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ON HARMONIC CONJUGATES WITH EXPONENTIAL MEAN GROWTH

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1. Introduction

Let $h_p(\varphi)$ denote the class of (complex-valued) functions harmonic in the unit disc Δ for which $M_p(u,r) = 0(\varphi(r))$, $r \to 1^-$, where φ is a positive, continuous function defined on some interval $[r_0, 1)$, $r_0 < 1$, and

$$M_p(u,r) = \Big\{\frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\theta})|^p \,d\theta\Big\}^{1/p}.$$

Following [8] we say that $h_p(\varphi)$ is *self-conjugate* if the Riesz projection maps $h_p(\varphi)$ into itself or, equivalently, if $f \in h_p(\varphi)$ whenever f is an analytic function such that $\operatorname{Re} f$ (= real part of f) is in $h_p(\varphi)$.

It follows from the Riesz projection theorem that $h_p(\varphi)$ is self-conjugate whenever $1 < p < \infty$, without additional restrictions on φ. That $h_p((1-r)^{-a})$ is self-conjugate for all $p > 0$, $a > 0$, was established by Hardy and Littlewood [3]. Shields and Williams [8, 9] were the first who studied the case where $\varphi(r)$ is different from $(1-r)^{-a}$. They proved that $h_p(\varphi)$ is self-conjugate provided

$$\text{(U)} \qquad (1-r)^{\beta}\varphi(r) \downarrow 0,\ r \to 1^-,\ \text{for some } \beta < \infty$$

and

$$\text{(L)} \qquad (1-r)^{\alpha}\varphi(r) \uparrow \infty,\ r \to 1^-,\ \text{for some } \alpha > 0.$$

(For the case $p < 1$ see [4, 6].)

The typical example of functions satisfying (U) + (L) is

$$\varphi(r) = (1-r)^{-a}\Big(\log\frac{1}{1-r}\Big)^{b},$$

where $a > 0$. It was also proved in [9] that if $(1-r)^{\beta}\varphi(r) \downarrow 0$ $(r \to 1^-)$ for *all* $\beta > 0$, then $h_\infty(\varphi)$ is not self-conjugate, which is true, e.g., if

$$\varphi(r) = \Big(\log\frac{1}{1-r}\Big)^p, \quad b > 0.$$

We are however interested in the case where $\varphi(r)$ grows *faster* than any positive power of $1/(1-r)$ and especially when

$$\varphi(r) = (1-r)^{-a}\Big(\log\frac{1}{1-r}\Big)^b \exp\frac{c}{1-r} \tag{1.1}$$

$(c > 0)$. We believe that condition (L) is sufficient for $h_p(\varphi)$ to be self-conjugate but we can prove it only under additional restrictions on the regularity of growth of φ. As a special case of our main result (Theorem 2.1) we have that $h_p(\varphi)$ is self-conjugate in the case of (1.1) ($c > 0$ or $c = 0$, $a > 0$).

Our proofs are surprisingly easy and are independent of any deeper fact from the theory of harmonic functions. The key is the inequality

$$|u(0)|^p \leqslant C_p \int_\Delta |u|^p \,\mathrm{d}A, \tag{1.2}$$

where u is harmonic in Δ, and $\mathrm{d}A$ is the normalized planar measure on Δ. If $p \geqslant 1$, then one can take $C_p = 1$ because of the subharmonicity of $|u|^p$. In the case of $p < 1$, in which $|u|^p$ need not be subharmonic, (1.2) is contained implicitly in another theorem of Hardy and Littlewood on harmonic conjugates [3], Theorem 5:

$$\int_\Delta |f|^p \,\mathrm{d}A \leqslant C_p \int_\Delta |\operatorname{Re} f|^p \,\mathrm{d}A, \tag{1.3}$$

where f is analytic and $\operatorname{Im} f(0) = 0$. Indeed, (1.2) follows from (1.3) and the subharmonicity of $|f|^p$. However, Hardy and Littlewood proved their theorems without mentioning the inequality (1.2) and this was the main reason for which their proofs were rather difficult and long.

