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ON L_w^2-QUASI-DERIVATIVES FOR SOLUTIONS OF PERTURBED GENERAL QUASI-DIFFERENTIAL EQUATIONS

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Abstract. This paper is concerned with square integrable quasi-derivatives for any solution of a general quasi-differential equation of nth order with complex coefficients $M[y]-\lambda wy = wf(t, y^{[0]}, \ldots, y^{[n-1]})$, $t \in [a, b)$ provided that all rth quasi-derivatives of solutions of $M[y]-\lambda wy = 0$ and all solutions of its normal adjoint $M^+[z]-\bar{\lambda}wz = 0$ are in $L_w^2(a, b)$ and under suitable conditions on the function f.

Keywords: quasi-differential operators, regular, singular, bounded and square integrable solutions

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1. Introduction

In [16] Anton Zettl proved, under suitable conditions on f, that $y^{(j)} \in L^2[0, \infty)$, $j = 0, 1, \ldots, n-1$ for any solution y of $M[y] = f(t, y)$ provided that all jth derivatives of solutions of $M[y] = 0$ and all solutions of $M^+[y] = 0$ are in $L^2[0, \infty)$, when M is a regular ordinary linear differential operator of order n with coefficients which are locally integrable on $[0, \infty)$ and M^+ is the formal adjoint of M. The case $j = 0$ was considered in [16] for a general nth order M and for $n = 2$ by Bradley [1]. Also, in [14] Wong proved that all solutions of a perturbed linear differential equation belong to $L^2[0, \infty)$ assuming that all solutions of the unperturbed equation possess the same property. These results were generalized by Ibrahim in [10] for general ordinary quasi-differential equations of nth order.

Our objective in this paper is to extend the results of Ibrahim, Wong and Zettl in [10], [14], [15] and [16] to a general ordinary quasi-differential expression M of order n with complex coefficients and to prove, under suitable conditions on f, that the quasi-derivatives $y^{[r]} \in L^2_w(a,b)$, $r = 0, 1, \ldots, n-1$ for any solution y of

$$M[y] - \lambda w y = w f(t, y^{[0]}, \ldots, y^{[n-1]}) \quad (\lambda \in \mathbb{C}) \quad \text{on } [a,b] \tag{1.1}$$

provided that all rth quasi-derivatives of solutions of the equation

$$M[y] - \lambda w y = 0 \tag{1.2}$$

and all solutions of

$$M^+[y] - \bar{\lambda} w y = 0 \tag{1.3}$$

are in $L^2_w(a,b)$. These results are extensions of those proved by Ibrahim in [10]. Our approach is based on an extension of Gronwall's inequality used by Bradley and due to Gollwitzer [6], on a technical lemma from Goldberg's book [5] and on an appropriate formulation of the variation of parameters formula.

We deal throughout with a quasi-differential expression M of an arbitrary order n defined by a Shin-Zettl matrix on the interval $I = (a,b)$. The left-hand end-point of I is assumed to be regular but the right-hand end-point may be either regular or singular.

2. Notation and preliminaries

The set $Z_n(I)$ of Shin-Zettl matrices on the interval I consists of $n \times n$-matrices $A = \{a_{rs}\}$ whose entries are complex-valued functions on I which satisfy the following conditions:

$$\begin{cases} a_{rs} \in L^1_{\mathrm{loc}}(I), & & (1 \leqslant r, s \leqslant n,\ n \geqslant 2), \\ a_{r,r+1} \neq 0 & \text{a.e. on } I, & (1 \leqslant r \leqslant n-1), \\ a_{rs} = 0 & \text{a.e. on } I, & (2 \leqslant r+1 < s \leqslant n). \end{cases} \tag{2.1}$$

For $A \in Z_n(I)$, the quasi-derivatives associated with A are defined by

$$\begin{cases} y^{[0]} := y, \\ y^{[r]} := a_{r,r+1}^{-1} \left\{ (y^{[r-1]})' - \displaystyle\sum_{s=1}^{r} a_{rs} y^{[s-1]} \right\}, \ (1 \leqslant r \leqslant n-1), \\ y^{[n]} := (y^{[n-1]})' - \displaystyle\sum_{s=1}^{n} a_{ns} y^{[s-1]} \end{cases} \tag{2.2}$$

where the prime $'$ denotes differentiation.

The quasi-differential expression M associated with A is given by

$$M[y] = \mathrm{i}^n y^{[n]}, \qquad (n \geqslant 2), \tag{2.3}$$

this being defined on the set

$$V(M) := \{y\colon\ y^{[r-1]} \in AC_{\mathrm{loc}}(I),\ r = 1, 2, \ldots, n\}, \tag{2.4}$$

where $AC_{\mathrm{loc}}(I)$ denotes the set of all functions which are absolutely continuous on every compact subinterval of I.

