Czechoslovak Mathematical Journal, 49 (124) (1999), 689–700

DISCRETE SPECTRUM AND PRINCIPAL FUNCTIONS OF NON-SELFADJOINT DIFFERENTIAL OPERATOR

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(Received April 23, 1996)

Abstract. In this article, we consider the operator L defined by the differential expression

$$\ell(y) = -y'' + q(x)y, \quad -\infty < x < \infty$$

in $L_2(-\infty,\infty)$, where q is a complex valued function. Discussing the spectrum, we prove that L has a finite number of eigenvalues and spectral singularities, if the condition

$$\sup_{-\infty<x<\infty} \Big\{ \exp\big(\varepsilon\sqrt{|x|}\big)|q(x)| \Big\} < \infty, \quad \varepsilon > 0$$

holds. Later we investigate the properties of the principal functions corresponding to the eigenvalues and the spectral singularities.

1. Introduction

Let us consider the non-selfadjoint one dimensional Schrödinger operator L_0 defined by the differential expression

$$\ell_0(y) = -y'' + q(x)y, \quad 0 \leqslant x < \infty$$

and the boundary condition $y(0) = 0$ in $L_2(0,\infty)$, where q is a complex valued function. The spectral analysis of L_0 was started by Naimark [10] in 1960. In his article he proved that some of the poles of the resolvent's kernel of L are not eigenvalues of the operator. Also he showed that those poles (which are called spectral singularities by Schwartz [15]) are on the continuous spectrum. Moreover he showed that the spectral singularities play an important role in the discussion of the spectral analysis of L_0, and if the condition

$$\int_0^\infty |q(x)| \exp(\varepsilon x)\,\mathrm{d}x < \infty, \quad \varepsilon > 0$$

holds then the eigenvalues and the spectral singularities are of finite number and each of them is of finite multiplicity. The effect of spectral singularities in the spectral expansion of the operator L_0, in terms of the principal functions, was investigated in [7]. Some problems related to spectral analysis of differential and some other types of operators with spectral singularities have been discussed by several authors [1], [4], [5], [8], [11]–[13].

Now let us consider an operator L defined by the differential expression

$$\ell(y) = -y'' + q(x)y, \quad -\infty < x < \infty$$

in $L_2(-\infty,\infty)$, where q is a complex valued function. L is called the one dimensional Schrödinger operator on the whole real axis. Since q is a complex valued function, the operator L is non-selfadjoint.

The above result of Naimark has been generalized to the operator L by Blashak [3]. Blashak has proved that the operator L has a finite number of eigenvalues and spectral singularities, if

$$\int_{-\infty}^{\infty} |q(x)| \exp(\varepsilon|x|)\,\mathrm{d}x < \infty, \quad \varepsilon > 0 \tag{1}$$

holds.

In the present article, we discuss the discrete spectrum of L and prove that this operator has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity, under the condition

$$\sup_{-\infty<x<\infty} \bigl\{\exp(\varepsilon\sqrt{|x|})|q(x)|\bigr\} < \infty, \quad \varepsilon > 0. \tag{2}$$

Afterwards, the properties of the principal functions corresponding to the eigenvalues and the spectral singularities of L are obtained.

Obviously the condition (2) is weaker than the condition (1). Under the condition (1) the finiteness of the eigenvalues and the spectral singularities of L is obtained by finite meromorphic continuation of the resolvent's kernel from the continuous spectrum [3]. But under the condition (2) there is no such finite meromorphic continuation. Hence the method of [3] can't be used in this case.

In the following, we use the notation

$$\mathbb{C}_+ = \{\lambda\colon\ \lambda \in \mathbb{C}, \operatorname{Im}\lambda > 0\}, \quad \overline{\mathbb{C}}_+ = \{\lambda\colon\ \lambda \in \mathbb{C}, \operatorname{Im}\lambda \geqslant 0\}.$$

Also, $\sigma_d(L)$ and $\sigma_{s,s}(L)$ will denote the eigenvalues and the spectral singularities of L, respectively.