A proof of (1.2) can be found in [2]. In order that the paper be self-contained we reproduce a very short and simple proof given in [7]. See Lemma 2.1.

2. Main result

A real function F is said to be *almost increasing* (almost decreasing) if there exists a constant $C > 0$ such that $F(x) \leqslant CF(y)$ ($F(y) \leqslant CF(x)$) whenever $x < y$. For a C^1-function F we say that it is *almost convex* if its derivative is almost increasing. An application of Lagrange's theorem shows that F is almost convex if and only if there is a constant $C > 0$ such that

$$F'(x)/C \leqslant \frac{F(y) - F(x)}{y - x} \leqslant CF'(y), \quad x < y. \tag{2.1}$$

By the term *majorant* we mean a function φ defined, positive and continuous on some interval $(r_0, 1)$, $0 < r_0 < 1$, and such that $\varphi(r) \to \infty$ as $r \to 1$. We say that a majorant φ satisfies condition (L^+) if it is C^1 and

$$\varphi^{-m} \text{ is almost convex on } (r_0, 1) \text{ for some } m > 0,\ r_0 < 1. \tag{L$^+$}$$

This is equivalent to the requirement that $\varphi'(r)/\varphi(r)^{m+1}$ is almost decreasing near 1, which implies that $\varphi' > 0$ near 1. Moreover, applying (2.1) to $F = \varphi^{-m}$ we obtain

$$-(m/C)\varphi'(r)\varphi(r)^{-m-1}(\varrho - r) \leqslant \varphi(\varrho)^{-m} - \varphi(r)^{-m}$$

for $r < \varrho < 1$, whence by letting ϱ tend to 1,

$$\varphi'(r) \geqslant \alpha(1 - r)^{-1}\varphi(r) \quad (r_0 < r < 1), \tag{2.2}$$

where $\alpha = C/m$. In particular, $\varphi'(r) \to \infty$ $(r \to 1)$. Thus if φ satisfies (L$^+$), then φ' is a majorant and φ satisfies (L). Further remarks are in Section 3.

Theorem 2.1. *Let φ be a majorant satisfying* (L$^+$) *and let $0 < p \leqslant \infty$. For a function f analytic in Δ the following assertions are equivalent:*

$$f \in h_p(\varphi), \tag{a}$$

$$\operatorname{Re} f \in h_p(\varphi), \tag{b}$$

$$f' \in h_p(\varphi'). \tag{c}$$

Recall that (c) means $M_p(f', r) = 0(\varphi'(r))$, $r \to 1^-$. Since the case $p \geqslant 1$ is somewhat easier (for instance, (a) is deduced from (c) by means of Minkowski's inequality) we shall assume from now on that $0 < p < 1$.

Lemma 2.1. *There is a constant $C_p < \infty$ such that*

$$\sup\{|u(w)|^p\colon\ w \in \Delta_{R/2}(z)\} \leqslant C_p \int_{\Delta_R(z)} |u|^p \,\mathrm{d}A \tag{2.2}$$

whenever u is harmonic in Δ and $\Delta_R(z) := \{w\colon\ |w-z| < R\} \subset \Delta$.

