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DISTINGUISHED EXTENSIONS OF AN MV-ALGEBRA

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1. Introduction

The notions of distinguished extension and distinguished completion of a lattice ordered group were investigated by R. N. Ball in [2], [4], [5].

The distinguished completion of a lattice ordered group G is denoted by $E(G)$. It is defined uniquely up to isomorphisms leaving all elements of G fixed.

In [4] and [5] it is proved that $E(G)$ is in a certain sense better than several other types of completions of the lattice ordered group G (cf. the diagram of such completions which is given in [4]).

In [5], $E(G)$ was described by applying the construction of the maximal essential extension in the category of distributive lattices which was dealt with by Balbes [1] and Ball [3].

Let $\mathcal{A}$ be an MV-algebra with the underlying set A. In view of the well-known result of Mundici [9] there exists an abelian lattice ordered group G with a strong unit u such that, under the notation as in [8] (cf. also Section 2 above) we have

(1) $$\mathcal{A} = \mathcal{A}_0(G, u).$$

This implies that we obtain a lattice order on the set A; the corresponding lattice will be denoted by $\ell(\mathcal{A})$.

If $\mathcal{B}$ is an MV-algebra such that $\mathcal{A}$ is a subalgebra of $\mathcal{B}$, then $\mathcal{B}$ is said to be an extension of $\mathcal{A}$. If, moreover, for each $0 < b \in B$ (= the underlying set of $\mathcal{B}$) there exists $0 < a \in A$ with $a \leqslant b$, then $\mathcal{A}$ is called a dense subalgebra of $\mathcal{B}$.

In analogy with [5] (p. 89) we introduce the following definitions:

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1.1. Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be MV-algebras such that $\mathcal{B}$ is an extension of $\mathcal{A}$. Suppose that

(i) $\mathcal{A}$ is dense in $\mathcal{B}$;
(ii) if $b_1, b_2 \in B$ and $b_1 < b_2$, then there are $a_1, a_2 \in A$ such that $a_1 < a_2$ and the interval $[a_1, a_2]$ of $\ell(\mathcal{B})$ is projective to a subinterval of $[b_1, b_2]$ in $\ell(\mathcal{B})$.

Then $\mathcal{B}$ is called a distinguished extension of $\mathcal{A}$.

1.2. Definition. An MV-algebra is called distinguished if it has no proper distinguished extension.

1.3. Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be MV-algebras such that

(i) $\mathcal{B}$ is a distinguished extension of $\mathcal{A}$;
(ii) the MV-algebra $\mathcal{B}$ is distinguished.

Then $\mathcal{B}$ is said to be a distinguished completion of $\mathcal{A}$.

If a lattice ordered group G is an ℓ-subgroup of a lattice ordered group H, then we write $G \preceq H$. Similarly, if an MV-algebra $\mathcal{A}$ is a subalgebra of an MV-algebra $\mathcal{B}$, then we express this fact by writing $\mathcal{A} \preceq \mathcal{B}$.

In the present paper we prove the following results.

1.4. Proposition. *Let G and H be abelian lattice ordered groups with $G \preceq H$. Suppose that u is a strong unit in both G and H. Further suppose that $\mathcal{A}$ and $\mathcal{B}$ are MV-algebras such that*

$$\mathcal{A} = \mathcal{A}_0(G, u), \quad \mathcal{B} = \mathcal{A}_0(H, u).$$

Then the following conditions are equivalent:

(i) *H is a distinguished extension of G;*
(ii) *$\mathcal{B}$ is a distinguished extension of $\mathcal{A}$.*

1.5. Proposition. *Let $\mathcal{A}$ be an MV-algebra; suppose that* (1) *is valid. Put*

$$G_1 = E(G), \quad \mathcal{B} = \mathcal{A}_0(G_1', u),$$

where G_1' is the convex ℓ-subgroup of G_1 which is generated by the element u. Then $\mathcal{B}$ is a distinguished completion of $\mathcal{A}$.

We also prove that the distinguished completion of an MV-algebra $\mathcal{A}$ is defined uniquely up to isomorphisms leaving all elements of A fixed.

2. Preliminaries

First we remark that if in Definitions 1.1, 1.2 and 1.3 above the MV-algebras are replaced by lattice ordered groups, then these modified definitions can be applied for lattice ordered groups (cf. $(*)$ in Section 3 below, and the corresponding definitions in [5]).

For MV-algebras we apply the terminology and the notation as in Gluschankof [7] (cf. also the author's paper [8]). Thus an MV-algebra is an algebraic system

$$\mathcal{A} = (A; \oplus, \neg, *, 0, 1),$$

where A is a nonempty set, $\oplus$ and $*$ are binary operations, $\neg$ is a unary operation, 0 and 1 are unary operations on A such that the indentities (m_1)–(m_9) from [7] are satisfied.