The formal adjoint M^+ of M is defined by the matrix $A^+ \in Z_n(I)$ given by

$$A^+ = -L^{-1} A^* L, \tag{2.5}$$

where A^* is the conjugate transpose of A and L is the non-singular $n \times n$-matrix,

$$L := \Big\{(-1)^r \delta_{r,n+1-s}\Big\}, \qquad (1 \leqslant r, s \leqslant n), \tag{2.6}$$

δ being the Kronecker delta. If $A^+ = \{a^+_{rs}\}$, then it follows that

$$a^+_{rs} = (-1)^{r+s+1} \bar{a}_{n-s+1,n-r+1} \quad \text{for each} \quad r \quad \text{and} \quad s. \tag{2.7}$$

The quasi-derivatives associated with A^+ are therefore

$$\begin{cases} y^{[0]}_+ := y, \\ y^{[r]}_+ := (\bar{a})^{-1}_{n-r,n-r+1} \Big\{ (y^{[r-1]}_+)' - \displaystyle\sum_{s=1}^{r} (-1)^{r+s+1} \bar{a}_{n-s+1,n-r+1} y^{[s-1]}_+ \Big\}, \\ \hfill (1 \leqslant r \leqslant n-1), \\ y^{[n]}_+ := (y^{[n-1]}_+)' - \displaystyle\sum_{s=1}^{n} (-1)^{n+s+1} \bar{a}_{n-s+1,1} y^{[s-1]}_+ \end{cases} \tag{2.8}$$

and

$$M^+[y] = \mathrm{i}^n y^{[n]}_+ \quad (n \geqslant 2) \tag{2.9}$$

for all y in

$$V(M^+) := \{y\colon\ y^{[r-1]}_+ \in AC_{\mathrm{loc}}(I),\ r = 1, 2, \ldots, n\}. \tag{2.10}$$

Note that $(A^+)^+ = A$ and so $(M^+)^+ = M$. We refer to [3], [9] and [17] for a full account of the above and the subsequent results on quasi-differential expressions.

Let the interval I have end-points a and b, $-\infty \leqslant a < b \leqslant \infty$, and let w be a function which satisfies

$$w > 0 \quad \text{almost everywhere on } I, \quad w \in L^1_{\text{loc}}(I). \tag{2.11}$$

The equation

$$M[y] = \lambda w y \quad (\lambda \in \mathbb{C}) \tag{2.12}$$

on I is said to be *regular* at the left end-point a if for all $X \in (a,b)$,

$$a \in \mathbb{R}, \quad a_{rs} \in L^1[a,X], \qquad (r,s = 1,2,\dots,n). \tag{2.13}$$

Otherwise (2.12) is said to be *singular* at a. Similarly we define the terms regular and singular at b. If $(2,12)$ is regular at both end-points, then it is said to be regular; in this case we have

$$a,\ b \in \mathbb{R}, \quad a_{rs} \in L^1(a,b) \quad (r,s = 1,2,\dots,n); \tag{2.14}$$

see [2], [8] and [9].

We shall be concerned with the case when a is a regular end-point for (2.12), the end-point b being allowed to be either regular or singular. Note that in view of (2.7), an end-point of the interval I is regular for $(2,12)$ if and only if it is regular for the equation

$$M^+[y] = \bar{\lambda} w y \quad (\lambda \in \mathbb{C}). \tag{2.15}$$

Let $L^2_w(a,b)$ denote the usual weighted L^2-space with the inner-product

$$(f,g) = \int_a^b f(x)\overline{g(x)}w(x)\,\mathrm{d}x \tag{2.16}$$

and the norm $\|f\| := (f,f)^{1/2}$; this is a Hilbert space provided we identify functions which differ only on a null space.

Denote by $S(M)$ and $S(M^+)$ the sets of all solutions of

$$M[y] - \lambda w y = 0 \quad \text{and} \quad M^+[y] - \bar{\lambda} w y = 0,$$

respectively, and let $S^r(M) = \{y^{[r]} | M[y] - \lambda w y = 0,\ r = 0,1,\dots,n-1\}$ denote the set of all quasi-derivatives of solutions of $M[y] - \lambda w y = 0$, etc. Let $\varphi_k(t,\lambda)$ for

$k = 1, 2, \ldots, n$ be the solutions of the homogeneous equation (1.2) determined by the initial conditions

$$\varphi_k^{[r]}(t_0, \lambda) = \delta_{k,r+1} \qquad \text{for all } t_0 \in [a, b), \tag{2.17}$$

$(k = 1, 2, \ldots, n;\ r = 0, 1, \ldots, n-1)$. Then $\varphi_k^{[r]}(t_0, \lambda)$ is continuous in (t, λ) for $a < t < b$, $|\lambda| < \infty$, and for fixed t it is entire in λ. Let $\varphi_k^+(t, \lambda)$ for $k = 1, 2, \ldots, n$ be the solutions of the homogeneous equation (1.3) determined by the initial conditions

$$(\varphi_k^+)^{[r]}(t_0) = (-1)^{k+r}\delta_{k,n-r}, \quad \text{for all } t_0 \in [a, b) \tag{2.18}$$

$(k = 1, 2, \ldots, n;\ r = 0, 1, \ldots, n-1)$.

