Czechoslovak Mathematical Journal, 49 (124) (1999), 891–899

A NEW CLASS OF NONEXPANSIVE TYPE MAPPINGS AND FIXED POINTS

Ljubomir B. Ćirić, Beograd

(Received April 4, 1997)

Abstract. In this paper a new class of self-mappings on metric spaces, which satisfy the nonexpensive type condition (3) below is introduced and investigated. The main result is that such mappings have a unique fixed point. Also, a remetrization theorem, which is converse to Banach contraction principle is given.

MSC 2000: 54H25, 47H09

Keywords: nonexpansive type mapping, asymptotically regular mapping, fixed point.

1. Introduction

Let (X, d) be a metric space, T a self-mapping on X and k a nonnegative real number such that the inequality $d(Tx, Ty) \leqslant kd(x, y)$ hold for any $x, y \in X$. If $k < 1$ then T is said to be a contractive mapping; if $k = 1$, then T is said to be a nonexpansive mapping. The well known Banach contraction mapping principle—already is obtained in particular situations by Liouville, Picard and Guorsat—states that if X is complete, then every contractive mapping has a unique fixed point and that point can be obtained as a limit of repeated iteration of the mapping at any point of X. However, a nonexpansive mapping need not have fixed points. Yet these mappings have a fixed point when X has a convex structure. There exists very abundant literature about contractive and nonexpansive type mappings, where the contractive and nonexpansive conditions are replaced with more general conditions. These mappings also have many applications, similarly as contractive or nonexpansive mappings ([2], [11], [20], [22–24]).

Bogin [1] proved the following result:

Theorem 1.1. *Let (X,d) be a nonempty complete metric space and $T\colon X\to X$ a mapping satisfying*

$$(1)\qquad d(Tx,Ty)\leqslant ad(x,y)+b\bigl[d(x,Tx)+d(y,Ty)\bigr]+c\bigl[d(d(x,Ty)+d(y,Tx))\bigr]$$

where $a\geqslant 0$, $b>0$, $c>0$ and

$$(2)\qquad a+2b+2c=1.$$

Then T has a unique fixed point.

This result was generalized by Rhoades [21], Ćirić [7–8] and Li [17]. K. Iseki [15] studied a family of commuting mappings $T_1,T_2,\dots,T_n$ which satisfy (1) with $a\geqslant 0$, $b\geqslant 0$, $c\geqslant 0$ and $a+2b+2c<1$.

Greguš [14] considered a class of self-mappings T on X which satisfy (1) with $c=0$. Indeed, he proved the following result:

Theorem 1.2. *Let C be a nonempty closed convex subset of a Banach space B and $T\colon C\to C$ a mapping that satisfies*

$$\|Tx-Ty\|\leqslant a\|x-y\|+b\bigl[\|x-Tx\|+\|y-Ty\|\bigr]$$

for all $x,y\in C$, where $a>0$, $b>0$ and $a+2b=1$. Then T has a unique fixed point.

Greguš's result has inspired many authors in further investigations: Ćirić [3–6], Delbosco, Ferrero and Rossati [9], Diviccaro, Fisher and Sessa [10], Fisher [12], Fisher and Sessa [13], Jungck [16], Li [17] and Mukherjee and Verma [19].

The purpose of this paper is to introduce and investigate a new class of self-mappings T on X which satisfy an inequality of type (1) with $b\geqslant 0$ and still have a fixed point. An example is constructed to show that if the mapping T satisfies (1) with $b=0$ and if a and c are such that (2) holds, then T need not have a fixed point. Therefore, a contractive condition for T, which shall guarantee a fixed point of T in the case $b=0$ and $a+2c=1$, must be stricter then (1).

Let (X,d) be a metric space and $T\colon X\to X$ a self-mapping of X. For $x,y\in X$ denote

$$M(x,y)=\max\{d(x,Ty),\ d(y,Tx)\},$$
$$m(x,y)=\min\{d(x,Ty),\ d(y,Tx)\}.$$

In this paper we shall investigate a new class of self-mappings T on X which satisfy the following contraction condition:

$$(3)\quad d(Tx,Ty)\leqslant a(x,y)\max\bigl\{d(x,y),d(x,Tx),d(y,Ty),(1/2)\bigl[M(x,y)+m(x,y)\bigr]\bigr\}$$
$$+\,c(x,y)\bigl[M(x,y)+hm(x,y)\bigr]$$

for all $x, y \in X$, where $0 < h < 1$,

$$(4) \qquad a(x,y) \geqslant 0, \quad \inf\{c(x,y)\colon\ x,y \in X\} > 0,$$

and

$$(5) \qquad a(x,y) + 2c(x,y) = 1.$$

2. Main results

We shall begin with three propositions in which some properties of T satisfying (3) are established.

