)=e$.

We will use the following substitution in $\overline{\mathfrak{w}}_{\alpha\beta}$. If $x_{\alpha\beta\gamma}^{\varepsilon_\gamma}\neq x_1$ then $b_{\alpha\beta\gamma}^{\varepsilon_\gamma}$ is not changed, if $x_{\alpha\beta\gamma}=x_1$ then $b_{\alpha\beta\gamma}=\overline{a}_1$, and in this case: If $\varepsilon_\gamma=1$, $\varepsilon_{ijk}=-1$, $s_\gamma=0$, then $m_{ijk}^{\varepsilon_{ijk}}$ in $b_{\alpha\beta\gamma}^{\varepsilon_\gamma}$ is substituted by $\bigvee M_{ijk}^{\varepsilon_{ijk}}$.

Similarly for

$$\begin{aligned}&\varepsilon_\gamma=1, && \varepsilon_{ijk}=1, && s_\gamma=1,\\&\varepsilon_\gamma=-1, && \varepsilon_{ijk}=1, && s_\gamma=0,\\&\varepsilon_\gamma=-1, && \varepsilon_{ijk}=-1, && s_\gamma=1.\end{aligned}$$

If ε_γ, $\varepsilon_{ijk}=1$, $s_\gamma=0$, then $m_{ijk}^{\varepsilon_{ijk}}$ in $b_{\alpha\beta\gamma}^{\varepsilon_\gamma}$ is substituted by $g^{\varepsilon_{ijk}}$, where $g^{\varepsilon_{ijk}}$ is any element in M_{ijk}.

Similarly for

$$\begin{aligned}
&\varepsilon_\gamma=1, && \varepsilon_{ijk}=-1, && s_\gamma=1,\\
&\varepsilon_\gamma=-1, && \varepsilon_{ijk}=-1, && s_\gamma=0,\\
&\varepsilon_\gamma=-1, && \varepsilon_{ijk}=1, && s_\gamma=1.
\end{aligned}$$

Denote the element obtained in this way from $b_{\alpha\beta j}$ by $c_{\alpha\beta j}$, and the element obtained from $\overline{\mathfrak{w}}_{\alpha\beta}$ by $\overline{u}_{\alpha\beta}$. Then always $c_{\alpha\beta j}^{\varepsilon_j}\leqslant b_{\alpha\beta j}^{\varepsilon_j}$, and hence $\overline{u}_{\alpha\beta}\leqslant\overline{\mathfrak{w}}_{\alpha\beta}$.

Now we have

$$\begin{aligned}
&\mathfrak{w}\big(a_1,\dots,a_n,\varphi(a_1),\dots,\varphi(a_n)\big)\\
&=\bigvee_{\alpha\in A}\bigwedge_{\beta\in B}\mathfrak{w}_{\alpha\beta}\big(a_1,\dots,a_n,\varphi(a_1),\dots,\varphi(a_n)\big),
\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{w}_{\alpha\beta}\big(a_1,\dots,a_n,\varphi(a_1),\dots,\varphi(a_n)\big)&=\varphi^{s_1}\left(a_{\alpha\beta 1}^{\varepsilon_1}\right)\dots\varphi^{s_k}\left(a_{\alpha\beta k}^{\varepsilon_k}\right)\\
&=\vee\,\overline{u}_{\alpha\beta}\big(a_1,\dots,a_n,\varphi(a_1),\dots,\varphi(a_n)\big),
\end{aligned}$$

and where the last supremum is meant all choices of elements in M_{ijk}.

Hence

$$\begin{aligned}
&\mathfrak{w}\big(a_1,\dots,a_n,\varphi(a_1),\dots,\varphi(a_n)\big)\\
&=\bigvee_{\alpha\in A}\bigwedge_{\beta\in B}\mathfrak{w}_{\alpha\beta}\big(a_1,\dots,a_n,\varphi(a_1),\dots,\varphi(a_n)\big)\\
&=\vee\left(\bigvee_{\alpha}\bigwedge_{\beta}\overline{u}_{\alpha\beta}\right)\leqslant\vee\left(\bigvee_{\alpha}\bigwedge_{\beta}\overline{\mathfrak{w}}_{\alpha\beta}\right)\\
&=\vee\,\mathfrak{w}\big(\overline{a}_1,a_2,\dots,a_n,\varphi(\overline{a}_1),\varphi(a_2),\dots,\varphi(a_n)\big),
\end{aligned}$$

that means

$$\mathfrak{w}\big(a_1,\dots,a_n,\varphi(a_1),\dots,\varphi(a_n)\big)\leqslant e.$$

By the same considerations applied to $\mathfrak{w}^{-1}$ we prove that

$$\mathfrak{w}^{-1}\big(a_1,\dots,a_n,\varphi(a_1),\dots,\varphi(a_n)\big)\leqslant e,$$

hence

$$\mathfrak{w}\big(a_1,\dots,a_n,\varphi(a_1),\dots,\varphi(a_n)\big)=e.$$

□

The following theorem is now an immediate consequence:

Theorem 3.7. *If a variety of m-groups $\mathcal{U}$ is a torsion class of m-groups, then the $\mathcal{U}$-radical $\mathcal{U}(G)$ of every m-group G is a closed m-ideal of G.*

Now, we can characterize the varieties $\mathcal{I}^n$, $n \geqslant 1$, by defining identities.

