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RADICALS OF GREEN'S RELATIONS

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Abstract. Some structural descriptions of semigroups in which the radicals of Green's relations are semilattice congruences will be given.

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A semigroup S is called an *epigroup* if for each element of S, some its power lies in a subgroup of S. L. N. Shevrin proved in [14] that in an epigroup $\sqrt{\mathscr{D}}$ is transitive if and only if it is a semilattice congruence. A more general result has been obtained by M. S. Putcha [11], who proved that in an epigroup, the transitive closure of $\sqrt{\mathscr{J}}$ is the smallest semilattice congruence. Since $\mathscr{D} = \mathscr{J}$ on any epigroup, the above Shevrin's result can be also derived from the one of M. S. Putcha. Various characterizations of regularity of semigroups by Green's relations have been investigated by A. H. Clifford and G. B. Preston [6], J. T. Sedlock [13], B. Pondělíček [10] and D. W. Miller [7].

In this paper we characterize semigroups in which $\sqrt{\mathscr{X}}(\tau(\mathscr{X}))$ ($\mathscr{X} \in \{\mathscr{J}, \mathscr{D}, \mathscr{L}, \mathscr{R}, \mathscr{H}\}$) is a semilattice congruence. We also describe semigroups in which $\tau(\mathscr{L})$ is band congruence.

Throughout this paper $\mathbb{Z}^+$ will denote the set of all positive integers. The *division relation* $\mid$ on a semigroup S is defined by

$$a \mid b \iff (\exists x, y \in S^1)\ b = xay$$

and the relation $\to$ is defined by

$$a \to b \iff (\exists n \in \mathbb{Z}^+)\ a \mid b^n.$$

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For an element a of a semigroup S we define the following set $\Sigma_1(a) = \{x \in S \mid a \to x\}$ and an equivalence relation σ_1 on S by

$$a\sigma_1 b \iff \Sigma_1(a) = \Sigma_1(b).$$

By $\mathscr{J}, \mathscr{D}, \mathscr{L}, \mathscr{R}, \mathscr{H}$ we denote the well known *Green's relations* and by σ we denote the *least semilattice congruence* on a semigroup S.

For a relation ϱ on a semigroup S, the *radical of* ϱ, in notation $\sqrt{\varrho}$, is a relation introduced by L. N. Shevrin [14] as follows:

$$(a,b) \in \sqrt{\varrho} \iff (\exists m,n \in \mathbb{Z}^+)\ (a^m, b^n) \in \varrho.$$

Here we also define *the radical* $\tau(\varrho)$ of a relation ϱ on S by

$$(a,b) \in \tau(\varrho) \iff (\exists n \in \mathbb{Z}^+)(a^n, b^n) \in \varrho.$$

For undefined notions and notations we refer to [1], [2], [6] and [8].

The first characterization of semilattices of Archimedean semigroups is due to M. S. Putcha [12] (see also T. Tamura [15]). Some other characterizations of these semigroups are given by M. Ćirić and S. Bogdanović [4] (see also the survey [3]). By the following theorem we give some new characterizations using the radicals of Green's relations.

Theorem 1. *Let $\mathscr{X} \in \{\mathscr{J}, \mathscr{L}, \mathscr{H}\}$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of Archimedean semigroups;*

(ii) *$\sqrt{\sigma_1}$ is a congruence on S;*

(iii) *$\tau(\sigma_1)$ is a semilattice (band) congruence on S;*

(iv) *$\sqrt{\sigma_1} = \sigma_1$;*

(v) *$\sqrt{\mathscr{X}} \subseteq \sigma_1$.*

P r o o f. (i) $\Longrightarrow$ (iv). Let $a,b \in S$ and $(a,b) \in \sqrt{\sigma_1}$. Then there exist $m,n \in \mathbb{Z}^+$ such that $(a^m, b^n) \in \sigma_1$. By Theorem 3 [5] we obtain that $a\sigma_1 a^m \sigma_1 b^n \sigma_1 b$. Thus $(a,b) \in \sigma_1$, so $\sqrt{\sigma_1} \subseteq \sigma_1$. Since the opposion inclusion also holds, we have the assertion (iv).

(iv) $\Longrightarrow$ (v). Let $a,b \in S$, and $(a,b) \in \sqrt{\mathscr{X}}$. By Lemma 5 [5], $(a,b) \in \sqrt{\sigma_1}$, so $(a,b) \in \sigma_1$. Thus $\sqrt{\mathscr{X}} \subseteq \sigma_1$.