We will systematically apply the following results which are due to Mundici [9] (Theorem 2.5 and 3.8); cf. also [8].

2.1. Proposition. *Let G be an abelian lattice ordered group with a strong unit u. Let A be the interval $[0, u]$ of G. For each a and b in A we put*

$$a \oplus b = (a + b) \wedge u, \quad \neg u = u - a, \quad 1 = u.$$

Further, let the operation $*$ on A be defined by (m_9). Then $\mathcal{A} = (A; \oplus, *, \neg, 0, u)$ is an MV-algebra.

If G and $\mathcal{A}$ are as in 2.1, then we denote $\mathcal{A} = \mathcal{A}_0(G, u)$.

2.2. Proposition. *Let $\mathcal{A}$ be an MV-algebra. Then there exists an abelian lattice ordered group G with a strong unit u such that $\mathcal{A} = \mathcal{A}_0(G, u)$.*

Let $\mathcal{A}$ be a given MV-algebra. From the construction of the lattice ordered group G with the property as in 2.2 performed in [9] we conclude that the following two lemmas are valid.

2.3. Lemma. *Let G and G' be abelian lattice ordered groups with strong units u and u'. Suppose that $\mathcal{A}$ is an MV-algebra such that $\mathcal{A} = \mathcal{A}_0(G, u)$ and $\mathcal{A} = (G', u')$. Then $u = u'$ and there is an isomorphism φ of G onto G' such that $\varphi(a) = a$ for each $a \in A$. Moreover, if $G \subseteq G'$ and $u = u'$, then $G = G'$.*

2.4. Lemma. *Let $\mathcal{A}$ and $\mathcal{B}$ be MV-algebras, $\mathcal{A} \preceq \mathcal{B}$, $\mathcal{A} = \mathcal{A}_0(G, u)$. Then* (i) *there exists an abelian lattice ordered group H such that u is a strong unit of H,*

$\mathcal{B} = \mathcal{A}_0(H,u)$ *and* (ii) *there is an isomorphism* φ *of* G *into* H *such that* $\varphi(a) = a$ *for each* $a \in A$.

If a lattice ordered group H is a distinguished extension of a lattice ordered group G, then we write

$$G \preceq_{\text{dist}} H.$$

An analogous notation will be applied for MV-algebras.

The following assertion is easy to verify.

2.5. Lemma. *Let* X, Y *and* Z *be lattice ordered groups such that* $X \preceq Z \preceq Y$. *If* $X \preceq_{\text{dist}} Y$, *then* $Z \preceq_{\text{dist}} Y$ *and* $X \preceq_{\text{dist}} Z$.

3. Proofs of 1.4 and 1.5

For the notion of projectivity of intervals in a lattice cf., e.g., Birkhoff [6].

Let L be a distributive lattice and let $[a,b]$ be an interval in L. For each $x \in L$ we put $x \ \pi \ ab = (x \vee a) \wedge b$. If, moreover, $[c,d]$ is an interval in L such that $c \ \pi \ ab = a$ and $d \ \pi \ ab = b$, then we say that a and b distinguish c from d. (Cf. [5].)

An easy calculation shows that

$(*)$ a and b distinguish c from d if and only if the interval $[a,b]$ is projective to subinterval of $[c,d]$.

The following lemma gives a deeper insight into the notion of projectivity of intervals in a distributive lattice. It seems to be folklore; the proof will be omitted.

3.1. Lemma. *Let* L *be a distributive lattice and let* $[a,b]$, $[c,d]$ *be intervals in* L. *Denote*

$$a \wedge c = u_1, \quad b \wedge d = v_1, \quad a \vee c = u_2, \quad b \vee d = v_2.$$

Then the following conditions are equivalent:

(i) *The intervals* $[a,b]$ *and* $[c,d]$ *are projective.*

(ii) *The relations*

$a \wedge v_1 = u_1$, $a \vee v_1 = b$, $c \wedge v_1 = u_1$, $c \vee v_1 = d$,
$b \wedge u_2 = a$, $b \vee u_2 = v_2$, $d \wedge u_2 = c$, $d \vee u_2 = v_2$ *are valid.*

The distributive law immediately yields

3.2. Lemma. *Let* L *be a distributive lattice. Let* $p, q \in L$ *and let* $[a,b]$, $[c,d]$ *be projective intervals in* L. *Denote* $x' = (x \vee p) \wedge q$ *for each* $x \in L$. *Then the intervals* $[a',b']$ *and* $[c',d']$ *are projective in* L.

