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SOME RESULTS ON SETS OF POSITIVE MEASURE IN A METRIC SPACE

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Introduction

H. Steinhaus [7] proved that if A is a measurable subset of the real line R with positive Lebesgue measure, then the distance set of A, i.e. $D(A) = \{|x-y|\colon\ x, y \in A\}$ contains an interval of the form (o, h) for a certain value of h. Many papers are devoted to the study of the set $D(A)$ for various A.

If (E, ϱ) is a metric space with a linear measure μ defined on E, then the following property is referred to as the Steinhaus property for distance sets:

If A is a measurable subset of E with a positive measure, then $D(A) = \{\varrho(x, y)\colon x, y \in A\}$ contains an interval with the origin as its end point.

A simple curve $C \subset E$ is the image of a continuous injective mapping $f\colon\ [0,1] \to E$ and if $\mu(C)$ is finite, then C is called a simple rectifiable curve. Also, since $[0,1]$ is compact and (C, ϱ) is Hausdorff, the surjective restriction of $f\colon\ [0,1] \to C$ is a homeomorphism.

Besicovitch and Taylor [1] showed that the Steinhaus property does not hold, in general, for all simple rectifiable curves. E. Boardman [2] proved that under certain conditions on the metric space (E, ϱ) all simple rectifiable curves in E have the Steinhaus property for distance sets.

M.S. Ruziewicz [6] proved the following theorem:

Theorem. *Let $A \subset R$ be a set of positive Lebesgue measure. For any set of m positive numbers $k_1, k_2, \ldots, k_m$ there exist a positive number d and $(m+1)$ points $x_1 < x_2 < \ldots < x_{m+1}$ of the set A such that $x_{i+1} - x_i = k_i d$ $(i = 1, 2, \ldots, m)$.*

In this note we prove a result similar to that of Ruziewicz for a subset with positive linear measure of a simple rectifiable curve in a metric space and also some other results related to sets of positive measure in a metric space.

Let (E,ϱ) be a metric space. For a set $A\subset E$, let

$$\Lambda^*(A)=\sup_{\delta>0}\left[\inf\left\{\sum_{i=1}^{\infty}d(A_i)\colon\ A_i\subset E, d(A_i)<\delta\quad\text{and}\quad\bigcup_{i=1}^{\infty}A_i\supset A\right\}\right],$$

where $d(A_i)$ stands for the diameter of A_i. Then Λ^* is a metric outer measure and the restriction of Λ^* to the measurable sets is known as the linear measure Λ. With respect to the outer measure all Borel sets are measurable.

The mapping f induces a linear ordering on the curve C such that for x, $y\in C$, $x<y$ if and only if $f^{-1}(x)<f^{-1}(y)$.

If a, $b\in C$ and $a<b$, the subarc $\langle a,b\rangle$ of C is defined by $\langle a,b\rangle=\{c\in C\colon a\leqslant c\leqslant b\}$. For a subarc $\langle a,b\rangle$ of C, we see that $\Lambda(\langle a,b\rangle)=l(\langle a,b\rangle)$ where $l(\langle a,b\rangle)$ denotes the length of the arc $\langle a,b\rangle$ defined by $l(\langle a,b\rangle)=\sup\Big\{\sum\limits_{r=1}^{n}\varrho(x_{r-1},x_r)\Big\}$ where the supremum is taken over all finite subdivisions $\{x_0,x_1,\ldots,x_n\}$ of $\langle a,b\rangle$ with $a=x_0<x_1<\ldots<x_n=b$.

It may be verified that Λ is continuous in the sense that if $b\in C$, $a_n\in C$, $a_n<a_{n+1}<b$ and $\lim\varrho(a_n,b)=0$, then

$$\lim_{n\to\infty}\Lambda(\langle a_n,b\rangle)=0. \tag{1}$$

Now we present some definitions which may be readily seen in Boardman [2] and Lahiri [3].

Definition 1. Let $B\subset C$ and $r>0$. Then $B(r)=\{z\in C\colon\exists u\in B$ such that $u<z$ and $\varrho(u,z)=r\}$ and $B(-r)=\{z\in C\colon\exists u\in B$ such that $z<u$ and $\varrho(u,z)=r\}$.

Definition 2. A simple rectifiable curve C is said to satisfy the condition (A), if there exist real numbers $c>0$ and $d_0>0$ such that for each subset $B\subset C$, $0<r<d_0$ implies $d(B)\geqslant c\big[d\big(B(-r)\big)\big]$.

Definition 3. Let G be a family of all linearly measurable subsets of C and let $A_r\in G$, $r=1,2,\ldots$. If there exists a set $A\in G$ such that $\Lambda[A_r\Delta A]\to 0$ as $r\to\infty$, then the sequence of sets $\{A_r\}$ is said to converge to the set A in G where the symbol Δ stands for the symmetric difference (Lahiri, 1981).