(v) $\Longrightarrow$ (i). Let (v) hold. Then for every $a \in S$ we have that $(a,a^2) \in \sqrt{\mathscr{X}}$. Thus $(a,a^2) \in \sigma_1$, and by Theorem 3 [5] we obtain (i).

(ii) $\Longrightarrow$ (i). Let $\sqrt{\sigma_1}$ be a semilattice congruence. Then for all $a, b \in S$ we have $(ab, a^2b) \in \sqrt{\sigma_1}$, whence $((ab)^k, (a^2b)^l) \in \sigma_1$, for some $k, l \in \mathbb{Z}^+$. Thus $(ab)^k \in Sa^2S$, and by Theorem 1 [4] we have that S is a semilattice of Archimedean semigroups.

(iv) $\Longrightarrow$ (ii). By Theorem 3 [5] we have that σ_1 is a semilattice congruence, and thus $\sqrt{\sigma_1}$ is also a semilattice congruence.

(iii) $\Longrightarrow$ (ii). If $\tau(\sigma_1)$ is a semilattice congruence, then $\tau(\sigma_1) = \sigma$. Since $\tau(\sigma_1) \subseteq \sqrt{\sigma_1}$, we then have that $\sigma = \tau(\sigma_1) \subseteq \sqrt{\sigma_1} \subseteq \sqrt{\sigma} = \sigma$. Thus $\sqrt{\sigma_1}$ is a semilattice congruence.

(i) $\Longrightarrow$ (iii). By Theorem 3 [5] we have that $(a, a^2) \in \sigma_1$, for all $a \in S$. If $(a, b) \in \sigma_1$, then for all $k \in \mathbb{Z}^+$ we have $(a^k, b^k) \in \sigma_1$, so $(a, b) \in \tau(\sigma_1)$. Thus $\sigma_1 \subseteq \tau(\sigma_1)$. Assume that $(a, b) \in \tau(\sigma_1)$. Then $(a^k, b^k) \in \sigma_1$ for some $k \in \mathbb{Z}^+$, whence $(a, b) \in \sigma_1$. Thus $\tau(\sigma_1) \subseteq \sigma_1$. Therefore, $\tau(\sigma_1) = \sigma_1$ and by (i) $\Longleftrightarrow$ (iv) we obtain that $\tau(\sigma_1) = \sqrt{\sigma_1} = \sigma_1$. Now by (iv) $\Leftrightarrow$ (ii) we have that $\tau(\sigma_1)$ is a semilattice congruence. $\square$

For an arbitrary Green's relation $\mathscr{X} \in \{\mathscr{J}, \mathscr{D}, \mathscr{L}, \mathscr{R}, \mathscr{H}\}$, we say that a semigroup S is *$\sqrt{\mathscr{X}}$-simple* if $(a, b) \in \sqrt{\mathscr{X}}$, for all $a, b \in S$.

Theorem 2. *Let $\mathscr{X} \in \{\mathscr{J}, \mathscr{L}, \mathscr{H}\}$. Then the following conditions on a semigroup S are equivalent:*

(i) *$\sqrt{\mathscr{X}}$ is a semilattice congruence;*

(ii) *$\sqrt{\mathscr{X}} = \sigma_1$;*

(iii) *S is a semilattice of $\sqrt{\mathscr{X}}$-simple semigroups.*

P r o o f. We will prove only the statement concerning the relation $\mathscr{J}$.

(i) $\Longrightarrow$ (ii). Let $\sqrt{\mathscr{J}}$ be a semilattice congruence. Then for all $a, b \in S$, $(ab, a^2b) \in \sqrt{\mathscr{J}}$. From this it follows that for every $a, b \in S$ there exists $n \in \mathbb{Z}^+$ such that $(ab)^n \in Sa^2S$, and by Theorem 1 [4], S is a semilattice of Archimedean semigroups. Now by Theorem 3 [5] we have that $\rightarrow$ is a quasi-order. Assume that $(a, b) \in \sqrt{\mathscr{J}}$. Then $a \in \Sigma_1(b)$ and $b \in \Sigma_1(a)$ and by transitivity of $\rightarrow$ we have that $\Sigma_1(a) = \Sigma_1(b)$. Thus $(a, b) \in \sigma_1$. Therefore, $\sqrt{\mathscr{J}} \subseteq \sigma_1 \subseteq \sigma$, and since σ is the smallest semilattice congruence on S, we obtain that $\sqrt{\mathscr{J}} = \sigma_1$.