Let us consider the following differential equation:

$$-y'' + q(x)y = \lambda^2 y, \quad x \in (-\infty, \infty) \tag{3}$$

where λ is a complex parameter. For the moment, we will assume that

$$\int_{-\infty}^{\infty} (1+|x|)|q(x)|\,\mathrm{d}x < \infty \tag{4}$$

holds and we introduce the notation

$$\begin{aligned}\sigma^+(x) &= \int_x^{\infty} |q(t)|\,\mathrm{d}t, \quad \sigma_1^+(x) = \int_x^{\infty} \sigma^+(t)\,\mathrm{d}t,\\ \sigma^-(x) &= \int_{-\infty}^{x} |q(t)|\,\mathrm{d}t, \quad \sigma_1^-(x) = \int_{-\infty}^{x} \sigma^-(t)\,\mathrm{d}t.\end{aligned}$$

Under the condition (4), the equation (3) has solutions [9]

$$e^+(x,\lambda) = \mathrm{e}^{\mathrm{i}\lambda x} + \int_x^{\infty} K^+(x,t)\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t \tag{5}$$

and

$$e^-(x,\lambda) = \mathrm{e}^{-\mathrm{i}\lambda x} + \int_{-\infty}^{x} K^-(x,t)\mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{d}t \tag{6}$$

for all $\lambda \in \overline{\mathbb{C}}_+$, where $K^\pm(x,t)$ are differentiable with respect to x and t, and satisfy the inequalities

$$\left|K^\pm(x,t)\right| \leqslant \frac{1}{2}\sigma^\pm\Big(\frac{x+t}{2}\Big)\exp\Big\{\sigma_1^\pm(x) - \sigma_1^\pm\Big(\frac{x+t}{2}\Big)\Big\}, \tag{7}$$

$$\Big|\frac{\partial}{\partial x_i}K^\pm(x_1,x_2) \mp \frac{1}{4}q\Big(\frac{x_1+x_2}{2}\Big)\Big| \leqslant \frac{1}{2}\sigma_1^\pm(x_i)\sigma^\pm\Big(\frac{x_1+x_2}{2}\Big)\exp\sigma_1^\pm(x_i). \tag{8}$$

Therefore the solutions $e^+(x,\lambda)$, $e^-(x,\lambda)$ are analytic in $\mathbb{C}_+$ with respect to λ and continuous on the real axis.

Now let us introduce

$$\alpha(\lambda) := W\{e^+(x,\lambda), e^-(x,\lambda)\},$$

where $W\{e^+(x,\lambda), e^-(x,\lambda)\}$ is the Wronskian of the solutions $e^+(x,\lambda)$ and $e^-(x,\lambda)$.

It is obvious that the function α is analytic in $\mathbb{C}_+$ and continuous on the real axis. So the following equalities are satisfied [3], [9]:

$$\alpha(\lambda) = -2\mathrm{i}\lambda + O(1), \quad \lambda \in \overline{\mathbb{C}}_+, \quad |\lambda| \to \infty, \tag{9}$$

$$\sigma_d(L) = \bigl\{\mu\colon\ \mu = \lambda^2, \lambda \in \mathbb{C}_+, \alpha(\lambda) = 0\bigr\}, \tag{10}$$

$$\sigma_{s,s}(L) = \bigl\{\mu\colon\ \mu = \lambda^2, \lambda \in (-\infty,\infty), \alpha(\lambda) = 0\bigr\}. \tag{11}$$

Definition 2.1. ([7]) The multiplicity of a zero of α in $\overline{\mathbb{C}}_+$ is called the multiplicity of the corresponding eigenvalue or spectral singularity of L.

We need the following uniqueness theorem obtained from [12]:

Theorem 2.2. ([12]) *Let us assume that the function g is analytic in $\mathbb{C}_+$, all of its derivatives are continuous on the real axis and there exists $N > 0$ such that*

$$|g^{(m)}(\lambda)| \leqslant c_m, \quad m = 0,1,2,\dots\ \lambda \in \overline{\mathbb{C}}_+, \quad |\lambda| < 2N, \tag{12}$$

and

$$\Bigl|\int_{-\infty}^{-N} \frac{\ln|g(x)|}{1+x^2}\,\mathrm{d}x\Bigr| < \infty, \quad \Bigl|\int_{N}^{\infty} \frac{\ln|g(x)|}{1+x^2}\,\mathrm{d}x\Bigr| < \infty \tag{13}$$

hold. If the set G with Lebesgue measure zero is the set of all zeros of the function g with infinite multiplicity and if

$$\int_0^h \ln F(s)\,\mathrm{d}\mu(G_s) = -\infty \tag{14}$$

holds then $g(\lambda) \equiv 0$, where $F(s) = \inf\limits_m \frac{c_m s^m}{m!}$, $m = 0,1,2,\dots,\mu(G_s)$ is the Lebesgue measure of the s–neighborhood of G and h is an arbitrary positive constant.