Theorem 1.1. *Let C be a simple rectifiable curve in a metric space (E,ϱ) satisfying the condition* (A). *If S is a linearly measurable subset of C with $\Lambda(S)>0$ and $\alpha_1,\alpha_2,\ldots,\alpha_m$ is any set of m positive real numbers then there exists an open* *interval* $(0, \delta)$ *such that to any* $x \in (0, \delta)$ *there correspond* $(m+1)$ *points* $a_0(x) < a_1(x) < \ldots < a_m(x)$ *of the set* S *for which* $\varrho(a_{i-1}(x), a_i(x)) = \alpha_i x$ $(i = 1, 2, \ldots, m)$.

Proof. Let C be determined by $f\colon [0,1] \to C \subset E$ and let $\Lambda(C) < \infty$. Then by (1) we assume that dist $(f(1), S) > 0$. Since S is measurable and C is rectifiable there exists a compact set K and an open set G such that $K \subset S \subset G \subset C$, $f(1) \in C \setminus G$ and $\Lambda(K) > \frac{1}{c}\Lambda(G \setminus K)$, which may be deduced from Theorem 13.5 of Munroe [4] as indicated by Boardman [2]. Let δ' be such that $\Lambda(K) - \delta' > \frac{1}{c}\Lambda(G \setminus K)$. By the definition of $\Lambda(K)$ there exists $\varepsilon_0 > 0$ such that for all ε' with $0 < \varepsilon' < \varepsilon_0$, all covers $\{A_i\}_{i=1}^{\infty}$ of K with $d(A_i) < \varepsilon'$ have the property

$$\sum_{i=1}^{\infty} d(A_i) \geqslant \Lambda(K) - \delta' > \frac{1}{c}\Lambda(G \setminus K). \tag{2}$$

Since the curve C satisfies condition (A), hence by Definition 2 there are positive real numbers c, d_0 such that for each subset $B \subset C$, $0 < r < d_0 \Rightarrow d(B) \geqslant c[d(B(-r))]$. Let $\alpha = \max(\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $d_1 = \operatorname{dist}(K, C \setminus G)$ so that $d_1 > 0$. Also let $\eta_1 = \min\{d_0, d_1, \varepsilon_0/3\}$. If we put $\delta_1 = \eta_{1/\alpha}$, then for any $x \in (0, \delta_1)$ we have $0 < \alpha x < \eta_1$. Then we have $0 < h_1 < \eta_1 \leqslant d_1$ where $h_1 = \alpha_1 x$. So,

$$K(h_1) \subset G. \tag{3}$$

First we show that $S \cap S(h_1) \neq \emptyset$.

Let ε be any number satisfying $0 < \varepsilon < \varepsilon_{0/3}$ and let $B'_i \subset C$ $(i = 1, 2, \ldots)$ be such that $d(B'_i) < \varepsilon$ and

$$K(h_1) \subset \bigcup_{i=1}^{\infty} B'_i. \tag{4}$$

As $f(1) \in C \setminus G$ and $h_1 < d_1$, so if $u_1 \in K$, then $\varrho(f(1), u_1) > h_1$. Also, since $\langle u_1, f(1)\rangle$ is connected there exists $z_1 \in C$ such that $u_1 < z_1$ and $\varrho(u_1, z_1) = h_1$. This implies that $u_1 \in B'_m(-h_1)$ for some m. Hence

$$K \subset \bigcup_{i=1}^{\infty} B'_i(-h_1). \tag{5}$$

If $s(x)$, $t(x) \in B'_i(-h_1)$, then there exist p, $q \in B'_i$ such that $\varrho(p, s(x)) = h_1$ and $\varrho(q, t(x)) = h_1$. Therefore $\varrho(s(x), t(x)) \leqslant 2h_1 + d(B'_i) < \frac{2\varepsilon_0}{3} + \frac{\varepsilon_0}{3} = \varepsilon_0$. Thus $d(B'_i(-h_1)) < \varepsilon_0$ and by (2),

$$\sum_{i=1}^{\infty} d(B'_i(-h_1)) \geqslant \Lambda(K) - \delta' > \frac{1}{c}\Lambda(G \setminus K).$$

Since $h_1 < d_0$, we have by the property (A)

$$\sum_{i=1}^{\infty} d(B_i') \geqslant c \sum_{i=1}^{\infty} d\big(B_i'(-h_1)\big) \geqslant c\big(\Lambda(K) - \delta'\big) > \Lambda(G \setminus K).$$

It follows from (4) that

$$\Lambda\big(K(h_1)\big) \geqslant c\big(\Lambda(K) - \delta'\big) > \Lambda(G \setminus K). \tag{6}$$