(ii) $\Longrightarrow$ (i). We have that $\sqrt{\mathscr{J}} = \sigma_1 = \sqrt{\sigma_1}$, so by Theorem 1 we obtain that $\sqrt{\mathscr{J}}$ is a semilattice congruence.

(i) $\Longrightarrow$ (iii). This is obvious.

(iii) $\Longrightarrow$ (ii). Let S be a semilattice Y of $\sqrt{\mathscr{J}}$-simple semigroups S_α, $\alpha \in Y$, and let ϱ be the corresponding semilattice congruence. Since the ϱ-classes are semilattice indecomposable semigroups, we then have that $\varrho = \sigma_1$ $(= \sigma)$. By Theorem 1 we have that $\sqrt{\mathscr{J}} \subseteq \sigma_1$. Let $(a, b) \in \sigma_1$, $a \in S_\alpha$, $b \in S_\beta$, $\alpha, \beta \in Y$. Then $a \rightarrow b$ and

$b \to a$, whence $\alpha = \beta$. Thus $a, b \in S_\alpha$ and so $(a,b) \in \sqrt{\mathscr{J}}$. Hence, $\sigma_1 \subseteq \sqrt{\mathscr{J}}$, so $\sqrt{\mathscr{J}} = \sigma_1$. $\square$

Theorem 3. *The following conditions on a semigroup S are equivalent:*
(i) *S is a semilattice (band) of nil-extensions of simple semigroups;*
(ii) *$\tau(\mathscr{J})$ is a semilattice (band) congruence;*
(iii) *$\tau(\mathscr{J}) = \sqrt{\mathscr{J}} = \sigma_1$; $(\tau(\mathscr{J}) = \sqrt{\mathscr{J}} \subseteq \sigma_1)$*
(iv) *$(\forall a \in S)(\forall b \in S^1)(ab, a^2b) \in \tau(\mathscr{J})$.*

P r o o f. (i) $\Longrightarrow$ (ii). Let S be a semilattice Y of nil-extensions of simple semigroups S_α, $\alpha \in Y$ and let ϱ be the corresponding semilattice congruence. Then for $(a,b) \in \varrho$ there exists $n \in \mathbb{Z}^+$ such that $a^n \mathscr{J} b^n$. Thus $\varrho \subseteq \tau(\mathscr{J})$. If $(a,b) \in \tau(\mathscr{J})$, then $a^n \mathscr{J} b^n$, for some $n \in \mathbb{Z}^+$. If $a \in S_\alpha$, $b \in S_\beta$, $\alpha, \beta \in Y$, then by Lemma 9 [4] we have that $\alpha = \beta$, so $a \varrho b$. Hence $\tau(\mathscr{J}) \subseteq \varrho$. Therefore, $\tau(\mathscr{J}) = \varrho$, i.e. $\tau(\mathscr{J})$ is a semilattice congruence.

(ii) $\Longrightarrow$ (iii). Let $\tau(\mathscr{J})$ be a semilattice congruence. Since $\tau(\mathscr{J}) \subseteq \sqrt{\mathscr{J}}$, we then have that S is a semilattice of $\sqrt{\mathscr{J}}$-simple semigroups. By Theorem 2 we obtain that

$$\tau(\mathscr{J}) \subseteq \sqrt{\mathscr{J}} = \sigma_1 (= \sigma),$$

whence we conclude that (iii) holds.

(iii) $\Longrightarrow$ (ii). By Theorem 2 we have that $\sqrt{\mathscr{J}}$ is a semilattice congruence. Therefore, $\tau(\mathscr{J})$ is also a semilattice congruence.

(ii) $\Longrightarrow$ (iv). This is obvious.