3. Eigenvalues and spectral singularities

It is clear from (10) and (11) that, in order to investigate the quantitative properties of the eigenvalues and the spectral singularities of L, we need to discuss the quantitative properties of the zeros of α in $\overline{\mathbb{C}}_+$.

Let P_1 denote the zeros of the function α in $\mathbb{C}_+$ and P_2 the zeros of the function α on the real axis.

Lemma 3.1. *Under the condition* (4)

a) *The set P_1 is bounded, has at most a countable number of elements and its limit points can lie in a bounded subinterval of the real axis.*

b) *The set P_2 is compact and its Lebesgue measure is zero.*

P r o o f. The boundedness of P_1 and P_2 is obtained from (9). Since α is analytic in $\mathbb{C}_+$, then the set P_1 has at most a countable number of elements. From the uniqueness of analytic functions and (9), it is deduced that the limit points of P_1 can lie only in a bounded subinterval of the real axis. The closedness and the property of having Lebesgue measure zero of the set P_2 can be obtained from the uniqueness theorem of the analytic functions due to Privalov [14]. $\square$

From (10), (11) and Lemma 3.1 we have

Remark 3.2. The sets of eigenvalues and spectral singularities of L are bounded, at most countable and their limit points can lie only in a bounded subinterval of the positive real axis if the condition (4) holds.

Lemma 3.3. *The function α satisfies*

$$\alpha(\lambda) = -2\mathrm{i}\lambda + \int_{-\infty}^{\infty} q(t)\,\mathrm{d}t + \int_0^{\infty} A(t)\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t, \tag{15}$$

where

$$\begin{aligned} A(t) = {} & K_t^+(0,t) + K^-(0,0)K^+(0,t) + K_x^-(0,-t) - K_x^+(0,t) \\ & - K_t^-(0,-t) + K^+(0,0)K^-(0,-t) + (K^+ * K_x^-)(t) - (K_x^+ * K^-)(t) \end{aligned} \tag{16}$$

in which $(*)$ is the convolution operation.

P r o o f. By the definition of the Wronskian of the solutions $e^+(x,\lambda)$, $e^-(x,\lambda)$ we have

$$\alpha(\lambda) = e^+(0,\lambda)e_x^-(0,\lambda) - e_x^+(0,\lambda)e^-(0,\lambda). \tag{17}$$

Substituting the values of $e^+(0,\lambda)$, $e_x^+(0,\lambda)$, $e^-(0,\lambda)$ and $e_x^-(0,\lambda)$ into (17) we obtain (15). $\square$

Now let us assume that

$$\sup_{-\infty<x<\infty} \Big\{ \exp\big(\varepsilon\sqrt{|x|}\big)|q(x)| \Big\} < \infty, \quad \varepsilon > 0 \tag{18}$$

holds. From (7) and (8) we find

$$|K^{\pm}(x,t)|, |K_x^{\pm}(x,t)|, |K_t^{\pm}(x,t)| \leqslant c\exp\Big\{-\frac{\varepsilon}{2}\sqrt{\frac{|x+t|}{2}}\Big\}, \tag{19}$$

where $c > 0$ is a constant, and also we get

$$(20)\qquad |A(t)| \leqslant c\exp\Big\{-\frac{\varepsilon}{2}\sqrt{\frac{|t|}{2}}\Big\}$$

by (16) and (19). This shows that the function α is analytic in $\mathbb{C}_+$, all of its derivatives are continuous up to the real axis and

$$(21)\qquad \Big|\frac{\mathrm{d}^m\alpha(\lambda)}{\mathrm{d}\lambda^m}\Big| \leqslant c_m, \quad \lambda\in\overline{\mathbb{C}}_+,\ m=1,2,\dots$$

hold where

$$\begin{aligned} c_1 &= 2+2^2c\int_0^\infty t\exp\Big\{-\frac{\varepsilon}{2}\sqrt{t}\Big\}\,\mathrm{d}t,\\ (22)\qquad c_m &= 2^{m+1}c\int_0^\infty t^m\exp\Big\{-\frac{\varepsilon}{2}\sqrt{t}\Big\}\,\mathrm{d}t, \quad m=2,3,\dots, \end{aligned}$$

in which $c > 0$ is a constant.