Proof. By dilatations and translations the proof reduces to the case where $z = 0$ and $R = 1$. We may also assume that u is continuous on the closed disc. Under this hypothesis we choose $z_0 \in \Delta$ such that the function

$$h(z) = (1-|z|)^2|u(z)|^p, \quad z \in \Delta,$$

attains its maximum for $z = z_0$. Then we apply the mean value property over the disc $\Delta_r(z_0)$, $r = (1-|z_0|)/2$ to get

$$|u(z_0)| \leqslant r^{-2} \int_{\Delta_r(z_0)} |u(z)| \,\mathrm{d}A(z). \tag{2.3}$$

On the other hand, we have that $(1-|z|)^{-1} \leqslant 2(1-|z_0|)^{-1}$ for $z \in \Delta_r(z_0)$ which, along with the inequality $h(z) \leqslant h(z_0)$, shows that $|u(z)| \leqslant 2^{2/p}|u(z_0)|$ for $z \in \Delta_r(z_0)$. Hence

$$|u(z)| \leqslant C|u(z)|^p|u(z_0)|^{1-p}, \quad z \in \Delta_r(z_0),$$

where C depends only on p. Combining this with (2.3) we obtain

$$h(z_0) \leqslant C_p \int_\Delta |u|^p \,\mathrm{d}A.$$

Now the desired result follows from the inequality $|u(z)|^p \leqslant 4h(z) \leqslant 4h(z_0)$, $|z| \leqslant 1/2$. □

Lemma 2.2. *If $u = \operatorname{Re} f$, where f is analytic in Δ, then there is a constant C_p such that*

$$M_p(f', r) \leqslant C_p(\varrho - r)^{-1} \sup_{0<t<\varrho} M_p(u,t) \tag{2.4}$$

whenever $0 < r < \varrho < 1$.

Proof. Using the simple, familiar estimate

$$|f'(z)| \leqslant CR^{-1} \sup_{\Delta_{R/2}(Z)} |u|$$

we deduce from (2.2) that

$$|f'(r)|^p \leqslant C(\varrho - r)^{-p-2} \int_{\Delta_R(r)} |u|^p \, \mathrm{d}A,$$

where $R = \varrho - r$, $0 < r < \varrho < 1$. Applying this to the functions $z \mapsto f(z\mathrm{e}^{\mathrm{i}\theta})$ we obtain

$$|f'(r\mathrm{e}^{\mathrm{i}\theta})|^p \leqslant C(\varrho - r)^{-p-2} \int_{\Delta_R(r)} |u(w\mathrm{e}^{\mathrm{i}\theta})|^p \, \mathrm{d}A(w),$$

where C depends only on p. Integrating this inequality over $0 \leqslant \theta \leqslant 2\pi$ we find that

$$\begin{aligned} M_p^p(f', r) &\leqslant (\varrho - r)^{-p-2} \int_{\Delta_R(r)} M_p^p(u, |w|) \, \mathrm{d}A(w), \\ &\leqslant C(\varrho - r)^{-p} \sup_{w \in \Delta_R(r)} M_p(u, |w|). \end{aligned}$$

The result follows because $\Delta_R(r) \subset \Delta_\varrho(0)$. □

Lemma 2.3. *There exists a constant C_p such that*

$$M_p^p(f, \varrho) - M_p^p(f, r) \leqslant C_p(\varrho - r)^p M_p^p(f', \varrho) \tag{2.5}$$

whenever $0 < r < \varrho < 1$ and f is analytic in Δ.

Proof. With these hypotheses let $s_j = \varrho - 2^{-j}(\varrho - r)$ and $t_j = (s_j + s_{j+1})/2$, $j \geqslant 0$. Using Lemma 2.1 ($u = f'$) we get

$$\begin{aligned} |f(s_{j+1}) - f(s_j)|^p &\leqslant (s_{j+1} - s_j)^p \sup_{s_j < x < s_{j+1}} |f'(x)|^p \\ &\leqslant C(s_{j+1} - s_j)^p (\varrho - t_j)^{-2} \int_{\Delta_j} |f'|^p \, \mathrm{d}A, \end{aligned}$$

where $\Delta_j = \{w\colon\ |w - t_j| < \varrho - t_j\}$. Now we apply this to the functions $z \mapsto f(z\mathrm{e}^{\mathrm{i}\theta})$ and then integrate with respect to θ. As a result we have

$$M_p^p(f, s_{j+1}) - M_p^p(f, s_j) \leqslant C(s_{j+1} - s_j)^p M_p^p(f', \varrho).$$

(We also have to use the "increasing property" of $M_p(f', \cdot)$.) Now (2.5) is obtained by summation from $j = 0$ to $j = \infty$. □

Remark. The proof can be made shorter by use of the Complex Maximal Theorem (see [5]).