The following lemma is a corollary of 3.1.

3.3. Lemma. *Let L_1 be a sublattice of a distributive lattice L. Let $[a,b]$, $[c,d]$ be intervals in L_1 and let $[a,b]^0$, $[c,d]^0$ be the corresponding intervals (with the endpoints a,b or c,d, respectively) in L. Then the following conditions are equivalent:*

(i) *The intervals $[a,b]$ and $[c,d]$ are projective in L_1.*
(ii) *The intervals $[a,b]^0$ and $[c,d]^0$ are projective in L.*

3.4. Lemma. *Let G and H be abelian lattice ordered groups with the same strong unit u. Suppose that H is a distinguished extension of G and that $\mathcal{A} = \mathcal{A}_0(G,u)$, $\mathcal{B} = \mathcal{A}_0(H,u)$. Then $\mathcal{B}$ is a distinguished extension of $\mathcal{A}$.*

Proof. In view of the definitions of $\mathcal{A}$ and $\mathcal{B}$ we conclude that $\mathcal{B}$ is an extension of $\mathcal{A}$. Let A or B be the underlying set of $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $0 < b \in B$. Then $b \in H^+$ and thus there exists $g \in G$ with $0 < g \leqslant b$. We obtain $g \leqslant u$, whence $g \in A$ and therefore $\mathcal{A}$ is a dense subalgebra of $\mathcal{B}$.

Let $b_1, b_2 \in B$, $b_1 < b_2$. Thus $b_1, b_2 \in H$. Hence there exist $b_1', b_2' \in H$ and $g_1, g_2 \in G$ such that

$$b_1 \leqslant b_1' < b_2' \leqslant b_2, \quad g_1 < g_2$$

and the intervals $[b_1', b_2']$, $[g_1, g_2]$ are projective in H. Denote

$$g_1' = (g_1 \vee 0) \wedge u, \quad g_2' = (g_2 \vee 0) \wedge u.$$

Then in view of 3.2, the intervals $[b_1', b_2']$ and $[g_1', g_2']$ are projective in H. Clearly $g_1', g_2' \in A$. According to 3.3, the intervals $[b_1', b_2']$, $[g_1', g_2']$ are projective in $\mathcal{B}$. Therefore $\mathcal{B}$ is a distinguished extension of $\mathcal{A}$. $\square$

3.5. Lemma. *Let G and H be abelian lattice ordered groups with the same strong unit u. Suppose that H is an extension of G, $\mathcal{A} = \mathcal{A}_0(G,u)$, $\mathcal{B} = \mathcal{A}_0(H,u)$ and that $\mathcal{B}$ is a distinguished extension of $\mathcal{A}$. Then H is a distinguished extension of G.*

Proof. a) First we verify that G is dense in H. Let $0 < h \in H$. There is a positive integer n with $h \leqslant nu$. Hence there are $b_1, b_2, \ldots, b_n \in H$ such that $0 \leqslant b_i \leqslant u$ for each $i \in \{1, 2, \ldots, n\}$ and $h = b_1 + b_2 + \ldots + b_n$. Then $b_{i(1)} > 0$ for some $i(1) \in \{1, 2, \ldots, n\}$. Since $b_{i(1)}$ belongs to B there is $a_{i(1)} \in A$ with $0 < a_{i(1)} \leqslant b_{i(1)}$. We get $a_{i(1)} \leqslant h$ and $a_{i(1)} \in G$; thus G is dense in H.

b) Now let $h_1, h_2 \in H$, $h_1 < h_2$. Since u is a strong unit in H there exists a positive integer n such that

$$-nu \leqslant h_1 < h_2 \leqslant nu. \tag{1}$$

Put $u_k = (-n+k)u$ for $k = 0, 1, 2, \ldots, 2n$. Hence we have a chain

$$-nu = u_0 < u_1 < u_2 < \ldots < u_{2n} = nu. \tag{2}$$

By considering the chains given in (1) and (2) and by applying the well-known theorem on refinements of finite chains in a modular lattice (cf. e.g., Birkhoff [6], Chapter V, Corollary to Theorem 5) we conclude that there is a chain

$$h_1 = y_0 \leqslant y_1 \leqslant y_2 \leqslant \ldots \leqslant y_{2n} = h_2$$

such that for each $k \in \{0, 1, 2, \ldots, 2n\}$ we have

$$[y_k, y_{k+1}] \sim [z_k, z'_k], \tag{3}$$

where the symbol $\sim$ denotes the projectivity of intervals in the lattice $\ell(H)$ and $[z_k, z'_k]$ is a subinterval of the interval $[u_k, u_{k+1}]$ in $\ell(H)$.