Let us denote the sets of all limit points of P_1 and P_2 by P_3 and P_4, respectively, and the set of all zeros of α with infinite multiplicity in $\overline{\mathbb{C}}_+$ by P_5. It is obvious from the uniqueness theorem of the analytic functions that

$$P_3\subset P_2,\ P_4\subset P_2,\ P_5\subset P_2.$$

Since all derivatives of the function α are continuous up to the real axis, we get

$$(23)\qquad P_3\subset P_5,\ P_4\subset P_5.$$

Lemma 3.4. $P_5=\emptyset$.

Proof. It is trivial from Lemma 3.1 and (2.1) that α satisfies the conditions (12), (13) of Theorem 2.2. Since $\alpha(\lambda)\not\equiv 0$, (14) yields

$$(24)\qquad \int_0^h \ln F(s)\,\mathrm{d}\mu(P_{5,s}) > -\infty$$

where $F(s)=\inf\limits_m \frac{c_m s^m}{m!}$, $\mu(P_{5,s})$ is the Lebesgue measure of the s-neighborhood of P_5 and c_m are constants defined by (21) and (22). Now we will obtain the following estimates for c_m:

$$\begin{aligned} (25)\quad c_m &= 2^{m+1}c\int_0^\infty t^m\exp\Big(-\frac{\varepsilon}{2}\sqrt{t}\Big)\,\mathrm{d}t \leqslant c2^{3m+4}\varepsilon^{-2(m+1)}(2m+2)^{2m+1}m!\\ &= c2^{4m+5}\varepsilon^{-2(m+1)}m^m\Big(1+\frac{1}{m}\Big)^m(m+1)m! \leqslant Bb^m m^m m! \end{aligned}$$

where $B = c2^5\mathrm{e}\varepsilon^{-2}$and $b = 2^4\mathrm{e}\varepsilon^{-2}$. Substituting (25) into the definition of $F(s)$, we arrive at

$$F(s) \leqslant B \inf_m \{b^m s^m m^m\} \leqslant B \exp\{-b^{-1}s^{-1}\mathrm{e}^{-1}\}$$

or by (24)

$$\int_0^h \frac{1}{s}\,\mathrm{d}\mu(P_{5,s}) < \infty. \tag{26}$$

The inequality (26) holds for an arbitrary s if and only if $\mu(P_{5,s}) = 0$ or $P_5 = \emptyset$. $\square$

Lemma 3.5. *α has a finite number of zeros with finite multiplicity in $\overline{\mathbb{C}}_+$.*

Proof. From (23) we get that

$$P_3 = P_4 = \emptyset. \tag{27}$$

We obtain the finiteness of the sets P_1 and P_2 by Lemma 3.1 and by (27). Since $P_5 = \emptyset$, all of the zeros of the function α have finite multiplicities. $\square$

Summarizing the above arguments we have

Theorem 3.6. *The operator L has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity if the condition* (18) *holds.*

4. Principal functions

In this section we assume that (18) holds. Let $\lambda_1, \ldots, \lambda_\ell$ denote the zeros of α in $\mathbb{C}_+$ (i.e. $\lambda_1^2, \ldots, \lambda_\ell^2$ are the eigenvalues of L) with multiplicities $m_1, \ldots, m_\ell$, respectively. Similarly let $\lambda_{\ell+1}, \ldots, \lambda_k$ be the zeros of α on the real axis (i.e. $\lambda_{\ell+1}^2, \ldots, \lambda_k^2$ are the spectral singularities of L) with multiplicities $m_{\ell+1}, \ldots, m_k$, respectively. It is trivial that

$$\left\{\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} W\big[e^+(x,\lambda), e^-(x,\lambda)\big]\right\}_{\lambda=\lambda_j} = \left\{\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\alpha(\lambda)\right\}_{\lambda=\lambda_j} = 0 \tag{28}$$

holds for $n = 0, 1, \ldots, m_j - 1$, $j = 1, 2, \ldots, \ell$. If $n = 0$ we get

$$e^+(x,\lambda_j) = a_0(\lambda_j)e^-(x,\lambda_j), \quad j = 1, \ldots, \ell. \tag{29}$$

So $a_0(\lambda_j) \neq 0$.