Suppose $a < t_0 < b$. According to Gilbert [4, Section 3] and Ibrahim [11, Section 3], a solution of $M[\varphi] - \lambda w\varphi = wf$, $f \in L^1_w(a, b)$ satisfying $\varphi^{[r]}(t_0, \lambda) = \alpha_{r+1}$, $r = 0, 1, \ldots, n-1$, is given by

$$\begin{aligned}\varphi(t, \lambda) = \sum_{j=1}^{n} \alpha_j(\lambda)\varphi_j(t, \lambda_0) + \big((\lambda - \lambda_0)/(\mathrm{i}^n)\big) \sum_{j,k=1}^{n} \xi^{jk}\varphi_j(t, \lambda_0)\\ \times \int_{t_0}^{t} \overline{\varphi_j^+(s, \lambda_0)} f(s) w(s)\,\mathrm{d}s\end{aligned} \tag{2.19}$$

where $\varphi_k^+(t, \lambda)$ stands for the complex conjugate of $\varphi_k(t, \lambda)$, and for each j, k, ξ^{jk} is a constant which is independent of t, λ (but does depend in general on t), also see [9, Corollary 3.10] and [17, Theorem 3].

Theorem 2.1 (Existence and Uniqueness Theorem). *Suppose $f \in L^1_w(a, b)$ and suppose that the conditions (2.1) are satisfied. Then given any complex numbers $\alpha_j \in \mathbb{C}$, $j = 0, \ldots, n-1$ and $t_0 \in (a, b)$ there exists a unique solution of $M[\varphi] - \lambda\varphi w = wf$ in (a, b) which satisfies*

$$\varphi^{[r]}(t_0, \lambda) = \alpha_{r+1}, \qquad r = 0, \ldots, n-1.$$

P r o o f. See [2, Theorem 3.10.1], [8, Theorem 16.2.2] and [9, Theorem 1.11]. $\square$

Our first lemma is a form of the variation of parameters formula. However it is different from the form of this formula generally found in textbooks and literature. For the variation of parameters formula for general quasi-differential equations, see [11, Section 3] and [17, Theorem 3]. These contain Lemma 3.1 as a special case, see [16].

Lemma 3.1. *Suppose f is a locally $L^1_w(a,b)$ function and $\varphi(t,\lambda)$ is the solution of $M[y]-\lambda wy = wf$ satisfying*

$$\varphi^{[r]}(t_0,\lambda) = \alpha_{r+1}(\lambda), \quad t_0 \in [a,b) \quad \text{for all } r = 0,1,\dots,n-1.$$

Then

$$\begin{aligned}(3.1)\qquad \varphi^{[r]}(t,\lambda) = \sum_{j=1}^{n} \alpha_j(\lambda)\varphi_j^{[r]}(t,\lambda_0) + \big((\lambda-\lambda_0)/(\mathrm{i}^n)\big) \sum_{j,k=1}^{n} \xi^{jk}\varphi_j^{[r]}(t,\lambda_0) \\ \times \int_a^t \overline{\varphi_j^{+}(s,\lambda_0)} f(s)w(s)\,\mathrm{d}s \quad \text{for } r = 0,1,\dots,n-1.\end{aligned}$$

Crucial in the study of boundedness of solutions of quasi-differential equations is the fundamental inequality of Gronwall, see [7]. Here, we will also need the following variant which may be found in [13].

Lemma 3.2. *Let $u(t)$, $v(t)$ be two non-negative functions, locally integrable on $[a,b)$. Then the inequality*

$$u(t) \leqslant c_0 + \int_a^t v(s)u^p(s)\,\mathrm{d}s, \quad c_0 > 0,$$

for $0 \leqslant p < 1$, implies that

$$(3.2)\qquad u(t) \leqslant \left(c_0^{1-p} + (1-p)\int_a^t v(s)\,\mathrm{d}s\right)^{1/(1-p)}.$$

In particular, if $v(t) \in L^1(a,b)$, then (3.2) implies that $u(t)$ is bounded.

The next lemma is a special case of an extension of the well-known Gronwall inequality due to Gollwitzer [6] (See also Willett [12] and Willet-Wong [13]).

Lemma 3.3. *Let u, z, g, h be continuous non-negative functions on $[a,b)$ and suppose that*

$$u(t) \leqslant z(t) + g(t)\left(\int_a^t u^2(s)h(s)\,\mathrm{d}s\right)^{\frac{1}{2}} \quad \text{for } t \geqslant a.$$

Then

$$u(t) \leqslant z(t) + g(t)\left(\int_a^t 2z^2(s)h(s)\exp\left[\int_a^s 2g^2(x)h(x)\,\mathrm{d}x\right]\mathrm{d}s\right)^{\frac{1}{2}} \text{ for } t \geqslant a.$$

The next two lemmas have been proved in [9] for the general case.