Proposition 2.1. *Let $T\colon X \to X$ be a mapping satisfying* (3) *with a and c satisfying* (4) *and* (5). *Then T has at most one fixed point.*

Proof. Let u and v be two fixed points of T. Then by (3) we have

$$d(u,v) = d(Tu,Tv) \leqslant a(u,v)d(u,v) + c(u,v)(1+h)d(u,v).$$

Hence by (5),

$$d(u,v) \leqslant \bigl[1 - c(u,v)(1-h)\bigr]d(u,v),$$

implying $u = v$ by (4). □

Recall that a self-mapping T on a metric space X is said to be asymptotically regular at $x_0 \in X$ if

$$\lim_{n\to\infty} d(T^n x_0, T^{n+1}x_0) = 0$$

Proposition 2.2. *Let $T\colon X \to X$ be a self-mapping satisfying* (3), *where $0 < h < 1$ and* (4) *and* (5) *hold. Then T is asymptotically regular at each point in X.*

Proof. Let $x_0 \in X$ and define the Picard iterates $x_n = Tx_{n-1} = T^n x_0$ for $n = 1, 2, \ldots$. Note that (3) implies the corresponding inequality with $h = 1$. So by (3) we have (for $n = 1, 2, \ldots$):

$$(6) \qquad \begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leqslant a\max\Bigl\{d(x_{n-1},x_n), d(x_{n-1},Tx_{n-1}), d(x_n,Tx_n), \\ &\qquad \frac{1}{2}\bigl[(M(x_{n-1},x_n) + m(x_{n-1},x_n))\bigr]\Bigr\} \\ &\quad + c\bigl[M(x_{n-1},x_n) + m(x_{n-1},x_n)\bigr], \end{aligned}$$

where $a = a(x_{n-1}, x_n)$, $c = c(x_{n-1}, x_n)$. Then $m(x_{n-1}, x_n) = 0$,

$$M(x_{n-1}, x_n) = d(x_{n-1}, x_n) \leqslant d(x_{n-1}, x_n) + d(x_n, x_{n+1}).$$

If we suppose that $d(x_n, x_{n+1}) > d(x_{n-1}, x_n)$ for some n, then by (6) and (4), (5) we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant ad(x_n, x_{n+1}) + c\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big] \\ &< ad(x_n, x_{n+1}) + 2cd(x_n, x_{n+1}) = d(x_n, x_{n+1}), \end{aligned}$$

which is a contradiction. Thus $d(x_n, x_{n+1}) \leqslant d(x_{n-1}, x_n)$. Hence

$$d(x_{n-1}, Tx_{n-1}) \leqslant d(x_0, Tx_0) \qquad (n = 1, 2, \ldots). \tag{7}$$

Using (3) again we have

$$\begin{aligned} d(x_1, Tx_2) &= d(Tx_0, Tx_2) \\ &\leqslant a \max\Big\{d(x_0, x_2), d(x_0, Tx_0), d(x_2, Tx_2), \frac{1}{2}\big[d(x_0, Tx_2) + d(x_2, Tx_0)\big]\Big\} \\ &\quad + c\big[M(x_0, x_2) + hm(x_0, x_2)\big], \end{aligned} \tag{8}$$

where $a = a(x_0, x_2)$, $c = c(x_0, x_2)$. From (7), (8) and the triangle inequality we get

$$\begin{aligned} d(x_1, Tx_2) &\leqslant 2ad(x_0, Tx_0) \\ &\quad + c\big[d(x_0, Tx_0) + d(Tx_0, Tx_1) + d(Tx_1, Tx_2) + hd(Tx_0, Tx_1)\big] \\ &\leqslant 2ad(x_0, Tx_0) + c(3 + h)d(x_0, Tx_0) \\ &= (2a + 4c)d(x_0, Tx_0) - c(1 - h)d(x_0, Tx_0) \\ &= [2 - c(1 - h)]d(x_0, Tx_0). \end{aligned} \tag{9}$$