Theorem 3.8. *For each $n \geqslant 1$, the variety $\mathcal{I}^n$ is defined by the identity*

$$\varphi\big(g^{2^{n-1}}\big) = g^{-2^{n-1}}.$$

P r o o f. For $n = 1$ it follows from the definition of the variety $\mathcal{I}$.

Let the assertion be proved for $n \geqslant 1$.

a) Let $(G, \varphi) \in \mathcal{I}^{n+1}$. Then there is an m-ideal B of G such that $(B, \varphi) \in \mathcal{I}$ and $(G/B, \overline{\varphi}) \in \mathcal{I}^n$. Hence, for each $g \in G$, $\overline{\varphi}(gB)^{2^{n-1}} = (gB)^{-2^{n-1}}$, thus $\varphi(g)^{2^{n-1}} \cdot B = g^{-2^{n-1}} \cdot B$, so $g^{2^{n-1}}\varphi(g)^{2^{n-1}} \in B$. Since $(B, \varphi) \in \mathcal{I}$, $\varphi(g)^{2^{n-1}} g^{2^{n-1}} = \varphi(g)^{-2^{n-1}} g^{-2^{n-1}}$, therefore $\varphi(g)^{2^n} = g^{-2^n}$.

b) Conversely, let an m-group (G, φ) satisfy the identity $\varphi(g)^{2^n} = g^{-2^n}$. Consider $\mathcal{I}(G)$, the $\mathcal{I}$-torsion radical of G. Let $g \in G$. Then

$$\varphi\Big(g^{2^{n-1}} \cdot \varphi\big(g^{2^{n-1}}\big)\Big) = \varphi\big(g^{2^{n-1}}\big) \cdot g^{2^{n-1}} = \varphi\big(g^{-2^{n-1}}\big) \cdot g^{-2^{n-1}} = \Big(g^{2^{n-1}} \cdot \varphi\big(g^{2^{n-1}}\big)\Big)^{-1},$$

thus $g^{2^{n-1}} \cdot \varphi\big(g^{2^{n-1}}\big) \in \mathcal{I}(G)$. Hence $\varphi\big(g^{2^{n-1}}\big) \cdot \mathcal{I}(G) = g^{-2^{n-1}} \cdot \mathcal{I}(G)$, i.e., $\overline{\varphi}\Big(\big(g \cdot \mathcal{I}(G)\big)^{2^{n-1}}\Big) = \big(g \cdot \mathcal{I}(G)\big)^{-2^{n-1}}$. That means $\big(G/\mathcal{I}(G), \overline{\varphi}\big) \in \mathcal{I}^n$, and so $(G, \varphi) \in \mathcal{I}^{n+1}$. $\square$

Remark 3.1. Since for any m-group (G, φ) belonging to the variety $\mathcal{C}$, $\varphi(x)^k = x^{-k}$ implies $\varphi(x) = x^{-1}$ for each $x \in G$ and $k \in \mathbb{Z}$, the assertion of Theorem 3.5 is now an immediate consequence of Theorem 3.8.

4. Free m-groups

If X is a non-empty se, denote by L_X the free ℓ-group over the free generating set X. Let $S = \{s_i^0;\ i \in I\}$. Set $S' = \{s_i^1;\ i \in I\}$, a disjoint copy of S (where $s_i^0 \to s_i^1$ is a bijection). Let $F_0\colon\ S \cup S' \to S \cup S'$ be a mapping such that $F_0(s_i^0) = s_i^1$, $F_0(s_i^1) = s_i^0$ $(i \in I)$.

Theorem 4.1. *The free m-group with the generating set S is $(L_{S\cup S'}, F)$ where F is defined as follows: If $\ell = \bigvee\limits_i \bigwedge\limits_j \prod\limits_k s_{ijk}^{\varepsilon_{ijk}} \in L_{S\cup S'}$ ($\varepsilon_{ijk} = 0$ or 1) then $F(\ell) = \bigwedge\limits_i \bigvee\limits_j \prod\limits_k F\big(s_{ijk}^{\varepsilon_{ijk}}\big)$.*

P r o o f. It is obvious that $F\colon L_{S\cup S'} \to L^*_{S\cup S'}$ such that $F\big(\bigvee_i \bigwedge_j \prod_k s_{ijk}^{\varepsilon_{ijk}}\big) = \bigwedge_i \bigvee_j \prod_k F_0\big(s_{ijk}^{\varepsilon_{ijk}}\big)$ is the unique ℓ-homomorphism of $L_{S\cup S'}$ onto $L^*_{S\cup S'}$ extending F_0. Moreover, F is a decreasing group automorphism of order 2 of $L_{S\cup S'}$, hence $(L_{S\cup S'}, F)$ is an m-group.