Proof of Theorem 2.1. The implication (a) $\Rightarrow$ (b) is obvious. To prove the rest we may assume that $\varphi' > 0$ on $[0,1)$ and $\varphi(0) = 1$. Then we define a sequence $\{r_j\}$ $(j \geqslant 0)$ by $\varphi(r_j) = 2^j$ and choose $t_j \in (R_j, r_{j+1})$ so that

$$\varphi(r_{j+1}) - \varphi(r_j) = \varphi'(t_j)(r_{j+1} - r_j)$$

i.e.,

$$r_{j+1} - r_j = \frac{2^j}{\varphi'(t_j)} \quad (j \geqslant 0). \tag{2.6}$$

Assuming that φ satisfies (L^+) we have the relation

$$\varphi'(t) \leqslant C\varphi'(r), \quad r_j \leqslant r \leqslant t \leqslant r_{j+2}, \tag{2.7}$$

where C is a constant independent of j, r, t. To show (2.7) choose $m > 0$ such that φ'/φ^{m+1} is almost decreasing on $[0,1)$. Then

$$\begin{aligned}\varphi'(t) &\leqslant C\varphi'(r)(\varphi(t)/\varphi(r))^{m+1}\\ &\leqslant C\varphi'(r)(\varphi(r_{j+2})/\varphi(r_j))^{m+1},\end{aligned}$$

which is implies (2.7).

Proof of (b) $\Rightarrow$ (c). Let $u = \operatorname{Re} f \in h_p(\varphi)$. Then $M_p(u, r_j)C\varphi(r_j) = C2^j$ and hence, by (2.4) and (2.6),

$$M_p(f', r_j) \leqslant C(r_{j+1} - r_j)^{-1}\varphi(r_{j+1}) = 2C\varphi'(t_j)$$

for some constant C. If $r \in (0,1)$ is arbitrary, we choose j such that $r_j \leqslant r \leqslant r_{j+1}$. Then

$$M_p(f, r) \leqslant M_p(f, r_{j+1}) \leqslant 2C\varphi'(r_{j+1}).$$

Now (c) follows from (2.7).

Proof of (c) $\Rightarrow$ (a). Let $f' \in h_p(\varphi')$. By (2.5), (2.7) and (2.6) we have that

$$\begin{aligned}M_p^p(f, r_{j+1}) - M_p^p(f, r_j) &\leqslant C(r_{j+1} - r_j)^p M_p^p(f', r_{j+1})\\ &\leqslant C(r_{j+1} - r_j)^p \varphi'(t_j)^p = C2^p.\end{aligned}$$

Now summation yields

$$M_p^p(f, r_{k+1}) - |f(0)|^p \leqslant C2^{kp} = C\varphi(r_k)^p,$$

which implies (a). This completes the proof of Theorem 2.1. $\square$

3. Examples of majorants

In this section we briefly discuss some classes of majorants for which the corresponding h_p-spaces are self-conjugate.

(i) (U) + (L) implies (L^+) (provided φ is C^1 near 1).

Indeed, (U) + (L) is equivalent to

$$\varphi'(r) \asymp (1-r)^{-1}\varphi(r), \quad r \to 1^-. \tag{3.1}$$

Since (L) implies that $(1-r)^{-1}\varphi(r)^{-m} \downarrow 0$ for some $m > 0$, we see that $\varphi'(r)/\varphi(r)^{m+1}$ is almost decreasing near 1.

Remark. We write $A(r) \asymp B(r)$, $r \to 1$, to denote that $A(r)/B(r)$ and $B(r)/A(r)$ remain bounded when r tends to 1.