There exists $k(1) \in \{0, 1, 2, \ldots, 2n\}$ such that $y_{k(1)} < y_{k(1)+1}$. Then $z_{k(1)} < z'_{k(1)}$. Denote

$$v_{k(1)} = z_{k(1)} - u_{k(1)}, \quad v'_{k(1)} = z'_{k(1)} - u_{k(1)}.$$

Thus

$$0 \leqslant v_{k(1)} < v'_{k(1)} \leqslant u.$$

Because $\mathcal{B}$ is a distinguished extension of $\mathcal{A}$, there exists a subinterval $[t_{k(1)}, t'_{k(1)}]$ of $[v_{k(1)}, v'_{k(1)}]$ such that

$$[t_{k(1)}, t'_{k(1)}] \sim [q_{k(1)}, q'_{k(1)}],$$

where $q_{k(1)}, q'_{k(1)}$ are elements of A and $q_{k(1)} < q'_{k(1)}$. Denote

$$p_{k(1)} = q_{k(1)} + u_{k(1)}, \quad p'_{k(1)} = q'_{k(1)} + u_{k(1)}.$$

Then $p_{k(1)}, p'_{k(1)}$ are elements of G, $p_{k(1)} < p'_{k(1)}$ and the interval $[p_{k(1)}, p'_{k(1)}]$ is projective in $\ell(H)$ with a subinterval of $[z_{k(1)}, z'_{k(1)}]$. Since the relation of projectivity is transitive, $[p_{k(1)}, p'_{k(1)}]$ is projective to a subinterval of $[y_{k(1)}, y_{k(1)+1}]$ (cf. (3)), and therefore $[p_{k(1)}, p'_{k(1)})]$ is projective to a subinterval of $[h_1, h_2]$ in H.

Hence H is a distinguished extension of G. $\square$

From 3.4 and 3.5 we conclude that 1.4 is valid.

For an abelian lattice ordered group X let $\overline{X}^{\omega}$ and X^c have the same meaning as in [5] (pp. 109–110). Thus $\overline{X}^{\omega}$ is the strong projectable completion of X and X^c is the cut completion of X.

In view of Theorem 4.2 in [5] we have

3.6. Proposition. *Let X be an abelian lattice ordered group. Then $E(X) = (\overline{X}^{\omega})^c$.*

From the definitions of $\overline{X}^{\omega}$ and X^c we immediately obtain

3.7. Lemma. *Let X and Y be abelian lattice ordered groups such that $X \preceq Y$. Then $\overline{X}^{\omega} \preceq \overline{Y}^{\omega}$ and $X^c \preceq Y^c$.*

Now, 3.6 and 3.7 yield

3.7.1. Corollary. *Let X and Y be as in 3.7. Then $E(X) \preceq E(Y)$.*

Proof of 1.5. We apply the assumptions and the notation as in 1.5.

a) Since $G \preceq_{\text{dist}} G_1$, from 2.5 we conclude that $G \preceq_{\text{dist}} G_1'$ and then, in view of 3.4, we obtain that

$$\mathcal{A} \preceq_{\text{dist}} \mathcal{B} \tag{4}$$

is valid.

b) Suppose that $\mathcal{B}_1$ is an MV-algebra with the underlying set B_1 such that $\mathcal{B} \preceq_{\text{dist}} \mathcal{B}_1$. Then in view of (4) we get

$$\mathcal{A} \preceq_{\text{dist}} \mathcal{B}_1. \tag{4'}$$

There exists an abelian lattice ordered group H with a strong unit u such that

$$\mathcal{B}_1 = \mathcal{A}_0(H, u).$$

According to 2.4 we can suppose, without loss of generality, that H is an extension of G (cf. Fig. 1).

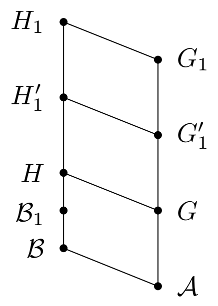

Fig. 1

Put $H_1 = E(H)$. In view of 3.6,

$$G_1 = (\overline{G}^{\omega})^c, \quad H_1 = (\overline{H}^{\omega})^c.$$

According to 3.7.1 we have

$$G_1 \preceq H_1. \tag{5}$$

c) In view of (4') and according to 1.4 we obtain

$$G \preceq_{\text{dist}} H.$$

Further, $H \preceq_{\text{dist}} H_1$, thus $G \preceq_{\text{dist}} H_1$. Hence by applying 2.5 and (5) we conclude that

$$G_1 \preceq_{\text{dist}} H_1.$$

However, since G_1 is distinguished we get $G_1 = H_1$. Then $G_1' = H_1'$, where H_1' is defined analogously to G_1'.