Lemma 3.4 ([9, Proposition 3.23]). *If all solutions of $M[\varphi] = \lambda_0 w\varphi$ are bounded on $[a,b)$ for some $\lambda_0 \in \mathbb{C}$ and $\varphi_j^+ \in L_w^1(a,b)$ for $j = 1,2,\dots,n$, then all solutions of $M[\varphi] = \lambda w\varphi$ are also bounded on $[a,b)$ for all $\lambda \in \mathbb{C}$.*

Lemma 3.5 ([9, Proposition 3.24]). *If all solutions of $M[\varphi] = \lambda_0 w\varphi$ and $M^+[\varphi] = \bar\lambda_0 w\varphi$ are in $L_w^2(a,b)$ for some $\lambda_0 \in \mathbb{C}$, then all solutions of $M[\varphi] = \lambda w\varphi$ and $M^+[\varphi] = \bar\lambda w\varphi$ are in $L_w^2(a,b)$ for all $\lambda \in \mathbb{C}$.*

4. Main results

Suppose there exist non-negative continuous functions $k(t)$ and $h_i(t)$, $i = 0,\dots, n-1$ such that

$$|f(t, y^{[0]},\dots,y^{[n-1]})| \leqslant k(t) + \sum_{i=0}^{n-1} h_i(t)|y^{[i]}|^\sigma \tag{4.1}$$

for $t \geqslant a$, $-\infty < y^{[i]} < \infty$, for each $i = 0,\dots,n-1$; $0 \leqslant \sigma \leqslant 1$.

Theorem 4.1. *Suppose f satisfies (4.1) with $\sigma = 1$, $s^r(M) \cup S(M^+) \subset L^\infty(a,b)$ for some $r = 0,1,\dots,n-1$ and some $\lambda_0 \in \mathbb{C}$ and*
(i) *$k(t) \in L_w^1(a,b)$ for all $t \in [a,b)$,*
(ii) *$h_i(t) \in L_w^1(a,b)$ for all $t \in [a,b)$, $i = 0,1,\dots,n-1$.*

Then $\varphi^{[r]}(t,\lambda)$ is bounded on $[a,b)$ for any solution $\varphi(t,\lambda)$ of the equation (1.1) for all $\lambda \in \mathbb{C}$.

P r o o f. Note that (4.1) and Theorem 2.1 imply that all solutions exist on the entire interval $[a,b)$, see [2, Chapter 3], [10], [14] and [15].

Let $\{\varphi_1(t,\lambda_0),\dots,\varphi_n(t,\lambda_0)\}$ and $\{\varphi_1^+(t,\lambda_0),\dots,\varphi_n^+(t,\lambda_0)\}$ be two sets of linearly independent solutions of (1.2) and (1.3), respectively and let $\varphi(t,\lambda)$ be any solution of (1.1) on $[a,b)$. Then by Lemma 3.1 we have

$$\begin{aligned}\varphi^{[r]}(t,\lambda) = {} & \sum_{j=1}^{n}\alpha_j(\lambda)\varphi_j^{[r]}(t,\lambda_0) + \bigl((\lambda-\lambda_0)/(\mathrm{i}^n)\bigr)\sum_{j,k=1}^{n}\xi^{jk}\varphi_j^{[r]}(t,\lambda_0)\\ & \times\int_a^t \overline{\varphi_j^+(s,\lambda_0)}f(s,\varphi^{[0]},\dots,\varphi^{[n-1]})w(s)\,\mathrm{d}s\\ & \text{for } r=0,\dots,n-1.\end{aligned}\tag{4.2}$$

Hence

$$\begin{aligned}|\varphi^{[r]}(t,\lambda)| \leqslant {} & \sum_{j=1}^{n}|\alpha_j(\lambda)||\varphi_j^{[r]}(t,\lambda_0)| + |\lambda-\lambda_0|\sum_{j,k=1}^{n}|\xi^{jk}||\varphi_j^{[r]}(t,\lambda_0)|\\ & \times\int_a^t |\overline{\varphi_k^+(s,\lambda_0)}|\biggl(k(s)+\sum_{i=0}^{n-1}h_i(s)|\varphi^{[i]}(s,\lambda)|\biggr)w(s)\,\mathrm{d}s,\\ & r=0,1,\dots,n-1.\end{aligned}\tag{4.3}$$