It is clear that (U) implies

$$\varphi(r) = 0(\varphi(r^2)), \quad r \to 1^-. \tag{3.2}$$

On the other hand, (L^+) + (3.2) implies (L) + (U). Indeed, as remarked before Theorem 2.1, (L^+) implies (L). Then we apply (2.1) to $F = \varphi^{-m}$ to get

$$\frac{\varphi(r)^{-m} - \varphi(r^2)^{-m}}{r - r^2} \leqslant -Cm\varphi(r)^{-m-1}\varphi'(r).$$

Using this and (3.2) we find that $\varphi'(r) \leqslant \gamma(1-r)^{-1}\varphi(r)$, $\gamma = \text{const.}$, which implies (U).

(ii) It is known [4, 6, 8, 9] that $h_p(\psi)$ is self-conjugate provided

$$\begin{aligned} &(1-r)^{\alpha}\psi(r) \quad \text{is almost increasing and } (1-r)^{\beta}\psi(r) \\ &\text{is almost decreasing near 1} \quad \text{for some } \alpha > 0,\ \beta > 0. \end{aligned} \tag{N}$$

This can be deduced from Theorem 2.1 by using the fact that (N) implies the existence of a majorant φ satisfying (3.1) (= (L) + (U)) and such that $\varphi(r) \asymp \psi(r)$, $r \to 1^-$.

To see the latter assume that ψ is defined and positive on $[0, 1)$ and let

$$\varphi(r) = \int_0^r (1-t)^{-1}\psi(t)\,\mathrm{d}t. \tag{3.2$'$}$$

Using (N) one shows that $\varphi \asymp \psi$ and since $\varphi'(r) = (1-r)^{-1}\psi(r)$ the result follows.

(iii) For a majorant ψ satisfying (L^+) let us choose $m > 0$ such that ψ'/ψ^{m+1} is almost decreasing near 1 and let

$$\eta(r) = \sup_{r<t<1} \psi'(t)/\psi(t)^{m+1}.$$

Then define φ by

$$\varphi(r)^{-m} = \int_r^1 \eta(t)\,\mathrm{d}t.$$

It is easily seen that $\varphi(r) \asymp \psi(r)$, $r \to 1^-$, and that φ^{-m} is convex near 1. By calculating the second derivative of φ^{-m} one concludes that the convexity of φ^{-m} for some $m > 0$, where φ is C^2, is equivalent to

$$\limsup_{r\to 1} \frac{\varphi''(r)\varphi(r)}{\varphi'(r)^2} < \infty. \tag{3.3}$$

Thus (3.3) ensures the conclusion of Theorem 2.1.

A slightly stronger condition

$$\limsup_{r\to 1} \frac{|\varphi''(r)|\varphi(r)}{\varphi'(r)^2} < \infty \tag{3.4}$$

means that there is a constant $m > 0$ such that both φ^m and φ^{-m} are convex near 1. If ψ satisfies (U) + (L) and φ is defined by $(3.2)'$, then φ satisfies (3.4). A consequence of this and (ii) is that every majorant satisfying (N) is "proportional" to one satisfying (3.4).

(iv) There is a large class of majorants, including (1.1), for which (3.4) holds. Sometimes it is convenient to represent φ as

$$\varphi(r) = F\Big(\frac{1}{1-r}\Big),$$

where F is a positive, continuous function defined on some $[A, \infty)$, $A > 0$, and such that $F(\infty) = \infty$. In [8], such an F is called a *weight.* With this notation we have