Theorem 4.1. *The formula*

$$\left\{\frac{\partial^n}{\partial\lambda^n}e^+(x,\lambda)\right\}_{\lambda=\lambda_j}=\sum_{i=0}^{n}\binom{n}{i}a_{n-i}\left\{\frac{\partial^i}{\partial\lambda^i}e^-(x,\lambda)\right\}_{\lambda=\lambda_j} \tag{30}$$

holds for $n=0,1,\dots,m_j-1$, $j=1,2,\dots,\ell$, where the constants $a_0,a_1,\dots,a_n$ depend on λ_j.

Proof. We will proceed by mathematical induction. For $n=0$, the proof is trivial from (29). Let us assume that for $0<n_0\leqslant m_j-2$, (30) holds; i.e.

$$\left\{\frac{\partial^{n_0}}{\partial\lambda^{n_0}}e^+(x,\lambda)\right\}_{\lambda=\lambda_j}=\sum_{i=0}^{n_0}\binom{n_0}{i}a_{n_0-i}\left\{\frac{\partial^i}{\partial\lambda^i}e^-(x,\lambda)\right\}_{\lambda=\lambda_j}. \tag{31}$$

Now we will prove that (30) holds for n_0+1, too. If $y(x,\lambda)$ is a solution of equation (3) then $\frac{\partial^n}{\partial\lambda^n}y(x,\lambda)$ satisfies

$$\left\{-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+q(x)-\lambda^2\right\}\frac{\partial^n}{\partial\lambda^n}y(x,\lambda)=2\lambda n\frac{\partial^{n-1}}{\partial\lambda^{n-1}}y(x,\lambda)+n(n-1)\frac{\partial^{n-2}}{\partial\lambda^{n-2}}y(x,\lambda). \tag{32}$$

Writing (32) for $e^+(x,\lambda_j)$ and $e^-(x,\lambda_j)$ and using (31) we find

$$\left\{-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+q(x)-\lambda_j^2\right\}f_{n_0+1}(x,\lambda_j)=0$$

where

$$f_{n_0+1}(x,\lambda_j)=\left\{\frac{\partial^{n_0+1}}{\partial\lambda^{n_0+1}}e^+(x,\lambda)\right\}_{\lambda=\lambda_j}-\sum_{i=1}^{n_0+1}\binom{n_0+1}{i}a_{n_0+1-i}\left\{\frac{\partial^i}{\partial\lambda^i}e^-(x,\lambda)\right\}_{\lambda=\lambda_j}.$$

From (28) we have

$$W\big[f_{n_0+1}(x,\lambda_j),e^-(x,\lambda_j)\big]=\left\{\frac{\mathrm{d}^{n_0+1}}{\mathrm{d}\lambda^{n_0+1}}W\big[e^+(x,\lambda),e^-(x,\lambda)\big]\right\}_{\lambda=\lambda_j}=0.$$

Hence there exists a constant $a_{n_0+1}(\lambda_j)$ such that

$$f_{n_0+1}(x,\lambda_j)=a_{n_0+1}(\lambda_j)e^-(x,\lambda_j).$$

This shows that (32) holds for $n=n_0+1$. $\square$

Let us introduce the functions

$$U_n(x,\mu_j)=\begin{cases}\displaystyle\sum_{i=0}^{n}\binom{n}{i}a_{n-i}(\lambda_j)\Big\{\frac{\partial^i}{\partial\lambda^i}e^-(x,\lambda)\Big\}_{\lambda=\lambda_j}, & -\infty<x<0,\\ \displaystyle\Big\{\frac{\partial^n}{\partial\lambda^n}e^+(x,\lambda)\Big\}_{\lambda=\lambda_j}, & 0\leqslant x<\infty\end{cases}$$

for $n=0,1,\ldots,m_j-1$, $j=1,2,\ldots,\ell$ where $\mu_j=\lambda_j^2$; using (5), (6) and (19) we arrive at