Since $k(t)\in L_w^1(a,b)$ and $\varphi_j^+(t,\lambda_0)$ is bounded on $[a,b)$ for some $\lambda_0\in\mathbb{C}$, we have $\varphi_j^+k\in L_w^1(a,b)$, $j=1,2,\dots,n$ for some $\lambda_0\in\mathbb{C}$. Setting

$$c_j = |\lambda-\lambda_0|\sum_{j,k=1}^{n}|\xi^{jk}|\int_a^b |\overline{\varphi_j^+(s,\lambda_0)}|k(s)w(s)\,\mathrm{d}s,\quad (j=1,2,\dots,n),\tag{4.4}$$

then

$$\begin{aligned}|\varphi^{[r]}(t,\lambda)| \leqslant {} & \sum_{j=1}^{n}\bigl(c_j+|\alpha_j(\lambda)|\bigr)|\varphi_j^{[r]}(t,\lambda_0)|\\ & + |\lambda-\lambda_0|\sum_{j,k=1}^{n}\sum_{i=0}^{n-1}|\xi^{jk}||\varphi_j^{[r]}(t,\lambda_0)|\\ & \times\int_a^t |\overline{\varphi_j^+(s,\lambda_0)}|h_i(s)|\varphi^{[i]}(s,\lambda)|w(s)\,\mathrm{d}s,\\ & r=0,1,\dots,n-1.\end{aligned}\tag{4.5}$$

By hypothesis, there exist positive constants K_0 and K_1 such that

$$|\varphi_j^{[r]}(t,\lambda_0)|\leqslant K_0 \quad\text{and}\quad |\varphi_j^+(t,\lambda_0)|\leqslant K_1 \quad\text{for all}\quad t\in[a,b),$$

$j=1,2,\ldots,n$ and some $r=0,1,\ldots,n-1$. Hence, by summing both sides of (4.5) from $r=0,\ldots,n-1$ we get

$$
\begin{aligned}
(4.6)\qquad \sum_{r=0}^{n-1}|\varphi^{[r]}(t,\lambda)| \leqslant (n-1)K_0\sum_{j=1}^{n}\big(c_j+|\alpha_j(\lambda)|\big)+(n-1)K_0K_1|\lambda-\lambda_0| \\
\times\sum_{j,k=1}^{n}|\xi^{jk}|\int_a^t\big(\max_{0\leqslant i\leqslant n-1}h_i(s)\big)\bigg(\sum_{i=0}^{n-1}|\varphi^{[i]}(s,\lambda)|\bigg)w(s)\,\mathrm{d}s.
\end{aligned}
$$

Applying Gronwall's inequality to (4.6) and using (ii), we deduce that $\sum\limits_{r=0}^{n-1}|\varphi^{[r]}(t,\lambda)|$ is finite and hence the result. $\square$

Remark. From [16, Section 3] and [9, Lemma 3.3], φ and $\varphi^{[j]}\in L^2_w(a,b)$ implies that $\varphi^{[r]}\in L^2_w(a,b)$ for any solution φ of the equation (1.1) for all $r=1,\ldots,j-1$, $1\leqslant j\leqslant n-1$.

Theorem 4.2. *Suppose f satisfies* (4.1) *with $\sigma=1$, $S^r(M)\cup S(M^+)\subset L^2_w(a,b)$ for some $\lambda_0\in\mathbb{C}$ and some $r=0,1,\ldots,n-1$ and that*

(i) $k(t)\in L^2_w(a,b)$,

(ii) $h_i(t)\in L^\infty(a,b)$, $i=0,1,\ldots,n-1$, *for all $t\in[a,b)$. Then $\varphi^{[r]}(t,\lambda)\in L^2_w(a,b)$ for any solution $\varphi(t,\lambda)$ of the equation* (1.1) *for all $\lambda\in\mathbb{C}$.*

Proof. Applying the Cauchy Schwartz inequality to the integral in (4.5) we get

$$
\begin{aligned}
(4.7)\qquad |\varphi^{[r]}(t,\lambda)| \leqslant{}& \sum_{j=1}^{n}\big(c_j+|\alpha_j(\lambda)|\big)|\varphi_j^{[r]}(t,\lambda_0)| \\
&+|\lambda-\lambda_0|\sum_{j,k=1}^{n}\sum_{i=0}^{n-1}|\xi^{jk}||\varphi_j^{[r]}(t,\lambda_0)| \\
&\times\bigg(\int_a^t\overline{|\varphi_j^+(s,\lambda_0)}|^2|h_i(s)|w\,\mathrm{d}s\bigg)^{\frac12}\bigg(\int_a^t|h_i(s)||\varphi^{[i]}(s,\lambda)|^2w\,\mathrm{d}s\bigg)^{\frac12}, \\
&r=0,1,\ldots,n-1.
\end{aligned}
$$

Since $\varphi_j^+(t,\lambda_0)\in L^2_w(a,b)$, $j=1,2,\ldots,n$ for some $\lambda_0\in\mathbb{C}$ and $h_i(t)\in L^\infty(a,b)$ by hypothesis, then $\varphi_j^+(t,\lambda_0)|h_i(t)|^{1/2}\in L^2_w(a,b)$, $j=1,2,\ldots,n$, $i=0,1,\ldots,n-1$. Let

$$
D_{ji}=\bigg(\int_a^t\overline{|\varphi_j^+(s,\lambda_0)}|^2|h_i(s)|w(s)\,\mathrm{d}s\bigg)^{\frac12},\qquad z(t)=\sum_{j=1}^{n}\big(c_j+|\alpha_j(\lambda)|\big)|\varphi_j^{[r]}(t,\lambda_0)|
$$

and $G(t)=|\lambda-\lambda_0|\sum\limits_{j,k=1}^{n}\sum\limits_{i=0}^{n-1}D_{ji}|\xi^{jk}||\varphi_j^{[r]}(t,\lambda_0)|$.