Proposition 3.1. *Let F be a weight such that F is C^2 and $F' > 0$. Then condition* (3.4) *is equivalent to*

$$\limsup_{x\to\infty} \frac{|F''(x)|F(x)}{F'(x)^2} < \infty. \tag{3.5}$$

P r o o f. The validity of the implication (3.4) $\Rightarrow$ (3.5) follows from the formula

$$\frac{F''(x)F(x)}{F'(x)^2} = \frac{\varphi''(r)\varphi(r)}{\varphi'(r)^2} - \frac{2\varphi(r)}{(1-r)\varphi'(r)}, \quad x = (1-r)^{-1}$$

and the facts that (3.4) implies (L^+) and (L^+) implies (2.2). In the opposite direction we use the formula

$$\frac{\varphi''(r)\varphi(r)}{\varphi'(r)^2} = \frac{F''(x)F(x)}{F'(x)^2} + \frac{2F(x)}{xF'(x)}$$

and the fact (3.5) means that there is a constant $m > 0$ such that F^m and F^{-m} are convex near ∞. In particular, if F satisfies (3.5), then there is a $c > 0$ such that

$$\frac{F(x)^m - F(c)^m}{x - c} \leqslant mF'(x)F(x)^{m-1}, \quad x > c.$$

Hence $\limsup_{x\to\infty} F(x)/xF'(x) \leqslant m$, which concludes the proof.

(v) A remarkable result of Hardy (cf. [1], Ch. V) makes the verification of (3.5) for a large class of weights almost trivial. Let $h(x)$ be an expression composed from $\{\mathrm{e}^x, \log x,$ constants$\}$ by successive applications of arithmetic operations and substitutions. We write $h \in (H)$ if $h(x)$ is defined in a neighbourhood of ∞. The result of Hardy states that sign $h(x)$, for $h \in (H)$, is constant near ∞. And since $h' \in (H)$ whenever $h \in (H)$ it follows that the limit $\lim_{x\to\infty} h(x)$ exists. Then it is easily shown that if a weight F belongs to (H), then the limit

$$\lim_{x\to\infty} \frac{F''(x)F(x)}{F'(x)^2} =: L(F)$$

exists (finite or not). Then by the L'Hospital rule

$$0 \leqslant \lim_{x\to\infty} \frac{F(x)/F'(x)}{x} = 1 - L(F).$$

This shows that when $\varphi(r) = F(1/(1-r))$, $F \in (H)$, conditions (L), (L^+), (3.3), (3.4) and (3.5) are equivalent. Moreover, each of them is implied by the existence of an $\alpha > 0$ such that

$$\lim_{x\to\infty} F(x)/x^\alpha = \infty.$$

A concrete example is

$$F(x) = x^a(\log x)^b \exp(cx^d + k(\log x)^m),$$

where $c > 0$, $d > 0$ or $c = 0$, $k > 0$, $m > 1$.

4. A problem

The "norm" in $h_p(\varphi)$ can be defined as follows. Choose $r_0 < 1$ such that $\varphi > 0$ on $[r_0, 1)$ and let

$$\|u\| = \sup_{r_0<r<1} M_p(u,r)/\varphi(r).$$

Then, using Lemma 2.1, one shows that the norm convergence in $h_p(\varphi)$ implies the uniform convergence on compact subsets of Δ. A consequence is that $h_p(\varphi)$ is norm complete. The space $H_p(\varphi)$ spanned by analytic functions is a closed subspace of $h_p(\varphi)$.

Problem. If φ satisfies (L^+) and $\limsup_{r\to 1} \varphi(r)/\varphi(r^2) = \infty$, is the space $h_p(\varphi)$ isomorphic to $H_p(\varphi)$?

This simplest case is that where $\varphi(r) = \exp(1/(1-r))$.

It is not hard to prove that if $h_p(\varphi)$ is self-conjugate, then the space $h_p(\varphi(r))$ is isomorphic to $H_p(\varphi(r^2))$ via the operator T defined by

$$(Tu)(z) = f(z^2) + zg(z^2),$$

where f, g are the unique analytic functions such that $u(z) = f(z) + g(\bar{z})$, $g(0) = 0$. $\square$

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