If $X = K \cap K(h_1)$, then $X = G \setminus \big[(G \setminus K) \cup \big(G \setminus K(h_1)\big)\big]$. So

$$\begin{aligned}\Lambda(X) &\geqslant \Lambda(G) - \Big[\Lambda(G \setminus K) + \Lambda\big(G \setminus K(h_1)\big)\Big] \\ &> \Lambda(G) - \Big[\Lambda\big(K(h_1)\big) + \Lambda(G) - \Lambda\big(K(h_1)\big)\Big] = 0.\end{aligned}$$

Therefore $\Lambda[K \cap K(h_1)] > 0$. Since $K \subset S$, $K(h_1) \subset S(h_1)$ for $h_1 \in (0, \eta_1)$, hence $\Lambda\big(S \cap S(h_1)\big) > 0$ and so $S \cap S(h_1)$ is non-empty.

Let $S \cap S(h_1) = S_1$. Since S_1 is a set of positive measure and $S_1 \subset S$, we can show in a similar manner that there exist an η_2 and a $\delta_2 = \eta_{2/\alpha}$ such that for any $x \in (0, \delta_2)$ we can select a positive number $h_2 (= \alpha_2 x) \in (0, \eta_2)$ with the property that $\Lambda[S_1 \cap S_1(h_2)] > 0$, i.e. $S_1 \cap S_1(h_2) \neq \emptyset$.

Proceeding in this way, after a finite number of steps we obtain an $\eta_m > 0$ and a set $S_{m-1} \subset S$ such that for any $x \in (0, \delta_m)(\delta_m = \eta_{m/\alpha})$ we can select a positive number $h_m (= \alpha_m x) \in (0, \eta_m)$ such that $\Lambda[S_{m-1} \cap S_{m-1}(h_m)] > 0$, i.e. $S_{m-1} \cap S_{m-1}(h_m) \neq \emptyset$. Therefore, in general, $S_{i-1} \cap S_{i-1}(h_i) \neq \emptyset$ $(i = 1, 2, \ldots, m)$ where $S_0 = S$.

Let $\delta = \min(\delta_1, \delta_2, \ldots, \delta_m)$. Then for every $x \in (0, \delta)$ there exists $a_i(x) \in S_{i-1} \cap S_{i-1}(h_i)$. Then there exists $a_{i-1}(x) \in S_{i-1}$ such that $a_{i-1}(x) < a_i(x)$ for which $\varrho\big(a_i(x), a_{i-1}(x)\big) = h_i = \alpha_i x$ $(i = 1, 2, \ldots, m)$.

Thus for every $x \in (0, \delta)$ there exist $m + 1$ points $a_0(x) < a_1(x) < \ldots < a_m(x)$ of the set S such that $\varrho\big(a_{i-1}(x), a_i(x)\big) = \alpha_i x$ $(i = 1, 2, \ldots, m)$. $\square$

Corollary. *Let S be a measurable subset of positive measure of a simple rectifiable curve satisfying the property* (A) *in a metric space (E, ϱ) and let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be any system of m positive real numbers. Then there exists an open interval $(0, \delta)$ such that for any $x \in (0, \delta)$ there are points $a_0(x)$, $a_1(x), \ldots, a_m(x)$ in S with the property*

$$x = \frac{\varrho\big(a_1(x), a_0(x)\big)}{\alpha_1} = \frac{\varrho\big(a_2(x), a_0(x)\big)}{\alpha_2} = \ldots = \frac{\varrho\big(a_m(x), a_0(x)\big)}{\alpha_m}.$$

P r o o f. We have established that

$$S_1 \cap S_1(h_2) = S \cap S(h_1) \cap S_1(h_2) \subset S \cap S(h_1) \cap S(h_2).$$

Proceeding in this way, after a finite number of steps we obtain a positive number $h_m = (\alpha_m x)$ and a set $S_{m-1}(\subset S)$ such that $\Lambda\big(S_{m-1} \cap S_{m-1}(h_m)\big) > 0$ and also

$$\begin{aligned} S_{m-1} \cap S_{m-1}(h_m) &= S \cap S(h_1) \cap S_1(h_2) \cap \ldots \cap S_{m-1}(h_m) \\ &\subset S \cap S(h_1) \cap S(h_2) \cap \ldots \cap S(h_m). \end{aligned}$$

Hence the set $S \cap S(h_1) \cap S(h_2) \cap \ldots \cap S(h_m)$ is of positive measure. Thus for every $x \in (0, \delta)$ there exist $a_0(x) \in S$ and $a_i(x) \in S$ $(i = 1, 2, \ldots, m)$ such that $\varrho\big(a_i(x), a_0(x)\big) = h_i = \alpha_i x$, i.e.