Let (G,φ) be an m-group generated, as an ℓ-group, by S. Then the ℓ-group G is also generated by $S \cup \{\varphi(s_i^0);\ s_i^0 \in S\}$. Thus there is a unique ℓ-homomorphism $p\colon L_{S\cup S'} \to G$ such that $p(s_i^0) = s_i^0$, $p(s_i^1) = \varphi(s_i^0)$ $(i \in I)$. Set $\delta_{ijk} = 1$ for $\varepsilon_{ijk} = 0$ and $\delta_{ijk} = 0$ for $\varepsilon_{ijk} = 1$. Then

$$\begin{aligned} pF\Big(\bigvee_i \bigwedge_j \prod_k s_{ijk}^{\varepsilon_{ijk}}\Big) &= p\Big(\bigwedge_i \bigvee_j \prod_k F(s_{ijk}^{\varepsilon_{ijk}})\Big) = p\Big(\bigwedge_i \bigvee_j \prod_k s_{ijk}^{\delta_{ijk}}\Big) \\ &= \bigwedge_i \bigvee_j \prod_k p(s_{ijk}^{\delta_{ijk}}) = \bigwedge_i \bigvee_j \prod_k \varphi^{\delta_{ijk}}(s_{ijk}) = \bigwedge_i \bigvee_j \prod_k \varphi\big(\varphi^{\varepsilon_{ijk}}(s_{ijk})\big) \\ &= \bigwedge_i \bigvee_j \prod_k \varphi\big(p(s_{ijk}^{\varepsilon_{ijk}})\big) = \varphi\Big(\bigvee_i \bigwedge_j \prod_k p(s_{ijk}^{\varepsilon_{ijk}})\Big) = \varphi p\Big(\bigvee_i \bigwedge_j \prod_k s_{ijk}^{\varepsilon_{ijk}}\Big), \end{aligned}$$

hence p is an m-homomorphism. □

Corollary 4.1. *The free m-group with one generator is not commutative.*

Corollary 4.2. *Let $\mathcal{V}$ be a reversible variety of ℓ-groups and $S = \{s_i^0;\ i \in I\}$ a non-empty set. Then the free m-group in the variety $\mathcal{V}_m$ of m-groups with the set of free generators S is $(L_{\mathcal{V},S\cup S'}, F)$, where $L_{\mathcal{V},S\cup S'}$ is the $\mathcal{V}$-free ℓ-group with the set of free generators $S \cup S'$ ($S' = \{s_i^1;\ i \in I\}$ is a disjoint copy of S) and F is the unique decreasing group automorphism with $F(s_i^0) = s_i^1$ and $F(s_i^1) = s_i^0$.*

Example 4.1. The $\mathscr{A}_m$-free m-group with one generator is (A_2, F) where A_2 is the free abelian ℓ-group over two generators s and s' and F is the unique decreasing group automorphism of order two such that $F(s) = s'$.

Proposition 4.1. *The $\mathcal{I}$-free m-group with one generator is $(\mathbb{Z} \times \mathbb{Z}, \mathrm{Inv})$ where $\mathbb{Z} \times \mathbb{Z}$ is the free ℓ-group with one generator* $(1,-1)$.

P r o o f. Let $(G, \mathrm{Inv}) \in \mathcal{I}$ be generated by an element a. Let $p\colon \mathbb{Z} \times \mathbb{Z} \to G$ be the unique ℓ-homomorphism such that $p((1,-1)) = a$. Consider any element $\bigvee_i \bigwedge_j \varepsilon_{ij}(1,-1)$ (where $\varepsilon_{ij} \in \mathbb{Z}$) in $\mathbb{Z} \times \mathbb{Z}$. Then

$$\begin{aligned} p\,\mathrm{Inv}\Big(\bigvee_i \bigwedge_j \varepsilon_{ij}(1,-1)\Big) &= p\Big(\bigwedge_i \bigvee_j \varepsilon_{ij}(-1,1)\Big) = \bigwedge_i \bigvee_j a^{-\varepsilon_{ij}} \\ &= \mathrm{Inv}\Big(\bigvee_i \bigwedge_j a^{\varepsilon_{ij}}\Big) = \mathrm{Inv}\, p\Big(\bigvee_i \bigwedge_j \varepsilon_{ij}(1,-1)\Big), \end{aligned}$$

hence p is an m-homomorphism. □

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