From (3), (7) and (9) we have

$$\begin{aligned} d(Tx_1, Tx_2) &\leqslant a' \max\big\{d(x_0, Tx_0), \tfrac{1}{2}d(x_1, Tx_2)\big\} + c'd(x_1, Tx_2) \\ &\leqslant a'd(x_0, Tx_0) + c'[2 - c(1 - h)]d(x_0, Tx_0) \\ &= [(a' + 2c') - c'c(1 - h)]d(x_0, Tx_0) \\ &= [1 - c'c(1 - h)]d(x_0, Tx_0), \end{aligned} \tag{10}$$

where $a' = a(x_1, x_2)$, $c' = c(x_1, x_2)$. Set

$$s = \inf\{c(x, y)\colon\ x, y \in X\}. \tag{11}$$

Then (10) yields

$$d(T^2x_0, T^3x_0) \leqslant [1 - s^2(1-h)]d(x_0, Tx_0).$$

Analogously,

$$d(T^3x_0, T^4x_0) \leqslant [1 - s^2(a-h)]d(Tx_0, T^2x_0) \leqslant [1 - s^2(1-h)]d(x_0, Tx_0).$$

Proceeding in this manner we obtain

$$d(T^nx_0, T^{n+1}x_0) \leqslant [1 - s^2(1-h)]^{[n/2]}d(x_0, Tx_0) \tag{12}$$

for all $n = 1, 2, \ldots$, where $[n/2]$ denotes the greatest integer not exceeding $n/2$. Since $0 < s \leqslant \frac{1}{2}$ and $h < 1$, we get that from (12)

$$\lim_{n\to\infty} d(T^nx_0, T^{n+1}x_0) = 0$$

which implies that T is regular at x_0. □

Note that, if T satisfies (3), where a and b are such that (4) and (5) hold, then T may be discontinuous at some points. Nevertheless we have the following result.

Proposition 2.3. *If T satisfies* (3), *where* (4) *and* (5) *hold, and if T has a fixed point, say p, then T is continuous at p.*

Proof. Let $x_n \to p = Tp$. Then (3) implies (with $h = 1$)

$$\begin{aligned} d(Tx_n, Tp) &\leqslant a \max\Big\{d(x_n, p), d(x_n, Tx_n), \tfrac{1}{2}\big[d(x_n, p) + d(p, Tx_n)\big]\Big\} \\ &\quad + c[d(x_n, p) + d(p, Tx_n)] \\ &\leqslant a[d(x_n, p) + d(p, Tx_n)] + c[d(x_n, p) + d(p, Tx_n)] \\ &= (a + 2c - c)d(x_n, p) + (a + 2c - c)d(p, Tx_n) \\ &\leqslant (1 - s)d(x_n, p) + (1 - s)d(p, Tx_n), \end{aligned}$$

where $s > 0$ is defined by (11). Hence we get

$$d(Tx_n, Tp) \leqslant \Big(\frac{1}{s} - 1\Big)d(x_n, p). \tag{13}$$

Hence $Tx_n \to Tp$ when $n \to \infty$. Therefore, T is continuous at p. □

Now we will prove the main result.

Theorem 2.1. *Let (X,d) be a nonempty complete metric space and $T\colon X\to X$ a self-mapping satisfying* (3)*, where a and b are such that* (4) *and* (5) *hold. Then T has a unique fixed point, say p, and at this point p the mapping T is continuous. Moreover, for each $x\in X$ a sequence $\{T^n x$ converges to p and*

$$d(T^n x,p)\leqslant \frac{2}{s^2(1-h)}\big[1-s^2(1-h)\big]^{[n/2]}d(x,Tx), \tag{14}$$

where s is given by (11) *and $[n/2]$ means the greatest integer not exceeding $n/2$.*

Proof. Let $x=x_0\in X$ be arbitrary and define the Picard iterates $x_n=Tx_{n-1}=T^n x_0$ for $n=1,2,\ldots$. Then, from (12), we conclude that $\{T^n x\}$ is a Cauchy sequence. Since X is complete, there is some p in X such that

$$\lim_{n\to\infty} T^n x=p. \tag{15}$$

From (3) (with $h=1$) it follows

$$\begin{aligned} d(T^n x,Tp)\leqslant a\max\Big\{&d(T^{n-1}x,p),d(T^{n-1}x,T^n x),d(p,Tp),\\ &\frac{1}{2}\big[d(T^{n-1}x,Tp)+d(p,T^n x)\big]\Big\}+c\big[d(T^{n-1}x,p)+d(p,T^n x)\big]\end{aligned}$$