From Lemma 3.3 we have

$$|\varphi^{[r]}(t,\lambda)| \leqslant Z(t) + G(t)\left(\int_a^t 2Z^2(s)|h_i(s)| \exp\left[\int_a^s 2G^2(x)|h_i(x)|w \,\mathrm{d}x\right] w \,\mathrm{d}s\right)^{\frac{1}{2}}.$$

Since $\int_a^b Z^2(s)|h_i(s)|w(s)\,\mathrm{d}s$ and $\int_a^b G^2(x)|h_i(x)|w(x)\,\mathrm{d}x$ are both finite, we conclude that $|\varphi^{[r]}(t,\lambda)|$ is bounded by a linear combination of $L^2_w(a,b)$ functions $Z(t)$ and $G(t)$. Therefore, by using Lemma 3.5, $\varphi^{[r]}(t,\lambda) \in L^2_w(a,b)$ for all $\lambda \in \mathbb{C}$. □

Remark. If we use the Cauchy-Schwartz inequality for the integral in (4.5) as

$$\int_a^t |\overline{\varphi_j^+}||h_i||\varphi| w \,\mathrm{d}s \leqslant \left(\int_a^t |\overline{\varphi_j^+}|^2 |h_i|^2 w \,\mathrm{d}s\right)^{\frac{1}{2}} \left(\int_a^t |\varphi^{[i]}|^2 w \,\mathrm{d}s\right)^{\frac{1}{2}}, \ i = 0,\dots,n-1,$$

we also get the result. We refer to [14] and [15] for more details.

Corollary 4.3. *Suppose that* $f(t, y^{[0]}, \dots, y^{[n-1]}) = \sum\limits_{i=1}^{n-1} h_i(t) y^{[i]}$, $S^r(M) \cup S(M^+) \subset L^2_w(a,b)$ *for some* $\lambda_0 \in \mathbb{C}$ *and some* $r = 0,1,\dots,n-1$ *and that* $h_i(t) \in L^p_w(a,b)$ *for some* $p \geqslant 2$, $t \in [a,b)$; $i = 0,1,\dots,n-1$. *Then* $\varphi^{[r]}(t,\lambda) \in L^1_w(a,b)$ *for any solution* $\varphi(t,\lambda)$ *of the equation* (1.1) *for all* $\lambda \in \mathbb{C}$ *and all* $r = 0,1,\dots,n-1$.

P r o o f. The proof is similar to Theorem 4.2 and therefore omitted.
The special case $h_i(t) \equiv 0$, $i = 0,\dots,n-1$ and $k \in L^2_w(a,b)$ yields □

Corollary 4.4. *If all solutions of* $M[\varphi] = \lambda_0 w\varphi$ *and* $M^+[\varphi] = \bar{\lambda}_0 w\varphi$ *are in* $\in L^2_w(a,b)$ *for some* $\lambda_0 \in \mathbb{C}$ *and* $k \in L^2_w(a,b)$, *then all solutions of* $M[\varphi] - \lambda w\varphi = wk$ *are in* $L^2_w(a,b)$ *for all* $\lambda \in \mathbb{C}$.

Next, we consider (4.1) with $0 \leqslant \sigma < 1$, and arrive at the following:

Theorem 4.5. *Suppose* f *satisfies* (4.1) *with* $0 \leqslant \sigma < 1$, $S^r(M) \cup S(M^+) \subset L^2_w(a,b)$ *for some* $\lambda_0 \in \mathbb{C}$ *and some* $r = 0,1,\dots,n-1$ *and*
(i) $k(t) \in L^2_w(a,b)$ *for all* $t \in [a,b)$,
(ii) $h_i(t) \in L_w^{2/(1-\sigma)}(a,b)$ $0 \leqslant \sigma < 1$, $i = 0,1,\dots,n-1$.
Then $\varphi^{[r]}(t,\lambda) \in L^2_w(a,b)$ *for any solution* $\varphi(t,\lambda)$ *of the equation* (1.1) *for all* $\lambda \in \mathbb{C}$.