(iv) $\Longrightarrow$ (i). From (iv) it follows that for every $a \in S$ there exists $n \in \mathbb{Z}^+$ such that $a^n \mathscr{J} a^{2n}$, i.e. S is an intra-π-regular semigroup. On the other hand, for any $a, b \in S$ there exist $n \in \mathbf{Z}^+$ such that $(ab)^n \in Sa^2S$. By Theorem 1, [4] S is a semilattice of Archimedean semigroups. Now by Theorem 2.12 [12] we have that S is a semilattice of nil-extensions of simple semigroups. $\square$

Corollary 1. *The following conditions on a semigroup S are equivalent:*
(i) *S is a semilattice of nil-extensions of left simple semigroups;*
(ii) *$\tau(\mathscr{L}) = \sqrt{\mathscr{L}} = \sigma_1$;*
(iii) *$\tau(\mathscr{L})$ is a semilattice congruence;*
(iv) *$(\forall a, b \in S)(ab, ba^2) \in \tau(\mathscr{L})$.*

Corollary 2. *The following conditions an a semigroup S are equivalent:*
(i) *S is a semilattice of nil-extensions of groups;*
(ii) *$\tau(\mathscr{H})$ is a semilattice congruence;*
(iii) *$\tau(\mathscr{H}) = \sqrt{\mathscr{H}} = \sigma_1$;*

(iv) $(\forall a, b \in S)(ab, (ba)^2) \in \tau(\mathscr{H})$.

By a *right regular band* we mean a band satisfying the identity $xyx = yx$ [7]. By the following theorem we describe semigroups in which $\tau(\mathscr{L})$ is a band congruence.

Theorem 4. *A semigroup S is a right regular band of nil-extensions of left simple semigroups if and only if $\tau(\mathscr{L})$ is a band congruence on S.*

Proof. Let S by a right regular band Y of nil-extensions of left simple semigroups S_α, $\alpha \in Y$. It is clear that $\tau(\mathscr{L})$ is reflexive and symmetric. If $(a, b) \in \tau(\mathscr{L})$, then $a, b \in S_\alpha$, for some $\alpha \in Y$, and by this it follows that $\tau(\mathscr{L})$ is a transitive relation. Also, for any $x \in S$ we have that $ax, bx \in S_\beta$, for some $\beta \in Y$. Thus $(ax, bx) \in \tau(\mathscr{L})$, so $\tau(\mathscr{L})$ is a right congruence. Similarly we obtain that $\tau(\mathscr{L})$ is a left congruence. Therefore, $\tau(\mathscr{L})$ is a congruence relation on S. Since S is left π-regular we have that $\tau(\mathscr{L})$ is a band congruence.

Conversely, let $\tau(\mathscr{L})$ be a band congruence. Then S is left π-regular and $\tau(\mathscr{L})$-classes are left Archimedean and left π-regular subsemigroups of S, so these are nil-extensions of left simple semigroups. Further, for every $x, y \in S$ there exists $n \in \mathbb{Z}^+$ such that

$$(xy)^n \mathscr{L} (xy)^{2n} \mathscr{L} (yx)^{2n-1} y.$$

Since $\mathscr{L} \subseteq \tau(\mathscr{L})$, we have that

$$(xy)^n \tau(\mathscr{L}) (yx)^{2n-1} y$$

whence

$$xy \tau(\mathscr{L}) (xy)^n \tau(\mathscr{L}) (yx)^{2n-1} y.$$

From this it follows that

$$xyx \tau(\mathscr{L}) (yx)^{2n} \tau(\mathscr{L}) yx.$$

Hence, $S/\tau(\mathscr{L})$ is a right regular band and S is a right regular band of nil-extensions of left simple semigroups. □

In a similar way we can prove the following two theorems:

Theorem 5. *$\sqrt{\mathscr{H}}$ is a band congruence on a semigroup S if and only if S is a band of $\sqrt{\mathscr{H}}$-simple semigroups.*

Theorem 6. *The following conditions on a semigroup S are equivalent:*
(i) *$\tau(\mathscr{H})$ is a band congruence;*
(ii) *S is a band of nil-extensions of groups;*
(iii) $(\forall a, b \in S) ab \tau(\mathscr{H}) a^2 b \tau(\mathscr{H}) ab^2$.

Problems. We state the following problems: (i) Desribe bands of $\sqrt{\mathscr{J}}$- ($\sqrt{\mathscr{L}}$-) simple semigroups, (ii) Describe semigroups in which $\sqrt{\mathscr{X}}$, $\mathscr{X} \in \{\mathscr{J}, \mathscr{D}, \mathscr{L}, \mathscr{R}, \mathscr{H}\}$, is a congruence.

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