If $Tp\neq p$, then because of (15) we may assume that n is chosen so large that we have

$$\begin{aligned} d(T^n x,Tp)&\leqslant ad(p,Tp)+c\big[d(p,Tp)+d(T^{n-1}x,p)+d(p,T^n x)\big]\\ &=(1-c)d(p,Tp)+c\big[d(T^{n-1}x,p)+d(p,T^n x)\big]\\ &\leqslant(1-s)d(p,Tp)+c\big[d(T^{n-1}x,p)+d(p,T^n x)\big],\end{aligned}$$

where s is defined by (11). Taking the limit when $n\to\infty$ yields

$$d(p,Tp)\leqslant(1-s)d(p,Tp).$$

Hence, $d(p,Tp)=0$, a contradiction. Thus $Tp=p$. By Proposition 2.1 and 2.3, p is a unique fixed point of T and at p the mapping T is continuous. Form (12) it is easy to obtain (14). $\square$

Next we state a simple example which shows that the condition (1) with $b=0$ and $a+2c=1$ (or (3) with $h=1$) does not guarantee the existence of a fixed point.

Example. Let $(X,d)=(-\infty,+\infty)$ and $T\colon X\to X$ be defined by $Tx=x+1$. Then T satisfies the condition

$$\begin{aligned} d(Tx,Ty)=d(x,y) &\leqslant ad(x,y)+c\bigl[d(x,Ty)+d(y,Tx)\bigr]=(a+2c)d(x,y)\\ &\leqslant ad(x,y)+c\bigl[d(x,y)+1+|d(x,y)-1|\bigr] \end{aligned}$$

for all $a\geqslant 0$, $c>0$ with $a+2c=1$, but T has no fixed points.

Finally we will prove a theorem which is closely related to Theorem 2.1 and belongs to remetrization theory.

Theorem 2.2. *Let T be a continuous mapping of a (nonempty) complete metric space (X,d) into itself satisfying (3) with a and c satisfying (4) and (5) for all $x,y\in X$. Then for any $\lambda\in(0,1)$, there exists a complete metric d_λ, topologically equivalent to d and such that*

$$d_\lambda(Tx,Ty)\leqslant \lambda d_\lambda(x,y)\qquad \text{for all}\quad x,y\in X.$$

P r o o f. We shall use the following theorem:

Theorem (Meyers [18])**.** *Let T be a continuous mapping of a (nonempty) metric space (X,d) into itself such that*
(i) *T has a unique fixed point p;*
(ii) *the sequence $\{T^n x\}$ of iterates converges to p for all $x\in X$;*
(iii) *there exist an open neighborhood U of p with the property*
that for any given open set V containing p there exists a positive integer N such that

$$T^n(U)\subset V\quad \text{for all } n\geqslant N.$$

Then for any $\lambda\in(0,1)$, there exist a metric d_λ, topologically equivalent to d such that $d_\lambda(Tx,Ty)\leqslant\lambda d_\lambda(x,y)$ for all $x,y\in X$. Further, if d is complete, then d_λ can be chosen complete.

Since in the proof of Theorem 2.1 the element x was arbitrary, the assumptions (i) and (ii) of the Meyers theorem follow by our Theorem 2.1. So we have to find an open neighborhood U of p such that $\{T^n(U)\}$ converges to p.

In Proposition 2.3 it is shown that

$$d(p,Tx)\leqslant[(1-s)/s]d(x,p).$$

So we have

$$d(x, Tx) \leqslant d(x, p) + d(p, Tx) \leqslant d(x, p)/s.$$

Therefore, from (14) we get

$$d(T^n x, p) \leqslant \frac{2}{s^2(1-h)}[1 - s^2(1-h)]^{n/2} \frac{1}{s} d(x, p). \tag{16}$$

Let

$$U = \{x \in X\colon\ d(x, p) < 1\}.$$

Then from (16) we conclude that $T^n(U) \to p$. □

References

[1] *J. Bogin*: A generalization of a fixed point theorem of Goebel, Kirk and Shimi. Canad. Math. Bull *19* (1976), 7–12.