P r o o f. For $0 \leqslant \sigma < 1$, the proof is the same up to (4.5). In this case (4.5) becomes

$$(4.8) \qquad |\varphi^{[r]}(t,\lambda)| \leqslant \sum_{j=1}^n \bigl(c_j + |\alpha_j(\lambda)|\bigr)|\varphi_j^{[r]}(t,\lambda_0)| + |\lambda-\lambda_0| \sum_{j,k=1}^n \sum_{i=0}^{n-1} |\xi^{jk}| \times |\varphi_j^{[r]}(t,\lambda_0)| \int_a^t |\overline{\varphi_j^+(s,\lambda_0)}| h_i(s)|\varphi^{[i]}(s,\lambda)|^{\sigma} w(s)\,\mathrm{d}s, \quad r = 0,1,\dots,n-1.$$

Applying Cauchy-Schwartz inequality to the integral in (4.8), we find

$$\begin{aligned}(4.9)\qquad &\int_a^t |\overline{\varphi_j^+(s,\lambda_0)}|h_i(s)|\varphi(s,\lambda)|^\sigma w(s)\,\mathrm{d}s\\ &\leqslant \left(\int_a^t |\overline{\varphi_j^+(s,\lambda_0)}h_i(s)|^\mu w(s)\,\mathrm{d}s\right)^{\frac{1}{\mu}} \left(\int_a^t |\varphi^{[i]}(s,\lambda)|^2 w(s)\,\mathrm{d}s\right)^{\frac{\sigma}{2}},\end{aligned}$$

where $\mu = 2/(2-\sigma)$. Since $\varphi_j^+(s,\lambda_0) \in L_w^2(a,b)$ for some $\lambda_0 \in \mathbb{C}$, $j = 1,2,\dots,n$ and $h_i(s) \in L_w^{2/(1-\sigma)}(a,b)$ by hypothesis, then we have $\varphi_j^+(s,\lambda_0)h_i(s) \in L_w^\mu(a,b)$ for some $\lambda_0 \in \mathbb{C}$, $j = 1,2,\dots,n$; $i = 0,1,\dots,n-1$. Using this fact and (4.9) in (4.8), we obtain

$$\begin{aligned}(4.10)\qquad |\varphi^{[r]}(t,\lambda)| &\leqslant \sum_{j=1}^n \bigl(c_j + |\alpha_j(\lambda)|\bigr)|\varphi_j^{[r]}(t,\lambda_0)|\\ &+ K_0|\lambda-\lambda_0| \sum_{j,k=1}^n \sum_{i=0}^{n-1} |\xi^{jk}||\varphi_j^{[r]}(t,\lambda_0)| \left(\int_a^t |\varphi^{[i]}(s,\lambda)|^2 w(s)\,\mathrm{d}s\right)^{\frac{\sigma}{2}},\\ &r = 0,1,\dots,n-1,\end{aligned}$$

where $K_0 = \|\varphi_j^+(t,\lambda_0)h_i(s)\|_\mu$, $\|\cdot\|_\mu$ denotes the norm in $L_w^\mu(a,b)$. The inequality,

$$(4.11)\qquad (u+v)^2 \leqslant 2(u^2+v^2),$$

implies that

$$\begin{aligned}(4.12)\qquad |\varphi^{[r]}(t,\lambda)|^2 &\leqslant 4\sum_{j=1}^n \bigl(c_j^2 + |\alpha_j(\lambda)|^2\bigr)|\varphi_j^{[r]}(t,\lambda_0)|^2\\ &+ 4K_0^2|\lambda-\lambda_0|^2 \sum_{j,k=1}^n \sum_{i=0}^{n-1} |\xi^{jk}|^2|\varphi_j^{[r]}(t,\lambda_0)|^2 \left(\int_a^t |\varphi^{[i]}(s,\lambda)|^2 w(s)\,\mathrm{d}s\right)^{\sigma},\\ &r = 0,1,\dots,n-1.\end{aligned}$$

Setting $K_1 = \int_a^b |\varphi_j^{[r]}(t,\lambda_0)|^2 w(s)\,\mathrm{d}s$ for some $\lambda_0 \in \mathbb{C}$ and some $r = 0,\dots,n-1$; $j = 1,2,\dots,n$ and integrating (4.12) we obtain

$$\begin{aligned}(4.13)\qquad \int_a^t |\varphi^{[r]}(s,\lambda)|^2 w(s)\,\mathrm{d}s &\leqslant K_2 + 4K_0^2|\lambda-\lambda_0|^2 \sum_{j,k=1}^n \sum_{i=0}^{n-1} |\xi^{jk}|^2 \int_a^t |\varphi_j^{[r]}(s,\lambda_0)|^2\\ &\times \left[\left(\int_a^s |\varphi^{[i]}(x,\lambda)|^2 w(x)\,\mathrm{d}x\right)^{\sigma}\right] w(s)\,\mathrm{d}s,\end{aligned}$$

where

$$K_2 = 4\sum_{j=1}^{n}\bigl(c_j^2 + \alpha_j(\lambda)|^2\bigr)K_1.$$

An application of Lemma 3.2 to (4.13) for $0 \leqslant \sigma < 1$ and of Gronwall's inequality to (4.13) for $\sigma = 1$ yields the result. $\square$

Theorem 4.6. *Suppose f satisfies (4.1) with $0 \leqslant \sigma < 1$, $S^r(M) \cup S(M^+) \subset L^2_w(a,b) \cap L^\infty(a,b)$ for some $\lambda_0 \in \mathbb{C}$ and some $r = 0,1,\dots,n-1$ and that*
(i) *$k(t) \in L^2_w(a,b)$ for all $t \in [a,b)$,*
(ii) *$h_i(t) \in L^p_w(a,b)$ for any p, $1 \leqslant p \leqslant 2/(1-\sigma)$, $i = 0,1,\dots,n-1$.*

Then $\varphi^{[r]}(t,\lambda) \in L^2_w(a,b) \cap L^\infty(a,b)$ for any solution $\varphi(t,\lambda)$ of the equation (1.1) for all $\lambda \in \mathbb{C}$.