$$\Big\{\frac{\partial^n}{\partial\lambda^n}e^+(\cdot,\lambda)\Big\}_{\lambda=\lambda_j}\in L_2(0,\infty),\ n=0,1,\ldots,m_j-1,\ j=1,2,\ldots,\ell,$$
$$\Big\{\frac{\partial^i}{\partial\lambda^i}e^-(\cdot,\lambda)\Big\}_{\lambda=\lambda_j}\in L_2(-\infty,0),\ i=0,1,\ldots,m_j-1,\ j=1,2,\ldots,\ell$$

or by (30),

$$U_n(\cdot,\mu_j)\in L_2(-\infty,\infty),\ n=0,1,\ldots,m_j-1,\ j=1,2,\ldots,\ell. \tag{33}$$

$$U_0(x,\mu_j),U_1(x,\mu_j),\ldots,U_{m_j-1}(x,\mu_j)$$

are called the principal functions corresponding to the eigenvalues $\mu_j=\lambda_j^2$, $j=1,2,\ldots,\ell$ of L. In the above $U_0(x,\mu_j)$ is an eigenfunction; $U_1(x,\mu_j),\ldots,U_{m_j-1}(x,\mu_j)$ are the associated functions of $U_0(x,\mu_j)$, [6].

If $\mu_{\ell+1}=\lambda_{\ell+1}^2,\ldots,\mu_k=\lambda_k^2$ are spectral singularities of L (i.e. $\lambda_{\ell+1},\ldots,\lambda_k$ are real zeros of α), then we can find

$$\Big\{\frac{\partial^v}{\partial\lambda^v}e^+(x,\lambda)\Big\}_{\lambda=\lambda_p}=\sum_{i=0}^{v}\binom{v}{i}b_{v-i}(\lambda_p)\Big\{\frac{\partial^i}{\partial\lambda^i}e^-(x,\lambda)\Big\}_{\lambda=\lambda_p} \tag{34}$$

for $v=0,1,\ldots,m_p-1$, $p=\ell+1,\ \ell+2,\ldots,k$, in a way similar to Theorem 4.1.

Let us introduce the following functions:

$$U_v(x,\mu_p)=\begin{cases}\displaystyle\sum_{i=0}^{v}\binom{v}{i}b_{v-i}(\lambda_p)\Big\{\frac{\partial^i}{\partial\lambda^i}e^-(x,\lambda)\Big\}_{\lambda=\lambda_p}, & -\infty<x<0,\\ \displaystyle\Big\{\frac{\partial^v}{\partial\lambda^v}e^+(x,\lambda)\Big\}_{\lambda=\lambda_p}, & 0\leqslant x<\infty\end{cases}$$

for $v=0,1,\ldots,m_p-1$, $p=\ell+1,\ \ell+2,\ldots,k$. It is trivial from (5) and (6) that

$$\Big\{\frac{\partial^v}{\partial\lambda^v}e^+(\cdot,\lambda)\Big\}_{\lambda=\lambda_p}\notin L_2(0,\infty),\ v=0,1,\ldots,m_p-1,\ p=\ell+1,\ \ell+2,\ldots,k,$$
$$\Big\{\frac{\partial^i}{\partial\lambda^i}e^-(\cdot,\lambda)\Big\}_{\lambda=\lambda_p}\notin L_2(-\infty,0),\ i=0,1,\ldots,m_p-1,\ p=\ell+1,\ \ell+2,\ldots,k$$

or by (34),

$$U_v(\cdot,\mu_p)\notin L_2(-\infty,\infty),\ v=0,1,\dots,m_p-1,\ p=\ell+1,\ \ell+2,\dots,k.$$

Now let us consider the Hilbert spaces

$$\begin{aligned}
H(-\infty,0;m)&=\Big\{f;\ \int_{-\infty}^{0}(1+|x|)^{2m}|f(x)|^2\,\mathrm{d}x<\infty\Big\},\\
H(-\infty,0;-m)&=\Big\{f;\ \int_{-\infty}^{0}(1+|x|)^{-2m}|f(x)|^2\,\mathrm{d}x<\infty\Big\},\\
H(0,\infty;m)&=\Big\{f;\ \int_{0}^{\infty}(1+|x|)^{2m}|f(x)|^2\,\mathrm{d}x<\infty\Big\},\\
H(0,\infty;-m)&=\Big\{f;\ \int_{0}^{\infty}(1+|x|)^{-2m}|f(x)|^2\,\mathrm{d}x<\infty\Big\},\\
H(-\infty,\infty;m)&=\Big\{f;\ \int_{-\infty}^{\infty}(1+|x|)^{2m}|f(x)|^2\,\mathrm{d}x<\infty\Big\},\\
H(-\infty,\infty;-m)&=\Big\{f;\ \int_{0}^{\infty}(1+|x|)^{-2m}|f(x)|^2\,\mathrm{d}x<\infty\Big\}.
\end{aligned}$$