[2] *S.K. Chatterjea*: Applications of an extension of a theorem of Fisher on common fixed point. Pure Math. Manuscript *6* (1987), 35–38.

[3] *Lj. B. Ćirić*: On a common fixed point theorem of a Greguš type. Publ. Inst. Math. (Beograd) (N.S.) *49 (63)* (1991), 174–178.

[4] *Lj. B. Ćirić*: On some discontinuous fixed point mappings in convex metric spaces. Czechoslovak Math. J. *43 (118)* (1993), 319–326.

[5] *Lj. B. Ćirić*: On Diviccaro, Fisher and Sessa open questions. Arch. Math. (Brno) *29* (1993), 145–152.

[6] *Lj. B. Ćirić*: Nonexpansive type mappings and fixed point theorems in convex metric spaces. Rend. Accad. Naz. Sci. XL Mem. Mat. *(5), Vol. XIX* (1995), 263–271.

[7] *Lj. B. Ćirić*: On some nonexpansive type mappings and fixed points. Indian J. Pure Appl. Math. *24 (3)* (1993), 145–149.

[8] *Lj. B. Ćirić*: On same mappings in metric space and fixed points. Acad. Roy. Belg. Bull. Cl. Sci. *(5) T.VI* (1995), 81–89.

[9] *D. Delbosco, O. Ferrero and F. Rossati*: Teoremi di punto fisso per applicazioni negli spazi di Banach. Boll. Un. Mat. Ital. A. *(6) 2* (1993), 297–303.

[10] *M. L. Diviccaro, B. Fisher and S. Sessa*: A common fixed point theorem of Greguš type. Publ. Math. (Debrecen) *34* (1987), 83–89.

[11] *Sh. T. Dzhabbarov*: A generalized contraction mapping principle and infinite systems of differential equations. Differentsialnye Uravneniya *26* (1990), no. 8, 1299–1309, 1467 (in Russian); translation in Differential Equations *26* (1990), no. 8, 944–952.

[12] *B. Fisher*: Common fixed points on a Banach space. Chung Yuan J. *11* (1982), 19–26.

[13] *B. Fisher and S. Sessa*: On a fixed point theorem of Greguš. Internat. J. Math. Sci. *9* (1986), 23–28.

[14] *M. Greguš*: A fixed point theorem in Banach spaces. Boll. Un. Mat. Ital. *A (5) 17* (1980), 193–198.

[15] *K. Iseki*: On common fixed point theorems of mappings. Proc. Japan Acad. Ser. A Math. Sci. *50* (1974), 408–409.

[16] *G. Jungck*: On a fixed point theorem of Fisher and Sessa. Internat. J. Math. Sci. *13* (1990), 497–500.

[17] *B. Y. Li*: Fixed point theorems of nonexpansive mappings in convex metric spaces. Appl. Math. Mech. (English Ed.) *10* (1989), 183–188.
[18] *P. R. Meyers*: A converse to Banach's contraction theorem. J. Res. Nat. Bur. Standards, Sect. B *71* (1967), 73–76.
[19] *R. N. Mukherjee and V. Verma*: A note on a fixed point theorem of Greguš. Math. Japon. *33* (1988), 745–749.
[20] *P. Omari, G. Villari and F. Zanolin*: A survey of recent applications of fixed point theory to periodic solutions of the Lienard equations, Fixed point theory and its applications. (Berkley, CA 1986), 171–178, Contemp. Math. *72*, Amer. Math. Soc., Providence, RI, 1988.
[21] *B. E. Rhoades*: A generalization of a fixed point theorem of Bogin. Math. Sem. Notes, Kobe Univ. *6* (1978), 1–7.
[22] *B. E. Rhoades*: Some applications of contractive type mappings. Math. Sem. Notes, Kobe Univ. *5* (1977), 137–139.
[23] *Md. Shahabuddin*: Study of applications of Banach's fixed point theorem. Chittagong Univ. Stud., Part II Sci. *11* (1987), 85–91.
[24] *A. Weiczorek*: Applications of fixed point theorems in game theory and mathematical economics (Polish). Wiadom. Mat. *28* (1988), 25–34.

Author's address: Faculty of Mechanical Engineering, 27. march 80, 11000 Belgrade, Yugoslavia.