$$x = \frac{\varrho\big(a_1(x), a_0(x)\big)}{\alpha_1} = \frac{\varrho\big(a_2(x), a_0(x)\big)}{\alpha_2} = \ldots = \frac{\varrho\big(a_m(x), a_0(x)\big)}{\alpha_m}.$$

□

Theorem 1.2. *Let C be a simple rectifiable curve in a metric space (E, ϱ) satisfying the condition* (A) *and let S be a linearly measurable subset of C with $\Lambda(S) > 0$. If $\{r_n\}$ is a sequence of positive real numbers converging to zero, then the set of points belonging to S for which for infinitely many n there exists $u_n \in S$ such that $\varrho(x, u_n) = r_n$, is a set of positive measure.*

P r o o f. When proving Theorem 1.1 we have shown that there exists $\eta(> 0)$ such that for any $r(0 < r < \eta)$, $\Lambda\big(S \cap S(r)\big) > 0$. Let $C_n = S \cap S(r_n)$. Since $r_n \to 0$, there exists a positive integer N_0 such that $0 < r_n < \eta$ for $n \geqslant N_0$ and hence $\Lambda(C_n) > 0$ whenever $n \geqslant N_0$. Let B be the set of all those points x in S for which there exist infinitely many $u_n \in S$ such that $\varrho(x, u_n) = r_n$. Then $B = \bigcap\limits_{N=1}^{\infty} \bigcup\limits_{n=N}^{\infty} C_n = \bigcap\limits_{N=1}^{\infty} D_N$ where $D_N = \bigcup\limits_{n=N}^{\infty} C_n$. Here $\Lambda(D_N) \geqslant \Lambda(C_N) > 0$ for $N \geqslant N_0$, while each set of the decreasing sequence $\{D_N\}$ is a subset of the set S. As the Hausdorff measure is regular, $\Lambda(B) = \lim\limits_{N \to \infty} \Lambda(D_N)$ [5]. It follows that $\Lambda(B) > 0$. □

Theorem 1.3. *Let C be a simple rectifiable curve in a metric space (E, ϱ) satisfying the condition* (A) *for $c > 1$. Suppose K is a compact subset of C with $\Lambda(K) > 0$. If $\{r_n\}$ is a sequence of positive real numbers converging to zero, then $K(r_n) \to K$ in G.*

In order to prove the theorem we require a lemma.

Lemma. *If $A_r \to A$ as $r \to \infty$, where A_r, $A \in G$, then $\Lambda(A_r) \to \Lambda(A)$ in G.*

The p r o o f of the lemma is easy and omitted. □

P r o o f o f t h e t h e o r e m. Since K is a compact subset of C, then given an ε (> 0), it is possible to find an open set G containing K such that $\Lambda(G \setminus K) < \frac{\varepsilon}{3}$. Let $d = \operatorname{dist}(K, C \setminus G)$. Then $d > 0$. Since $r_n \to 0$, it is possible to find a positive integer N_1 such that $0 < r_n < d$ for $n \geqslant N_1$. Consequently, $K(r_n) \subset G$ for $n \geqslant N_1$. Let $X_n = K \cap K(r_n)$. Then $X_n = G \setminus \bigl[(G \setminus K) \cup \bigl(G \setminus K(r_n)\bigr)\bigr]$. So, for $n \geqslant N_1$, $\Lambda(X_n) \geqslant \Lambda(G) - \bigl[\Lambda(G \setminus K) + \Lambda\bigl(G \setminus K(r_n)\bigr)\bigr]$.

Let $\eta = \min\bigl\{d_0, d, \frac{\varepsilon}{3}\bigr\}$. Then we can find positive integer N_2 such that $0 < r_n < \eta$ for $n \geqslant N_2$. Then, proceeding in the same manner as in Theorem 1.1, we have $\Lambda\bigl(K(r_n)\bigr) > \Lambda(K) - \frac{\varepsilon}{3}$ for $n \geqslant N_2$. Let $N = \max(N_1, N_2)$. Then $\Lambda(X_n) > \Lambda(K) - 2\frac{\varepsilon}{3} > \Lambda(G) - \varepsilon$ for $n \geqslant N$. Consequently, $\Lambda\bigl(K(r_n) \Delta K\bigr) < \varepsilon$ for $n \geqslant N$. Hence $K(r_n) \to K$ in G. $\square$

Corollary. *For any measurable set B in C satisfying the condition* (A) *with $c > 1$, $K(r_n) \cap B \to K \cap B$ in G.*

P r o o f. We have

$$\Lambda\Bigl[\bigl(K(r) \cap B\bigr)\Delta(K \cap B)\Bigr] \leqslant \Lambda\bigl[K(r) \setminus K\bigr] + \Lambda\bigl[K \setminus K(r)\bigr] = \Lambda\bigl[K(r)\Delta K\bigr] \to 0 \text{ as } r \to \infty.$$

$\square$

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