It is evident that

$$\begin{gathered}
H(-\infty,0;0)=L_2(-\infty,0),\ H(0,\infty;0)=L_2(0,\infty),\ H(-\infty,\infty;0)=L_2(-\infty,\infty),\\
H(-\infty,0;m)\underset{\neq}{\subseteq}L_2(-\infty,0)\underset{\neq}{\subseteq}H(-\infty,0;-m),\quad m=1,2,\dots,\\
H(0,\infty;m)\underset{\neq}{\subseteq}L_2(0,\infty)\underset{\neq}{\subseteq}H(0,\infty;-m),\quad m=1,2,\dots,\\
H(-\infty,\infty;m)\underset{\neq}{\subseteq}L_2(-\infty,\infty)\underset{\neq}{\subseteq}H(-\infty,\infty;-m),\quad m=1,2,\dots.
\end{gathered}$$

Let $H'(-\infty,0;m)$, $H'(0,\infty;m)$ and $H'(-\infty,\infty;m)$ denote the duals of

$$H(-\infty,0;m),\ H(0,\infty;m)\text{ and }H(-\infty,\infty;m),$$

respectively. Obviously $H'(-\infty,0;m)$, $H'(0,\infty;m)$ and $H'(-\infty,\infty;m)$ are isomorphic to $H(-\infty,0;-m)$, $H(0,\infty;-m)$ and $H(-\infty,\infty;-m)$, respectively [2].

Using (5) and (6) we arrive at

$$\begin{aligned}
&\Big\{\frac{\partial^v}{\partial\lambda^v}e^+(\cdot,\lambda)\Big\}_{\lambda=\lambda_p}\in H(0,\infty-(v+1)),\ v=0,1,\dots,m_p-1,\ p=\ell+1,\ell+2,\dots,k,\\
&\Big\{\frac{\partial^i}{\partial\lambda^i}e^-(\cdot,\lambda)\Big\}_{\lambda=\lambda_p}\in H(-\infty,0;-(i+1)),\ i=0,1,\dots,m_p-1,\ p=\ell+1,\ell+2,\dots,k,
\end{aligned}$$

or by (34)

(35) $U_v(\cdot,\mu_p)\in H(-\infty,\infty;-(v+1)),\ v=0,1,\dots,m_p-1,\ p=\ell+1,\ell+2,\dots,k.$

Let us choose so that

$$m_0 = \max\{m_{\ell+1}, m_{\ell+2}, \dots, m_k\}.$$

We will use the notation

$$H_+ = H(-\infty, \infty; m_0 + 1), \ \ H_- = H(-\infty, \infty; -(m_0 + 1)).$$

It is trivial that the dual of H_+ is isomorphic to H_- ($H'_+ \sim H_-$) and

$$H_+ \underset{\neq}{\subseteq} L_2(-\infty, \infty) \underset{\neq}{\subseteq} H_-.$$

Thus, from (35) we have

Theorem 4.2. *$U_v(\cdot, \mu_p) \in H_-$ for $v = 0, 1, \dots, m_p - 1$, $p = \ell + 1, \ \ell + 2, \dots, k$.*

$$U_0(x, \mu_p), U_1(x, \mu_p), \dots, U_{m_p-1}(x, \mu_p)$$

are called the principal functions corresponding to the spectral singularities $\mu_p = \lambda_p^2$, $p = \ell + 1, \ \ell + 2, \dots, k$ of L. In the above $U_0(x, \mu_p)$ is the generalized eigenfunction, $U_1(x, \mu_p), \dots, U_{m_p-1}(x, \mu_p)$ are the generalized associated functions of $U_0(x, \mu_p)$, [4].

The spectral expansion in terms of the principal functions of the operator L will be the subject of another article.

Acknowledgement. E. Bairamov would like to thank Scientific and Technical Research Council of Turkey for financial support.

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