Let $b_1 \in B_1$. Then $b_1 \in H$ and $0 \leqslant b_1 \leqslant u$, whence $b_1 \in H_1' = G_1'$, thus $b_1 \in B$. Therefore $B_1 = B$. We have verified that $\mathcal{B}$ is distinguished. □

3.8. Lemma. *Let $\mathcal{B}^0$ be an MV-algebra which is distinguished. Suppose that H^0 is an abelian lattice ordered group with a weak unit u such that $\mathcal{B}^0 = \mathcal{A}_0(H^0, u)$. Put $H_1^0 = E(H^0)$ and let $G_0^{1\prime}$ be the convex ℓ-subgroup of H_1^0 which is generated by the element u. Then $G_0^{1\prime} = H^0$.*

P r o o f. We have

$$H^0 \preceq G_1^{0\prime} \preceq H_1^0, \quad H^0 \preceq_{\text{dist}} H_1^0,$$

whence in view of 2.5,

$$H_0 \preceq_{\text{dist}} G_1^{0\prime} \preceq_{\text{dist}} H_1^0.$$

Put $\mathcal{B}^1 = \mathcal{A}_0(G_1^{0\prime}, u)$. Then $\mathcal{B}^0 \preceq \mathcal{B}^1$, and according to 1.4,

$$\mathcal{B}^0 \preceq_{\text{dist}} \mathcal{B}^1.$$

Since $\mathcal{B}^0$ is distinguished, we obtain $\mathcal{B}^0 = \mathcal{B}^1$. From this and from 2.3 we infer that $H_0 = G_1^{0\prime}$. □

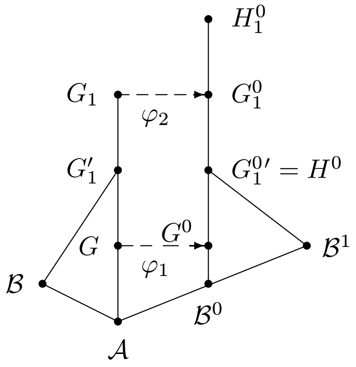

Fig. 2

Again, consider the lattice ordered groups and the MV-algebras from Fig. 1. In the proof of 1.5 we have verified that $H_1' = G_1'$ and $H_1 = G_1$.

Now let $\mathcal{B}^0$ be an arbitrary distinguished extension of $\mathcal{A}$. In view of 2.4 there exists an abelian lattice ordered group H^0 with the strong unit u such that

(i) $\mathcal{B}^0 = \mathcal{A}_0(H^0, u)$,

(ii) there exists an isomorphisms φ_1 of G into H^0 such that $\varphi_1(a) = (a)$ for each $a \in A$ and $G^0 \preceq H^0$, where $G^0 = \varphi_1(G)$. (Cf. Fig. 2).

Put $H_1^0 = E(H^0)$ and $G_1^0 = E(G^0)$. Then according to 3.7, $G_1^0 \preceq H_1^0$. Further, (ii) yields that there exists an isomorphism φ_2 of G_1 onto G_1^0 such that

$$\varphi_2(g) = \varphi_1(g) \quad \text{for each } g \in G.$$

Let $G_1^{0\prime}$ be as in 3.8. Then

$$\varphi_2(G_1') = G_1^{0\prime}.$$

Moreover, in view of 3.8, $G_1^{0\prime} = H^0$, hence $\varphi_2(G_1') = H^0$.

This yields that φ_2 maps isomorphically the MV-algebra $\mathcal{B}$ onto $\mathcal{B}^0$. Also, $\varphi_2(a) = a$ for each $a \in A$. Therefore we have

3.9. Proposition. *Let $\mathcal{A}$ be an MV-algebra. Then the distinguished completion of $\mathcal{A}$ is defined uniquely up to isomorphisms leaving all elements of $\mathcal{A}$ fixed.*

3.10. Proposition. *Let $\mathcal{A}$, G, G_1 and G_1' be as in 1.5. Then the following conditions are equivalent:*

(i) *The MV-algebra $\mathcal{A}$ is distinguished.*

(ii) $G = G_1'$.

P r o o f. In view of 3.8, the implication (i)$\Rightarrow$(ii) is valid. Assume that (ii) holds. Then (under the notation as in 1.5) we have $\mathcal{B} = \mathcal{A}$, whence 1.5 yields that $\mathcal{A}$ is distinguished. $\square$

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