Proof. Since $S^r(M) \cup S(M^+) \subset L^2_w(a,b)$ for some $\lambda_0 \in \mathbb{C}$ and some $r = 0,1,\dots,n-1$, then $\varphi_j^{[r]}(t,\lambda_0)$, $\varphi_j^+(t,\lambda_0) \in L^q_w(a,b)$, $j = 1,2,\dots,n$ for every $q \geqslant 2$ and for some $\lambda_0 \in \mathbb{C}$, $r = 0,1,\dots,n-1$.

First, suppose that $h_i(t) \in L^p_w(a,b)$ for some p, $1 \leqslant p \leqslant 2$. Setting

$$K_0 = \|\varphi^{[r]}(t,\lambda_0)\|_\infty \quad\text{and}\quad K_1 = \|\varphi_j^+(t,\lambda_0)\|_\infty,\ j = 1,2,\dots,n,$$

for some $\lambda_0 \in \mathbb{C}$ and some $r = 0,1,\dots,n-1$, we have from (4.8)

$$\begin{aligned}(4.14)\quad |\varphi^{[r]}(t,\lambda)| \leqslant K_0 &\sum_{j=1}^{n}(c_j + |\alpha_j(\lambda)|)\\ &+ K_0K_1|\lambda-\lambda_0|\biggl(\sum_{j,k=1}^{n}\sum_{i=0}^{n-1}|\xi^{jk}|\int_a^t h_i(s)|\varphi^{[i]}(s,\lambda)|^\sigma w(s)\,\mathrm{d}s\biggr).\end{aligned}$$

Since $h_i(t) \in L^p_w(a,b)$, $1 \leqslant p \leqslant 2$, then Lemma 3.2 together with Gronwall's inequality implies that $\varphi^{[r]}(t,\lambda) \in L^\infty(a,b)$ for all $\lambda \in \mathbb{C}$, i.e., there exists a positive constant K_2 such that

$$(4.15)\qquad |\varphi^{[r]}(t,\lambda)| \leqslant K_2 \text{ for all } \lambda \in \mathbb{C},\ t \in [a,b),\ r = 0,1,\dots,n-1.$$

From (4.8) and (4.15) we obtain

$$|\varphi^{[r]}(t,\lambda)| \leqslant \sum_{j=1}^{n}\bigl(c_j + |\alpha_j(\lambda)| + K_3\bigr)|\varphi_j^{[r]}(t,\lambda_0)|$$

for an appropriate constant K_3. Since $\varphi_j^{[r]}(t,\lambda_0) \in L^2_w(a,b)$ for some $\lambda_0 \in \mathbb{C}$ and some $r = 0,1,\dots,n-1$, this proves $\varphi^{[r]}(t,\lambda) \in L^p_w(a,b)$ for all $\lambda \in \mathbb{C}$, $1 \leqslant p \leqslant 2$.

Next, suppose that $h_i(t) \in L^p_w(a,b)$, $2 < p \leqslant 2/(1-\sigma)$; $i = 0, 1, \dots, n-1$. Define $q \geqslant 2$ by

$$\frac{1}{q} = \frac{2-\sigma}{2} - \frac{1}{p}$$

(which is possible because of the restriction on p). Thus $\varphi_j^{[r]}(t,\lambda_0)\varphi_j^{+}(t,\lambda_0) \in L^q_w(a,b)$ and $\varphi_j^{+}(t,\lambda_0)h_i(t) = L^{\mu}_w(a,b)$, $\mu = 2/(2-\sigma)$.

Repeating the same argument in the proof of Theorem 4.5, from (4.8) to (4.13), we obtain that $\varphi^{[r]}(t,\lambda) \in L^2_w(a,b)$. Returning to (4.9), we find that the integral on the left-hand side is bounded, which implies, by (4.8), that

$$|\varphi^{[r]}(t,\lambda)| \leqslant \sum_{j=1}^{n} \big(c_j + |\alpha_j(\lambda)| + K_3\big)|\varphi_j^{[r]}(t,\lambda_0)|,$$

for an appropriate constant K_3. Since $\varphi_j^{[r]}(t,\lambda_0) \in L^{\infty}(a,b)$, this completes the proof. We refer to [10], [14] and [16] for more details. □

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